SOME CLEBSCH-GORDAN TYPE LINEARIZATION RELATIONS AND OTHER POLYNOMIAL EXPANSIONS ASSOCIATED WITH A CLASS OF GENERALIZED MULTIPLE HYPERGEOMETRIC SERIES ARISING IN PHYSICAL AND QUANTUM CHEMICAL APPLICATIONS

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ABSTRACT

The multivariable hypergeometric function

$$F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix},$$

studied recently by Niukkanen and Srivastava, provides an interesting and useful unification of the generalized hypergeometric ${}_pF_q$ function of one variable (with p numerator and q denominator parameters), Appell's and Kampé de Fériet's hypergeometric functions of two variables, and Lauricella's hypergeometric functions of n variables, and indeed also of many other classes of hypergeometric series which are encountered naturally in various physical and quantum chemical applications. In fact, as already observed by Srivastava, this multivariable hypergeometric function is an obvious special case of the generalized Lauricella hypergeometric function of n variables, which was first introduced and studied systematically by Srivastava and Daoust. By employing such fruitful connections of this function with much more general multiple hypergeometric functions studied in the literature rather systematically and widely, Srivastava presented several interesting and useful properties and characteristics of this multivariable hypergeometric function, most of which did not appear in the work of Niukkanen. The object of this sequel to Srivastava's work is to derive a number of new expansions (in series of various classes of hypergeometric polynomials) for the multivariable hypergeometric function as useful consequences of substantially more general results available in the literature. Some interesting special cases of the polynomial expansions presented here are also indicated. Furthermore, by suitably specializing some of these polynomial expansions, Clebsch–Gordan type linearization relations are deduced for the products of several Jacobi or Laguerre polynomials.

1. INTRODUCTION, NOTATIONS, AND DEFINITIONS

It is fairly well known that hypergeometric series (and hypergeometric polynomials) in one and more variables arise rather frequently in a wide variety of problems in theoretical physics and applied mathematics, and indeed also in engineering sciences, statistics, and operations research (see, for examples, Srivastava and Karlsson 1985, §1.7, and the various references cited there). In fact, a considerably vast field of physical and quantum mechanical situations (such as Schrödinger's wave mechanics) lead naturally to such hypergeometric polynomials as the Bessel polynomials (see, for generalized hypergeometric ${}_pF_q$ notations, Slater 1966, Chapter 2):

$$y_n(x,\alpha,\beta) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k$$

$$= {}_2F_0\left[\begin{matrix}-n, \ \alpha+n-1; \\ \rule{3em}{0.4pt};\end{matrix} \ -\frac{x}{\beta}\right], \tag{1}$$

and the classical orthogonal polynomials including, for example, the Hermite polynomials:

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, n!}{k!(n-2k)!} (2x)^{n-2k}$$

$$= (2x)^n \, {}_2F_0\left[\begin{matrix}-\tfrac{1}{2}n, \ -\tfrac{1}{2}n+\tfrac{1}{2}; \\ \rule{3em}{0.4pt};\end{matrix} \ -\frac{1}{x^2}\right], \tag{2}$$

the Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \binom{\beta+n}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}$$

$$= \binom{\alpha+n}{n} {}_2F_1\left[\begin{matrix} -n,\ \alpha+\beta+n+1; \\ \alpha+1; \end{matrix}\ \frac{1-x}{2}\right], \tag{3}$$

and the Laguerre polynomials:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{(-x)^k}{k!}$$

$$= \binom{\alpha+n}{n} {}_1F_1\left[\begin{matrix} -n; \\ \alpha+1; \end{matrix}\ x\right], \tag{4}$$

and also to such special cases of the Jacobi polynomials as the Gegenbauer (or ultraspherical) polynomials:

$$C_n^{\nu+\frac{1}{2}}(x) = \binom{\nu+n}{n}^{-1} \binom{2\nu+n}{n} P_n^{(\nu,\nu)}(x), \tag{5}$$

the Legendre polynomials:

$$P_n(x) = P_n^{(0,0)}(x), \tag{6}$$

and the Tchebycheff polynomials (of the first and second kinds):

$$T_n(x) = \begin{bmatrix} n-\frac{1}{2} \\ n \end{bmatrix}^{-1} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{7}$$

and

$$U_n(x) = \frac{1}{2}\begin{bmatrix} n+\frac{1}{2} \\ n+1 \end{bmatrix}^{-1} P_n^{(\frac{1}{2},\frac{1}{2})}(x). \tag{8}$$

Since

$$y_n(x,\alpha,\beta) = n!\left[-\frac{x}{\beta}\right]^n L_n^{(1-\alpha-2n)}\left[\frac{\beta}{x}\right] \tag{9}$$

and

$$H_{2n+\epsilon}(x) = (-1)^n\, 2^{2n+\epsilon}\, n!\, x^\epsilon\, L_n^{(\epsilon-\frac{1}{2})}(x) \qquad (\epsilon = 0 \text{ or } 1), \tag{10}$$

all of the aforementioned orthogonal polynomials are easily recoverable from the classical Jacobi and Laguerre polynomials.

The Jacobi and Laguerre polynomials also play a significant rôle in approximate variational solutions of complex many–electron systems; indeed, in such variational methods, the basis functions are quite frequently connected with these two classes of orthogonal polynomials (for example, rotator functions or the Wigner D functions via the Jacobi polynomials, and hydrogen–like functions via the Laguerre polynomials).

