A FACTORIZATION THEOREM FOR MATRICES

by

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ABSTRACT

It is shown that a nonscalar invertible square matrix can be written as a product of two square matrices with prescribed eigenvalues subject only to the obvious determinant condition. As corollaries, we give short proofs of some known results such as Ballantine's characterization of products of four or five positive definite matrices, the commutator theorem of Shoda-Thompson for fields with sufficiently many elements and other results.

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INTRODUCTION

Let A be an invertible square matrix over a field F. When can A be factored as a product of square matrices B and C with prescribed eigenvalues, counting multiplicity? Obvious necessary conditions are that A be nonscalar and that the product of the prescribed eigenvalues of B and C be equal to $\det A$. Our main result is to prove that these necessary conditions are also sufficient.

Using our factorization theorem, we give short proofs of Ballantine's results ([1] and [2]) which state that every nonscalar real or complex matrix with positive determinant is the product of four positive-definite matrices and that every matrix with positive determinant is the product of five positive-definite matrices. We also give short proofs of special cases (when the underlying field contains sufficiently many elements) of the Shoda-Thompson commutator theorem (if $\det A = 1$, then A is a commutator) and the Gustafson-Halmos-Radjavi Theorem (if $\det A = \pm 1$, then A is the product of four involutions).

We now fix some notation and terminology. All matrices in this paper are square matrices. As usual $M_n(F)$ denotes the set of all $n \times n$ matrices over a field F. The determinant of a matrix is denoted by $\det A$. The eigenvalues of a matrix are always repeated according to algebraic multiplicity, i.e. multiplicity as zeros of the characteristic polynomial.

1. THE MAIN THEOREM

THEOREM 1. *Let* A *be a nonscalar invertible* $n \times n$ *matrix over a field* F *and let* β_j *and* γ_j $(1 \le j \le n)$ *be elements of* F *such that* $\prod_{j=1}^{n} \beta_j \gamma_j = \det A$. *There exists* $n \times n$ *matrices* B *and* C *with eigenvalues* $\beta_1, \ldots, \beta_n$ *and*

$\gamma_1, \ldots, \gamma_n$ respectively such that $A = BC$. Furthermore B and C can be chosen so that B is lower triangularizable and C is simultaneously upper triangularizable.

Proof. We use induction. The result is vacuously true for $n = 1$. Now let us assume that the conclusion of the theorem is true for all square matrices of size less than n, $n \geq 2$, and let A, β_j and γ_j be as in the statement of the theorem. First, we show that A is similar to a matrix whose $(1,1)$ entry is $\beta_1\gamma_1$. To prove this, let e_1 be a vector which is not an eigenvector of $A - \beta_1\gamma_1 I$ and then choose an ordered basis $\mathcal{B}$ of F^n whose first two members are e_1 and $e_2 = (A - \beta_1\gamma_1 I)e_1$. If $\tilde{A}$ is the linear transformation on F^n given by $\tilde{A}(x) = Ax$, then the matrix A_1 of $\tilde{A}$ relative to the basis $\mathcal{B}$ has a first column: $(\beta_1\gamma_1, 1, 0, \ldots 0)^t$. We conclude that A is similar to the matrix

$$A_1 = \begin{pmatrix} \beta_1\gamma_1 & y' \\ x & R \end{pmatrix}$$

where x is a nonzero column vector, y' is a row vector and $R \in M_{n-1}(F)$.

In the case $n = 2$, we have that x, y' and $R = r$ are merely elements of F. Using the fact that $\det A_1 = \beta_1\beta_2\gamma_1\gamma_2$, we have

$$\begin{pmatrix} \beta_1\gamma_1 & y' \\ x & r \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ \gamma_1^{-1}x & \beta_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \beta_1^{-1}y' \\ 0 & \gamma_2 \end{pmatrix}.$$

This proves the conclusion of the theorem for $n = 2$.