Given two sequences of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$, encountered (for example) in quantum mechanical applications, it is often convenient (and, sometimes, necessary) to express the product $p_\ell(x)p_m(x)$ as a linear combination of the polynomials $p_n(x)$ or $q_n(x)$, that is, to make use of a linearization relation of the Clebsch–Gordan type:

$$P_\ell(x)p_m(x) = \sum_n \lambda_n \, p_n(x) \tag{11}$$

or of the (modified) Clebsch–Gordan type:

$$p_\ell(x)p_m(x) = \sum_n \mu_n \, q_n(x). \tag{12}$$

Much more general linearization relations than those characterized by (11) and (12) above (involving, for example, the products of three or more Jacobi or Laguerre polynomials) are becoming increasingly important in atomic and nuclear shell theories. In particular, the hydrogen–like functions (or, equivalently, the Laguerre polynomials) have been frequently encountered in recent years as perspective basis functions for variational calculations of molecular electron wave–functions. With this point in view, Niukkanen (1985) developed a linearization relation for the product

$$t^k \, L_{m_1}^{(\alpha_1)}(x_1 t) \cdots L_{m_n}^{(\alpha_n)}(x_n t) \tag{13}$$

and discussed a number of particular cases of practical interest. For non–negative integer values of the parameter k (and this is how k is implicitly constrained in Niukkanen's work), the additional factor t^k in (13) seems to serve no purpose whatsoever, since

$$t^k = (-1)^k \, k! \, L_k^{(k)}(t) \qquad (k = 0,1,2,\ldots), \tag{14}$$

which follows readily from the familiar relationship (cf., e.g., Szegö 1975, p. 102, Equation

(5.2.1)):

$$L_n^{(-k)}(t) = (-t)^k \frac{(n-k)!}{n!} L_{n-k}^{(k)}(t) \quad (k = 0,1,\ldots,n). \tag{15}$$

The main object of this paper is to present a number of substantially more general expansion formulas (in series of various classes of generalized hypergeometric polynomials) and to show how linearization relations for products like in (13), but with unrestricted k, would result rather systematically from some of our general polynomial expansions. Many of these general polynomial expansions may be applied also to deduce a fairly wide variety of multiplication theorems involving classical orthogonal polynomials.

We find it to be convenient to begin by introducing a number of useful notations, conventions, and definitions which will be employed throughout this paper. First of all, we set

$$\underset{\sim}{a} = (a^1,\ldots,a^p), \qquad \underset{\sim}{b} = (b^1,\ldots,b^q), \tag{16}$$

and

$$\underset{\sim}{a}_j = (a_j^1,\ldots,a_j^{p_j}), \qquad \underset{\sim}{b}_j = (b_j^1,\ldots,b_j^{q_j}), \tag{17}$$

so that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors with dimensions p and q, respectively, and

$$\underset{\sim}{a}_j \quad \text{and} \quad \underset{\sim}{b}_j \qquad (j = 0,1,\ldots,n)$$

are vectors with dimensions p_j and q_j, respectively. Secondly, in terms of the Pochhammer symbol defined by

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, \text{ if } m = 0, \\ \lambda(\lambda+1)\cdots(\lambda+m-1), \text{ if } m = 1,2,3,\ldots, \end{cases} \tag{18}$$

we let

$$(\underset{\sim}{a})_m = \prod_{k=1}^{p} (a^k)_m, \qquad (\underset{\sim}{b})_m = \prod_{k=1}^{q} (b^k)_m, \tag{19}$$

and

$$(\underset{\sim}{a}_j)_m = \prod_{k=1}^{p_j} (a_j^k)_m, \qquad (\underset{\sim}{b}_j)_m = \prod_{k=1}^{q_j} (b_j^k)_m. \tag{20}$$

Next we define a generalized hypergeometric function of n variables by

$$F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\equiv F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\begin{bmatrix} \underset{\sim}{a}_0 : \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; & \\ & x_1,\ldots,x_n \\ \underset{\sim}{b}_0 : \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n; & \end{bmatrix}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\underset{\sim}{a}_0)_{m_1+\ldots+m_n}}{(\underset{\sim}{b}_0)_{m_1+\ldots+m_n}} \prod_{j=1}^{n} \left\{ \frac{(\underset{\sim}{a}_j)_{m_j}}{(\underset{\sim}{b}_j)_{m_j}} \frac{x_j^{m_j}}{m_j!} \right\} \tag{21}$$

where, for (absolute) convergence of the multiple hypergeometric series,

$$1 + q_0 + q_k - p_0 - p_k \geqq 0 \qquad (k = 1,\ldots,n). \tag{22}$$

It should be remarked in passing that the equality in (22) holds true provided that, in addition, we have <u>either</u>

$$p_0 > q_0 \quad \text{and} \quad |x_1|^{1/(p_0-q_0)} + \ldots + |x_n|^{1/(p_0-q_0)} < 1 \tag{23}$$

<u>or</u>

$$p_0 \leqq q_0 \quad \text{and} \quad \max\left\{|x_1|,\ldots,|x_n|\right\} < 1. \tag{24}$$

Furthermore, under certain parametric constraints, the multiple hypergeometric series in (21) would converge also when

$$x_k = \pm 1 \qquad (k = 1,\ldots,n) \tag{25}$$

together, of course, with the equality in (22).

Many of the recent studies by, for example, Niukkanen (1983, 1984) and Srivastava (1985a, b; 1987) on the multivariable hypergeometric function (21) are motivated by a remarkably vast field of physical and quantum chemical applications of such multiple hypergeometric series [see, for numerous other applications, Exton (1976, Chapters 7 and 9; 1978, Chapter 7), Carlson (1977), Srivastava and Kashyap (1982), and Srivastava and Karlsson (1985, §1.7)]. Indeed, as already observed by Srivastava (1985a), the multivariable hypergeometric function (21) is an obvious special case of the generalized Lauricella hypergeometric function of n variables, which was first introduced and studied by Srivastava and Daoust (1969, p. 454 <u>et</u> <u>seq</u>.), and a book by Srivastava and Karlsson (1985, p. 37 <u>et</u> <u>seq</u>.); note also that a <u>further</u> special case of the multivariable hypergeometric function (21) when

$$p_1 = \ldots = p_n \quad \text{and} \quad q_1 = \ldots = q_n \tag{26}$$

was considered earlier by Karlsson (1973). Srivastava (1985a, b; 1987) employed these fruitful connections of (21) with much more general multiple hypergeometric functions (studied in the literature rather systematically and widely) in order to present several interesting and useful properties of (21) (including, for example, regions of convergence, reduction and summation formulas, expansion and multiplication theorems, generating functions, operational formulas, and Neumann expansions in series of the Bessel functions $J_\nu(z)$ and $I_\nu(z)$ and of their various products), most of which did not appear in the work of Niukkanen (1983, 1984). For the multivariable hypergeometric function (21), we first derive a number of general expansion formulas in series of such classes of generalized hypergeometric polynomials as

$$B_m(t) = {}_{2+p+s}F_{q+r}\left[\begin{matrix} -m,\ \lambda+m,\ \underset{\sim}{a}-\mu,\ \underset{\sim}{d}; \\ \underset{\sim}{b}-\mu,\ \underset{\sim}{c}; \end{matrix}\ t\right], \tag{27}$$

$$B_m^*(t) = {}_{1+p+s}F_{q+r}\left[\begin{matrix} -m,\ \underset{\sim}{a}-\mu,\ \underset{\sim}{d}; \\ \underset{\sim}{b}-\mu,\ \underset{\sim}{c}; \end{matrix}\ t\right], \tag{28}$$

and

$$B_m^{(\alpha)}(t) = {}_{2+p+s}F_{2+q+r}\left[\begin{matrix} -m,\ 1+\beta/(1-\alpha),\ \underset{\sim}{a}-\mu,\ \underset{\sim}{d}; \\ \beta/(1-\alpha),\ \beta-\alpha m+1,\ \underset{\sim}{b}-\mu,\ \underset{\sim}{c}; \end{matrix}\ t\right], \tag{29}$$

where, by analogy with the abbreviations introduced in (16) and (17),

$$\underset{\sim}{c} = (c^1,\ldots,c^r) \quad \text{and} \quad \underset{\sim}{d} = (d^1,\ldots,d^s), \tag{30}$$

so that $\underset{\sim}{c}$ and $\underset{\sim}{d}$ are vectors with dimensions r and s, respectively. Indeed, as already indicated above, each of these expansion formulas contains, as its specialized or limiting cases, scores of linearization relations of the types (11) and (12) for products of several Jacobi or Laguerre polynomials like in (13) considered by Niukkanen (1985). We also prove a general expansion (or multiplication) theorem involving multiple series with essentially arbitrary terms and relate this theorem with various interesting classes of polynomial expansions for the multivariable hypergeometric function (21), which were discussed by Srivastava (1987). Finally, some interesting addition theorems for the classical Laguerre polynomials are considered briefly.

2. GENERAL EXPANSION FORMULAS

For a (real or complex) parameter μ, let $B_m(t)$, $B_m^*(t)$, and $B_m^{(\alpha)}(t)$ denote the hypergeometric polynomials defined by (27), (28), and (29), respectively. Then, from the work of Srivastava and Panda (1974, 1976) containing several general classes of polynomial expansions for multivariable functions defined by multiple series or multiple Mellin–Barnes type contour integrals, it is not difficult to derive the following expansions for the generalized multiple hypergeometric function defined by (21):

$$t^{\mu}\, F^{p+p_0:p_1;\ldots;p_n}_{q+q_0:q_1;\ldots;q_n}\begin{bmatrix} x_1 t \\ \vdots \\ x_n t \end{bmatrix}$$

$$\equiv t^{\mu}\, F^{p+p_0:p_1;\ldots;p_n}_{q+q_0:q_1;\ldots;q_n}\left[\begin{matrix} \underset{\sim}{a},\, \underset{\sim}{a}_0:\, \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \underset{\sim}{b},\, \underset{\sim}{b}_0:\, \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n; \end{matrix}\quad x_1 t,\ldots,x_n t\right]$$

$$= \frac{(\underset{\sim}{a})_{-\mu}\ (\underset{\sim}{c})_{\mu}}{(\underset{\sim}{b})_{-\mu}\ (\underset{\sim}{d})_{\mu}} \sum_{m=0}^{\infty} \frac{(\lambda+2m)\ (-\mu)_m}{m!\,(\lambda+m)_{\mu+1}}\ B_m(t)$$

$$\cdot\ F^{1+r+p_0:p_1;\ldots;p_n}_{2+s+q_0:q_1;\ldots;q_n}\left[\begin{matrix} \mu+1,\ \underset{\sim}{c}+\mu,\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \mu-m+1,\ \lambda+\mu+m+1,\ \underset{\sim}{d}+\mu,\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n; \end{matrix}\ x_1,\ldots,x_n\right], \tag{31}$$

$p+s+1 = q + r$ (and $0 < t \leqq 1$);

$$t^{\mu}\, F^{p+p_0:p_1;\ldots;p_n}_{q+q_0:q_1;\ldots;q_n}\left[\begin{matrix} x_1 t \\ \vdots \\ x_n t \end{matrix}\right]$$

$$= \frac{(\underset{\sim}{a})_{-\mu}\ (\underset{\sim}{c})_{\mu}}{(\underset{\sim}{b})_{-\mu}\ (\underset{\sim}{d})_{\mu}} \sum_{m=0}^{\infty} \frac{(-\mu)_m}{m!}\ B^{*}_m(t)$$

$$\cdot\ F^{1+r+p_0:p_1;\ldots;p_n}_{1+s+q_0:q_1;\ldots;q_n}\left[\begin{matrix} \mu+1,\ \underset{\sim}{c}+\mu,\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \mu-m+1,\ \underset{\sim}{d}+\mu,\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n; \end{matrix}\ x_1,\ldots,x_n\right], \tag{32}$$