We now assume that $n \geq 3$. We will show that the matrix A_1 is similar to a matrix of the form

$$A_2 = \begin{pmatrix} \beta_1\gamma_1 & z' \\ x & S \end{pmatrix}$$

where $S - \beta_1^{-1}\gamma_1^{-1}\, xz'$ is not a scalar. If $R - \beta_1^{-1}\gamma_1^{-1}\, xy'$ is not a scalar, there is nothing to prove, so we assume that $R - \beta_1^{-1}\gamma_1^{-1}\, xy' = \alpha I$ for some $\alpha \in F$. Since rank $A > 2$, the linear span of the columns of R is not contained in the linear span of $\{x\}$ and so there exists a row vector w' of size $n - 1$ such that $w'x = 0$ but $w'R \neq 0$. Let $P = \begin{pmatrix} 1 & w' \\ 0 & I \end{pmatrix}$ so

$$P^{-1}A_1P = \begin{pmatrix} \beta_1\gamma_1 & z' \\ x & S \end{pmatrix}$$

where $z' = y' + \beta_1\gamma_1 w' - w'R$ and $S = R + xw'$ and so

$$\begin{aligned} S - \beta_1^{-1}\gamma_1^{-1}\, xz' &= R - \beta_1^{-1}\gamma_1^{-1}\, xy' + \beta_1^{-1}\gamma_1^{-1}\, xw'R \\ &= \alpha I + \beta_1^{-1}\gamma_1^{-1}\, xw'R. \end{aligned}$$

Since $x \neq 0$ and $w'R \neq 0$, the matrix $\beta_1\gamma_1\, xw'R$ is of rank one and so $S - \beta_1^{-1}\gamma_1^{-1}\, xz'$ is not a scalar.

We now apply the induction hypothesis to the $(n-1) \times (n-1)$ matrix $S - \beta_1^{-1}\gamma_1^{-1}\, xz'$. Since

$$\begin{pmatrix} \beta_1\gamma_1 & z' \\ x & S \end{pmatrix} = \begin{pmatrix} \beta_1\gamma_1 & 0 \\ x & S - \beta_1^{-1}\gamma_1^{-1}\, xz' \end{pmatrix} \begin{pmatrix} 1 & \beta_1^{-1}\gamma_1^{-1}\, z' \\ 0 & I \end{pmatrix},$$

we have

$$\det\left(S - \beta_1^{-1}\gamma_1^{-1}\, xz'\right) = \beta_1^{-1}\gamma_1^{-1} \det A = \prod_{j=2}^{n} \beta_j\gamma_j .$$

By the induction hypothesis, there exist B_0 and C_0 in $M_{n-1}(F)$ such that the eigenvalues of B_0 (respectively C_0) are $\beta_2, \ldots, \beta_n$ (respectively $\gamma_2, \ldots, \gamma_n$) and $S - \beta_1^{-1}\gamma_1^{-1} xz' = B_0C_0$. It follows that

$$A_2 = \begin{bmatrix} \beta_1 & 0 \\ \gamma_1^{-1}x & B_0 \end{bmatrix} \begin{bmatrix} \gamma_1 & \beta_1^{-1} z' \\ 0 & C_0 \end{bmatrix}.$$

Again, by the induction hypothesis, there exists an invertible matrix $Q_0 \in M_{n-1}(F)$ such that $B_1 := Q_0^{-1}B_0Q_0$ is lower triangular and $C_1 := Q_0^{-1}C_0Q_0$ is upper triangular. If $Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_0 \end{pmatrix}$, then

$$Q^{-1}A_2Q = \begin{pmatrix} \beta_1 & 0 \\ \xi & B_1 \end{pmatrix} \begin{pmatrix} \gamma_1 & \eta' \\ 0 & C_1 \end{pmatrix}$$

where ξ is a column vector and η' is a row vector. Since $Q^{-1}A_2Q$ is similar to A, the conclusion of the theorem has been verified. □

A special case of Theorem 1 (with $F = C$ and A cyclic) is contained in [5; Theorem 1]. It was also shown in [5; Theorem 2] that every square matrix over C with determinant 1 is a product of three unipotent matrices. (A square matrix is called unipotent if it is of the form $I + N$ where N is nilpotent.) The following generalization of [5; Theorem 2] is a special case of Theorem 1 (with $\beta_j = \gamma_j = 1$ for $1 \le j \le n$).