$p + s + 1 = q + r$ (and $0 < t < \infty$);

$$t^{\mu}\, F^{p+p_0:p_1;\ldots;p_n}_{q+q_0:q_1;\ldots;q_n}\left[\begin{matrix} x_1 t \\ \vdots \\ x_n t \end{matrix}\right]$$

$$= \frac{\beta(\underset{\sim}{a})_{-\mu}\ (\underset{\sim}{c})_{\mu}}{(\underset{\sim}{b})_{-\mu}\ (\underset{\sim}{d})_{\mu}} \sum_{m=0}^{\infty} \frac{(\beta-\alpha m+1)_{\mu-1}\ (-\mu)_m}{m!}\ B^{(\alpha)}_m(t)$$

$$\cdot\ F^{2+r+p_0:p_1;\ldots;p_n}_{1+s+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\mu+1,\ \mu-\alpha m+\beta,\ \underset{\sim}{c}+\mu,\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \mu-m+1,\ \underset{\sim}{d}+\mu,\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ x_1,\ldots,x_n\right], \tag{33}$$

$p + s + 1 = q + r$ (and $0 < t < \infty$).

It is understood in every case that

$$1 + q_0 + q_k - p_0 - p_k \geqq p - q \qquad (k = 1,\ldots,n), \tag{34}$$

where the equality holds true when the variables t and $x_1, \ldots, x_n$ are appropriately constrained in accordance with (23) and (24). Furthermore, exceptional parameter values which would render either side invalid or undefined are tacitly excluded. Thus, for example, the expansion formula (31) remains valid also for $t = 0$, provided that $\text{Re}(\mu) > 0$; on the other hand, when μ in any of these expansion formulas takes on a non–negative integer value N, the right–hand side will have to be modified by a suitable limit process in order to (tacitly) avoid division by zero for the summation index (cf. Srivastava and Panda 1976, p. 142)

$$m = N,\ N + 1,\ N + 2,\ \ldots \qquad (N = 0,1,2,\ldots). \tag{35}$$

We now turn to certain other classes of polynomial expansions which were applied recently (Srivastava 1987) with a view to deducing various Neumann expansions for the multivariable hypergeometric function (21) in series of the Bessel functions $J_\nu(z)$ and $I_\nu(z)$ or of their suitable products. Many of these classes of multivariable polynomial expansions can indeed be unified (and generalized) to the following form:

THEOREM. For bounded complex coefficients

$$\Lambda(m_1,\ldots,m_n) \text{ and } \Omega_m \quad (\forall m,\ m_i \in \{0,1,2,\ldots\},\ i = 1,\ldots,n), \tag{36}$$

let $\Phi(x_1,\ldots,x_n)$ be defined by

$$\Phi(x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \Lambda(m_1,\ldots,m_n)\Omega_L\, x_1^{m_1} \cdots x_n^{m_n}, \tag{37}$$

where L is given by

$$L = \ell_1 m_1 + \ldots + \ell_n m_n \tag{38}$$

for arbitrary positive integers $\ell_1, \ldots, \ell_n$.

Then

$$\Phi(x_1\omega^{\ell_1},\ldots,x_n\omega^{\ell_n}) = \sum_{m=0}^{\infty} \frac{(-\mu)_m}{(\lambda+m)_m} \frac{\omega^m}{m!} \sum_{k=0}^{\infty} \Omega_{m+k} \frac{(\mu)_k}{(\lambda+2m+1)_k} \frac{\omega^k}{k!}$$

$$\cdot \sum_{\substack{m_1,\ldots,m_n=0 \\ L \leq m}} \frac{(-m)_L\, (\lambda+m)_L\, (\lambda+\mu+m+k+L)_{m-L}}{(\mu-m+1)_L\, (\lambda+\mu+2L+1)_{m-L}} \Lambda(m_1,\ldots,m_n) x_1^{m_1} \cdots x_n^{m_n}, \tag{39}$$

provided that the parameters λ and μ, and the variables $\omega, x_1, \ldots, x_n$, are so constrained that each side of the expansion formula (39) exists.

Proof. Denoting, for convenience, the right–hand side of the expansion formula (39) by $\mathcal{S}$, and applying the definition (18), we observe that

$$\mathcal{S} = \sum_{m,k=0}^{\infty} \sum_{m_1,\ldots,m_n=0}^{L \leq m} \frac{(-\mu)_{m-L}(\mu)_k(\lambda)_{m+L}(\lambda+1)_{2m}(\lambda+\mu)_{2m+k}(\lambda+1)_{2L}}{(m-L)!(\lambda)_{2m}(\lambda+1)_{2m+k}(\lambda+\mu)_{m+k+L}(\lambda+\mu+1)_{m+L}}$$

$$\cdot \frac{\omega^{m+k}}{k!} \Lambda(m_1,\ldots,m_n)\Omega_{m+k}\, x_1^{m_1} \cdots x_n^{m_n}$$

$$= \sum_{k,m_1,\ldots,m_n=0}^{\infty} \frac{(\mu)_k\, \omega^{k+L}}{k!(\lambda+2L+1)_k} \Lambda(m_1,\ldots,m_n)\Omega_{k+L}\, x_1^{m_1} \cdots x_n^{m_n}$$

$$\cdot\, {}_5F_4\left[\begin{matrix}\lambda+2L,\ \tfrac{1}{2}\lambda+L+1, & -\mu,\ \lambda+\mu+k+2L, & -k; \\ & & 1 \\ \tfrac{1}{2}\lambda+L,\ \lambda+\mu+2L+1, & 1-\mu-k,\ \lambda+k+2L+1; & \end{matrix}\right], \tag{40}$$

where L is given by (38).

The hypergeometric ${}_5F_4$ series involved in the last member of (40) is well–poised, and by appealing to a terminating version of a well–known summation theorem (cf. Slater 1966, p. 244, Equation (III.13)) its sum can readily be found to be the Kronecker delta δ_{k0}. This immediately yields

$$\mathcal{S} = \sum_{m_1,\ldots,m_n=0}^{\infty} \Lambda(m_1,\ldots,m_n)\Omega_L\, (x_1\omega^{\ell_1})^{m_1} \cdots (x_n\omega^{\ell_n})^{m_n}, \tag{41}$$

which is precisely the left–hand side of (39) as given by (37).

This evidently completes the proof of the theorem under the hypothesis that the various interchanges of the order of summation are permissible by absolute convergence of the series involved, and (by the principle of analytic continuation) the final result (39) holds true as asserted above.

In view of the principle of confluence exhibited, for example, by

$$\lim_{\lambda\to\infty}\left\{(\lambda)_m\left[\frac{z}{\lambda}\right]^m\right\} = z^m = \lim_{\mu\to\infty}\left\{\frac{(\mu z)^m}{(\mu)_m}\right\} \tag{42}$$

for bounded z and $m = 0, 1, 2, \ldots,$ our expansion formula (39) would yield a known multivariable polynomial expansion (Srivastava 1981, p. 300, Equation (1.4)) if in (39) we replace

$$\omega \text{ by } \omega/\mu \text{ and } x_i \text{ by } x_i/\mu^{\ell_i} \quad (i = 1,\ldots,n),$$

and then let $\mu \to \infty$. Furthermore, a limiting case of (39) when ω is replaced by $\lambda\omega/\mu$, and

$$x_i \text{ by } x_i(\mu/\lambda)^{\ell_i} \quad (i = 1,\ldots,n), \text{ and } \lambda, \mu \to \infty$$

yields another known multivariable polynomial expansion (Srivastava 1981, p. 300, Equation (1.5)). Yet another limiting case of the expansion formula (39) when ω is replaced by $\lambda\omega$, and

$$x_i \text{ by } x_i/\lambda^{\ell_i} \quad (i = 1,\ldots,n), \text{ and } \lambda \to \infty$$

would lead us to the following multivariable polynomial expansion:

$$\Phi(x_1\omega^{\ell_1},\ldots,x_n\omega^{\ell_n}) = \sum_{m=0}^{\infty} (-\mu)_m \frac{\omega^m}{m!} \sum_{k=0}^{\infty} \Omega_{m+k} (\mu)_k \frac{\omega^k}{k!}$$

$$\cdot \sum_{m_1,\ldots,m_n=0}^{L \leq m} \frac{(-m)_L}{(\mu-m+1)_L} \Lambda(m_1,\ldots,m_n) x_1^{m_1} \ldots x_n^{m_n}, \tag{43}$$

which naturally provides an interesting generalization of the aforementioned known multivariable polynomial expansion (Srivastava 1981, p. 300, Equation (1.5)).

In terms of the multivariable hypergeometric function defined by (21), this last result (43) with suitable special values of the coefficients listed in (36) and with

$$\ell_i = \ell \;\; (i = 1,\ldots,n;\; \ell = 1,2,3,\ldots)$$

yields the polynomial expansion:

$$F^{\ell p+p_0:p_1;\ldots;p_n}_{\ell q+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;\underset{\sim}{a}),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \Delta(\ell;\underset{\sim}{b}),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\; x_1\omega^\ell \ell^{\ell(p-q)},\ldots,x_n\omega^\ell \ell^{\ell(p-q)}\right]$$

$$= \sum_{m=0}^{\infty} (-\mu)_m \Gamma_m(\underset{\sim}{a},\underset{\sim}{c};\underset{\sim}{b},\underset{\sim}{d}) \frac{\omega^m}{m!} {}_{1+p+r}F_{q+s}\left[\begin{matrix}\mu,\ \underset{\sim}{a}+m,\ \underset{\sim}{c}+m; \\ \\ \underset{\sim}{b}+m,\ \underset{\sim}{d}+m;\end{matrix}\; \omega\right]$$

$$\cdot F^{\ell(1+s)+p_0:p_1;\ldots;p_n}_{\ell(1+r)+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\underset{\sim}{d}),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \Delta(\ell;\mu-m+1),\ \Delta(\ell;\underset{\sim}{c}),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\; x_1\ell^{\ell(s-r)},\ldots,x_n\ell^{\ell(s-r)}\right], \tag{44}$$

where $p + r \leqq q + s$ (the equality holds true when $|\omega| < 1$),

$$1 + q_0 + q_k - p_0 - p_k \geqq \ell(p-q) \qquad (k = 1,\ldots,n), \tag{45}$$

$\Delta(\ell;\lambda)$ abbreviates the array of ℓ parameters:

$$\frac{\lambda}{\ell}, \frac{\lambda+1}{\ell}, \ldots, \frac{\lambda+\ell-1}{\ell} \qquad (\ell = 1,2,3,\ldots)$$

so that, for example, $\Delta(\ell;\underset{\sim}{a})$ abbreviates the array of ℓp parameters (cf. Equation (16)):

$$\frac{a^j}{\ell}, \frac{a^j+1}{\ell}, \ldots, \frac{a^j+\ell-1}{\ell} \qquad (j = 1,\ldots,p;\ \ell = 1,2,3,\ldots),$$

and

$$\Gamma_m(\underset{\sim}{a},\underset{\sim}{c};\underset{\sim}{b},\underset{\sim}{d}) = \frac{(\underset{\sim}{a})_m\,(\underset{\sim}{c})_m}{(\underset{\sim}{b})_m\,(\underset{\sim}{d})_m} \qquad (m = 0,1,2,\ldots), \tag{46}$$

it being understood that the equality in (45) holds true when the variables $|\omega|$ and $|x_1|, \ldots, |x_n|$ are appropriately constrained in accordance with (23) and (24).