COROLLARY. *Let* $A \in M_n(F)$ *and assume that* $\det A = 1$. *Then* A *is a product of three unipotent matrices. If* A *is nonscalar, then it is a product of two unipotent matrices.*

REMARK. An additive version of Theorem 1 is contained in Fillmore [4].

2. PRODUCTS OF POSITIVE DEFINITE MATRICES

In [1] and [2], Ballantine proved that every real or complex $n \times n$ matrix A with positive determinant is a product of five positive-definite matrices and, unless A is a nonpositive scalar, it is a product of four positive-definite matrices, (see also Taussky [9]). We give very short proofs of these facts using our Theorem 1. (Ballantine also characterized products of three positive definite matrices.)

THEOREM 2 (Ballantine). _Let_ A _be a real or complex_ $n \times n$ _matrix_. _Then_

(a) A _is a product of four positive-definite matrices if and only if_ $\det A > 0$ _and_ A _is not a scalar_ αI _where_ α _is not positive_.

(b) A _is a product of five positive-definite matrices if and only if_ $\det A > 0$.

Proof. (a) Assume that $\det A > 0$ and A is not a scalar. (The case $A = \alpha I$, $\alpha > 0$ is trivial.) There exist distinct positive numbers $\beta_1, \ldots, \beta_n$ and distinct positive numbers $\gamma_1, \ldots, \gamma_n$ such that $\prod_{i=1}^{n} \beta_i\gamma_i = \det A$. By Theorem 1, there exists an $n \times n$ matrix B with eivenvalues $\beta_1, \ldots, \beta_n$ and an $n \times n$ matrix C with eigenvalues $\gamma_1, \ldots, \gamma_n$ such that $A = BC$. Each of B and C is similar to a positive diagonal matrix, so $B = R^{-1}DR$ where R is invertible and D is diagonal and positive. Therefore $B = R^{-1}R^{*-1}R^{*}DR$, a product of the two positive definite matrices $R^{-1}R^{*-1}$ and $R^{*}DR$. Similarly C can be written as a product of two positive-definite matrices and hence A is a product of four positive-definite matrices. (The fact that a matrix is a product of two positive-definite matrices if and only if it is similar to a positive-definite matrix has been observed by Taussky [9].)

Conversely, if $A = P_1P_2P_3P_4$ where each P_j, $1 \le j \le 4$, is positive-definite, then obviously $\det A > 0$. Furthermore, if A is a scalar αI, then $P_1P_2 = \alpha P_4^{-1}P_3^{-1}$. However, $P_1P_2 = P_1^{1/2}\left(P_1^{1/2}P_2P_1^{1/2}\right)P_1^{-1/2}$, so P_1P_2 is similar to a positive-definite matrix and hence has positive spectrum. Similarly $P_4^{-1}P_3^{-1}$ has positive spectrum. Therefore $\alpha > 0$.

(b) We need only consider the case $A = \alpha I$. Let P be a nonscalar positive-definite matrix, and write $A = \alpha P^{-1}P$. Applying the result of part (a) to αP^{-1}, we get that A is a product of five positive-definite matrices. □

3. COMMUTATORS

Let $GL(n,F)$ denote the group of invertible $n \times n$ matrices over a field F and let $SL(n,F)$ be the subgroup of matrices with determinant 1. Shoda [8] showed that if F is algebraically closed, then the set of commutators of $GL(n,F)$, i.e. $\left\{BCB^{-1}C^{-1} : B, C \in GL(n,F)\right\}$ coincides with $SL(n,F)$. Thompson [10] showed that Shoda's theorem holds for all fields F with the exception of the case when $n = 2$ and $|F| = 2$. He also characterized the commutators of matrices with prescribed determinant [11]. (See [3] for related results.) We show that when F contains sufficiently many elements, these results follow easily from Theorem 1. (See also [6] for another short proof.)