Just as its companions considered elsewhere (cf. Srivastava 1987, p. 850, Equations (16), (17), and (18)), the expansion formula (44) can be suitably applied in order to deduce for the multivariable hypergeometric function (21) a number of additional Neumann expansions in series of the Bessel functions:

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left[\begin{matrix} \text{——}; \\ \nu+1; \end{matrix}\ -\tfrac{1}{4}z^2\right] \tag{47a}$$

$$= \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, e^{\pm iz}\, {}_1F_1\left[\begin{matrix} \nu+\frac{1}{2}; \\ 2\nu+1; \end{matrix}\ \mp 2iz\right] \tag{47b}$$

and

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left[\begin{matrix} \text{———}; \\ \nu+1; \end{matrix}\ \tfrac{1}{4}z^2\right] \tag{48a}$$

$$= \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, e^{\pm z}\ {}_1F_1\left[\begin{matrix} \nu+\frac{1}{2}; \\ 2\nu+1; \end{matrix}\ \mp 2z\right], \tag{48b}$$

and of their such products as (cf., e.g., Watson 1944, p. 147, Equation (1))

$$J_\mu(z)J_\nu(z) = \frac{(\frac{1}{2}z)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\, {}_2F_3\left[\begin{matrix} \Delta(2;\mu+\nu+1); \\ \mu+1,\ \nu+1,\ \mu+\nu+1; \end{matrix}\ -z^2\right] \tag{49}$$

or, equivalently,

$$I_\mu(z)I_\nu(z) = \frac{(\frac{1}{2}z)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\, {}_2F_3\left[\begin{matrix} \Delta(2;\mu+\nu+1); \\ \mu+1,\ \nu+1,\ \mu+\nu+1; \end{matrix}\ z^2\right], \tag{50}$$

and (cf. Luke 1962, p. 25, Equation (20))

$$J_\nu(z)I_\nu(z) = \frac{(\frac{1}{2}z)^{2\nu}}{\{\Gamma(\nu+1)\}^2}\ {}_0F_3\left[\begin{matrix} \text{————}; \\ \Delta(2;\nu+1),\ \nu+1; \end{matrix}\ -\frac{z^4}{64}\right]. \tag{51}$$

The details are provided reasonably fully by Srivastava (1987) and are, therefore, omitted here.

3. APPLICATIONS TO JACOBI AND LAGUERRE POLYNOMIALS

Each of our expansion formulas (31), (32), and (33), and their specialized or limiting cases indicated in the preceding section, can be applied to derive scores of linearization relations analogous to (11) and (12) for the products of several generalized hypergeometric polynomials of the types defined by (27), (28), and (29). In particular, for the classical Jacobi polynomials defined by (3), the expansion formula (31) with

$$\begin{cases} \lambda=\alpha+\beta+1\,, \quad p=q=p_0=q_0=r-1=s=0, \quad c^1=\alpha+1, \\ p_k-1=q_k=1, \quad a_k^1=-m_k, \quad a_k^2=\alpha_k+\beta_k+m_k+1\,, \quad b_k^1=\alpha_k+1 \; (k=1,\ldots,n) \end{cases}$$

yields the following linearization relation of the Clebsch–Gordan type (11):

$$t^\mu P_{m_1}^{(\alpha_1,\beta_1)}(1-2x_1t)\cdots P_{m_n}^{(\alpha_n,\beta_n)}(1-2x_nt)$$

$$= (\alpha+1)_\mu \binom{\alpha_1+m_1}{m_1}\cdots\binom{\alpha_n+m_n}{m_n}\sum_{m=0}^{\infty}\frac{(\alpha+\beta+2m+1)\,(-\mu)_m}{(\alpha+1)_m(\alpha+\beta+m+1)_{\mu+1}}\,P_m^{(\alpha,\beta)}(1-2t)$$

$$\cdot F^{2:2;\ldots;2}_{2:1;\ldots;1}\left[\begin{matrix} \mu+1\,, \; \alpha+\mu+1: -m_1,\, \alpha_1+\beta_1+m_1+1;\ldots;-m_n,\, \alpha_n+\beta_n+m_n+1; \\ \mu-m+1,\; \alpha+\beta+\mu+m+2: \quad \alpha_1+1;\ldots; \quad \alpha_n+1; \end{matrix}\; x_1,\ldots,x_n\right]. \tag{52}$$

In a similar manner, if we apply the definition (4) for the classical Laguerre polynomials, the expansion formula (32) with

$$\begin{cases} p=q=p_0=q_0=r-1=s=0, \quad c^1=\alpha+1, \\ p_k=q_k=1, \quad a_k^1=-m_k, \quad b_k^1=\alpha_k+1 \quad (k=1,\ldots,n) \end{cases}$$

would reduce to the linearization relation:

$$t^{\mu} L_{m_1}^{(\alpha_1)}(x_1 t) \cdots L_{m_n}^{(\alpha_n)}(x_n t)$$

$$= (\alpha+1)_{\mu} \begin{bmatrix} \alpha_1+m_1 \\ m_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n+m_n \\ m_n \end{bmatrix} \sum_{m=0}^{\infty} \frac{(-\mu)_m}{(\alpha+1)_m} L_m^{(\alpha)}(t)$$

$$\cdot F_{1:1;\ldots;1}^{2:1;\ldots;1} \begin{bmatrix} \mu+1,\ \alpha+\mu+1: & -m_1;\ldots; & -m_n; & \\ & & & x_1,\ldots,x_n \\ \mu-m+1: & \alpha_1+1;\ldots; & \alpha_n+1; & \end{bmatrix}, \tag{53}$$

which is also of the Clebsch–Gordan type (11).

The multivariable hypergeometric polynomials involved in the coefficients of each of the (Clebsch–Gordan type) linearization relations (52) and (53) can easily be rewritten in descending powers of $x_1, \ldots, x_n$. Notice also that, in view of the familiar limit relationship (cf. Szegö 1975, p. 103, Equation (5.3.4))

$$L_n^{(\alpha)}(x) = \lim_{\beta \to \infty} P_n^{(\alpha,\beta)}\left[1 - \frac{2x}{\beta}\right], \tag{54}$$

the linearization relation (53) would follow directly from (52) when we replace t by t/β, and

$$x_k \text{ by } \beta x_k/\beta_k \qquad (k = 1,...,n),$$

and let $\beta, \beta_1, ..., \beta_n \to \infty$. More importantly, since

$$ {}_2F_1\left[\begin{matrix} \alpha+\mu+M+1, & -m; \\ & & 1 \\ & \alpha+1; \end{matrix}\right] = \frac{(-\mu)_m \ (\mu+1)_M}{(\alpha+1)_m \ (\mu-m+1)_M} \qquad (m,M=0,1,2,...), \ (55)$$

by a well–known special case of the Gaussian summation theorem (cf. Slater 1966, p. 243, Equation (III.4)), it is not difficult to rewrite the linearization relation (53) in the elegant form:

$$t^\mu L_{m_1}^{(\alpha_1)}(x_1 t) \cdots L_{m_n}^{(\alpha_n)}(x_n t) = \sum_{m=0}^{\infty} \gamma_m(\mu;x_1,...,x_n) L_n^{(\alpha)}(t), \qquad (56)$$

where, for convenience,

$$\gamma_m(\mu;x_1,...,x_n) = (\alpha+1)_\mu \begin{bmatrix} \alpha_1+m_1 \\ m_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n+m_n \\ m_n \end{bmatrix}$$

$$\cdot F_A^{(n+1)}\left[\alpha+\mu+1, -m_1,...,-m_n, -m; \alpha_1+1,...,\alpha_n+1, \alpha+1; x_1,...,x_n, 1\right] \qquad (57)$$

in terms of one of Lauricella's hypergeometric functions of $n + 1$ variables (cf. Lauricella 1893, p. 113; see also Appell et Kampé de Fériet 1926, p. 114, Equation (1)).

The linearization relation (53) corresponding to the restricted product in (13) was given by Niukkanen (1985). On the other hand, the equivalent expansion (56) with $\mu = 0$

immediately yields the following result due to Erdélyi (1938):

$$L_{m_1}^{(\alpha_1)}(x_1t) \cdots L_{m_n}^{(\alpha_n)}(x_nt) = \begin{bmatrix}\alpha_1+m_1\\ m_1\end{bmatrix} \cdots \begin{bmatrix}\alpha_n+m_n\\ m_n\end{bmatrix} \sum_{m=0}^{m_1+\ldots+m_n} L_m^{(\alpha)}(t)$$

$$\cdot F_A^{(n+1)}\left[\alpha+1, -m_1,\ldots,-m_n, -m; \alpha_1+1,\ldots,\alpha_n+1, \alpha+1; x_1,\ldots,x_n,1\right], \tag{58}$$

which has since been reproduced in numerous subsequent works.

We should like to mention here that Erdélyi's linearization relation (58) can be extended fairly easily to an expansion (or multiplication) theorem for a class of multivariable hypergeometric polynomials in the form:

$$F_{q:\ q_1;\ldots;\ q_n}^{p:1+p_1;\ldots;1+p_n}\left[\begin{matrix}\underset{\sim}{a}: -m_1, \underset{\sim}{a}_1;\ldots;-m_n, \underset{\sim}{a}_n;\\ \underset{\sim}{b}: \qquad \underset{\sim}{b}_1;\ldots; \qquad \underset{\sim}{b}_n;\end{matrix}\ x_1t,\ldots,x_nt\right]$$

$$= \sum_{m=0}^{m_1+\ldots+m_n} \begin{bmatrix}\alpha+m-1\\ m\end{bmatrix} {}_{1+p}F_{1+q}\left[\begin{matrix}-m, \underset{\sim}{a};\\ \alpha, \underset{\sim}{b};\end{matrix}\ t\right]$$

$$\cdot F_{0:\ q_1;\ldots;\ q_n;1}^{1:1+p_1;\ldots;1+p_n;1}\left[\begin{matrix}\alpha: -m_1, \underset{\sim}{a}_1;\ldots;-m_n, \underset{\sim}{a}_n;\\ -: \qquad \underset{\sim}{b}_1;\ldots; \qquad \underset{\sim}{b}_n;\end{matrix}\ x_1,\ldots,x_n, 1\right], \tag{59}$$

which is derivable from a more general result involving the generalized Lauricella function (cf. Srivastava 1971, p. 114; see also Srivastava and Manocha 1984, p. 262, Problem 5). As a matter of fact, in view of the hypergeometric identity (55), this last result (59) corresponds to an obvious (terminating) version of the special case $\mu = 0$ of the following consequence of our expansion formula (32):

$$t^{\mu}\, F^{p:p_1;\ldots;p_n}_{q:q_1;\ldots;q_n}\left[\begin{matrix} \underset{\sim}{a}:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \underset{\sim}{b}:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n; \end{matrix}\ x_1 t,\ldots,x_n t\right]$$

$$= \frac{(\alpha)_{\mu}\,(\underset{\sim}{a})_{-\mu}}{(\underset{\sim}{b})_{-\mu}} \sum_{m=0}^{\infty} \begin{bmatrix}\alpha+m-1 \\ m\end{bmatrix} {}_{1+p}F_{1+q}\left[\begin{matrix} -m,\ \underset{\sim}{a}-\mu; \\ \\ \alpha,\ \underset{\sim}{b}-\mu; \end{matrix}\ t\right]$$

$$\cdot\ F^{1:p_1;\ldots;p_n;1}_{0:q_1;\ldots;q_n;1}\left[\begin{matrix} \alpha+\mu:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\ -m; \\ \\ \text{——}:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\quad \alpha; \end{matrix}\ x_1,\ldots,x_n,\ 1\right], \tag{60}$$

which evidently holds true for an essentially arbitrary parameter μ.