THEOREM 3 (Shoda-Thompson). *Let* $A \in SL(n,F)$.

(a) *If* F *has at least* $n + 1$ *elements, then* A *is a commutator of matrices in* $GL(n,F)$.

(b) *If* F *has at least* $n + 2$ *elements and* A *is nonscalar, then* A *is a commutator of matrices in* $SL(n,F)$.

(c) _If_ F _has at least_ $n + 3$ _elements and_ A _is nonscalar, then_ A _is a commutator of matrices with arbitrarily prescribed nonzero determinants_.

Proof. (a) If A is not a scalar, then, by Theorem 1, we can write A as a product BD where B has distinct nonzero eigenvalues $\beta_1, \ldots, \beta_n$ and where D has eigenvalues $\beta_1^{-1}, \ldots, \beta_n^{-1}$. So D is similar to B^{-1}, i.e. $D = CB^{-1}C^{-1}$ and hence $A = BCB^{-1}C^{-1}$. In the case $A = \alpha I$, $\alpha^n = 1$, we take $B = \mathrm{diag}\{\alpha,\alpha^2,\ldots,\alpha^n\}$ and $D = \mathrm{diag}\{1,\alpha^{-1},\ldots,\alpha^{1-n}\}$.

(b) If F has at least $n + 2$ elements, we show that $\beta_1, \ldots, \beta_n$ may be chosen to satisfy $\beta_1 \ldots \beta_n = 1$, in addition to being distinct. If n is odd, take $\beta_1 = 1$, then take $\frac{n-1}{2}$ distinct pairs of the form $\{\beta,\beta^{-1}\}$ with $\beta \neq \pm 1$. If n is even, take $\frac{n}{2}$ pairs of the form $\{\beta,\beta^{-1}\}$ with $\beta \neq \pm 1$ (this may seem to require $n + 3$ elements in F since 0, 1 and -1 are excluded, but if $|F| = n + 2$ then the characteristic of F is 2 and $1 = -1$). Therefore $\det B = 1$. The matrix C may be replaced by CE for any matrix E commuting with B. Since B is diagonalizable, there exists a diagonalizable E which commutes with B and which has arbitrary nonzero determinant. So we can replace C by a matrix with any arbitrarily assigned nonzero determinant, in particular with a matrix in $SL(n,F)$.

(c) The proof is similar to (b). □

4. PRODUCTS OF INVOLUTIONS

An involution is a square matrix whose square is the identity. In [7], Gustafson, Halmos and Radjavi showed that every $n \times n$ matrix, over an arbitrary field F, with determinant ± 1 is the product of at most four involutions. We give a short proof of the special case when F has at least $n+2$ elements.

THEOREM 5 (Gustafson-Halmos-Radjavi). Let A be an $n \times n$ matrix over a field F containing at least $n+2$ elements. If $\det A = \pm 1$, then A is the product of at most four involutions.

Proof. First consider the case $\det A = 1$. As in the proof of part (b) of Theorem 4, we may write A as a product BC where each of B and C has distince eigenvalues of the form: $\{\beta_1, \beta_1^{-1}, \ldots, \beta_m, \beta_m^{-1}\}$ or $\{1, \beta_1, \beta_1^{-1}, \ldots, \beta_m, \beta_m^{-1}\}$ according as n is even or odd. Since each of B and C is diagonalizable, it suffices to show that the matrix $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ is a product of two involutions. This follows easily since $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta^{-1} \\ \beta & 0 \end{pmatrix}$.

Now assume $\det A = -1$ and that $1 \neq -1$. If n is odd, the result follows from the first part by considering $-A$, so we assume that n is even. We may write $A = BC$ with B and C as above except that β_1 and β_1^{-1} in the list of eigenvalues of B, but not in C, are replaced by 1 and -1. This contributes a direct summand $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is itself an involution. $\square$

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