In conclusion, we remark that the literature contains a considerably wide variety of addition theorems for various special functions (cf., e.g., Srivastava et al. 1983). In the case of the classical Laguerre polynomials, a few of the available addition theorems were discussed by Niukkanen (1985). A particularly elegant result for these polynomials is the following addition theorem of Srivastava (1972, p. 6, Equation (10)):

$$L_m^{(\alpha)}(x)\, L_m^{(\beta)}(y) = \begin{bmatrix}\alpha+m \\ m\end{bmatrix}\begin{bmatrix}\beta+m \\ m\end{bmatrix}\begin{bmatrix}\gamma+m \\ m\end{bmatrix}^{-1}$$

$$\cdot \sum_{r,\,s=0}^{r+s\leq m} \frac{(\gamma+1)_r\ (\beta-\gamma)_s\ x^r\ y^{r+s}}{r!\ \ s!\ (\alpha+1)_r\ (\beta+1)_{r+s}}\ \xi_{rs}\ L_{m-r-s}^{(\gamma+2r+s)}(x+y), \tag{61}$$

where, for convenience,

$$\xi_{rs} = {}_3F_2\left[\begin{matrix} -s\,,\ \alpha-\gamma,\ -\beta-r-s; \\ \\ \alpha+r+1\,,\ \gamma-\beta-s+1; \end{matrix} \ -\frac{x}{y}\right]. \tag{62}$$

It may be of interest to observe from (62) that $\xi_{rs} = 1$ when $\gamma = \alpha$, and thus (61) reduces immediately to the significantly simpler form:

$$L_m^{(\alpha)}(x)\,L_m^{(\beta)}(y) = \begin{bmatrix}\beta+m \\ m\end{bmatrix} \sum_{r,s=0}^{r+s\leq m} \frac{(\beta-\alpha)_s\, x^r\, y^{r+s}}{r!\ s!\ (\beta+1)_{r+s}}\, L_{m-r-s}^{(\alpha+2r+s)}(x+y), \tag{63}$$

whose special case when $\beta = \alpha$ would yield one of several such addition theorems considered extensively by Bailey (1936, p. 219, Equation (5.4); 1939, p. 60, Equation (1.1)).

Since

$${}_1F_1\left[\begin{matrix}\alpha; \\ \beta;\end{matrix}\ z\right] = e^z\, {}_1F_1\left[\begin{matrix}\beta-\alpha; \\ \beta;\end{matrix}\ -z\right], \tag{64}$$

by Kummer's first theorem (cf., e.g., Srivastava and Kashyap 1982, p. 24, Equation (7)), the definition (4) may be rewritten at once as

$$L_n^{(\alpha)}(x) = \begin{bmatrix}\alpha+n \\ n\end{bmatrix} e^x\, {}_1F_1\left[\begin{matrix}\alpha+n+1; \\ \alpha+1;\end{matrix}\ -x\right]. \tag{65}$$

Now recall an expansion formula recorded already by Srivastava (1985a, p. L230, Equation (20)) which indeed follows readily from a more general result due to Srivastava and Daoust (1969, p. 456, Equation (4.3)). Making use of (65), it is fairly straightforward to deduce from the aforementioned expansion formula the following linearization relation for the

Laguerre polynomials:

$$L_{m_1}^{(\alpha_1)}(x_1t)\cdots L_{m_n}^{(\alpha_n)}(x_nt)=\binom{\alpha_1+m_1}{m_1}\cdots\binom{\alpha_n+m_n}{m_n}\exp[-(x-x_1-\cdots-x_n)t]$$

$$=\sum_{m=0}^{s+M}\frac{\alpha+2m}{\alpha+m}\binom{\alpha+s+M}{s+M-m}^{-1}\frac{(xt)^m}{m!}L_{s+M-m}^{(\alpha+2m)}(xt)$$

$$\cdot F_{1:1;\ldots;1}^{2:1;\ldots;1}\left[\begin{matrix}-m,\quad \alpha+m:\ \alpha_1+m_1+1;\ldots;\ \alpha_n+m_n+1;\\ \alpha+s+M+1:\qquad \alpha_1+1;\ldots;\qquad \alpha_n+1;\end{matrix}\ \frac{x_1}{x},\ldots,\frac{x_n}{x}\right], \tag{66}$$

where, for convenience,

$$M=m_1+\cdots+m_n \quad\text{and}\quad s=\alpha_1+\cdots+\alpha_n-\alpha+1,$$

α being so constrained that s is a non–negative integer. Formula (66) with $n=2$ would provide the corrected[1] (and modified) version of a result proved, in a markedly different and involved manner, by Niukkanen (1985, p. 1413, Equation (48)). Furthermore, (66) with

$$x=x_1+\cdots+x_n$$

would immediately yield a class of addition theorems for Laguerre polynomials, which (for

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[1]Niukkanen's error can be traced back to the missing factor $(\frac{1}{2}z)^{\gamma-\alpha_1-\alpha_2}$ on the left–hand side of a well–known result reproduced and used incorrectly by him (cf. Niukkanen 1985, p. 1412, Equation (47)).

n = 2) would correspond essentially to the corrected version of another known result (Niukkanen 1985, p. 1414, Equation (52)).

Numerous further applications of each one of the expansion (or multiplication) formulas and linearization relations presented in this paper to various other families of orthogonal polynomials (or to simpler special functions of one and more variables) can indeed be given in a manner outlined above fairly completely. Moreover, these multivariable polynomial expansions are also capable of yielding various desired linearization relations of the modified Clebsch–Gordan type (12) for each of the classical orthogonal polynomials as well as for other hypergeometric polynomials considered in this paper.

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