Paired Eternal Domination in Grid Graphs and k-Trees

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Introduction: Graphs and Graph Classes

A *graph* consists of a set of *vertices* and a set of connections between pairs of vertices, called *edges*. Graphs can be organised into classes by their properties. In the following, we consider two types of graph:

Grid Graphs

Grid graphs are exactly what you'd expect. An $m \times n$ grid graph looks like a rectangular subset of the plane.

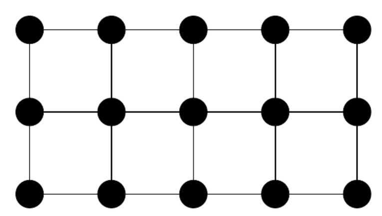

Figure 1: The 3×5 grid

k-Trees

k-Trees are a result of generalising the construction of a tree. k can be any natural number, allowing these graphs to become quite complex despite their simple construction.

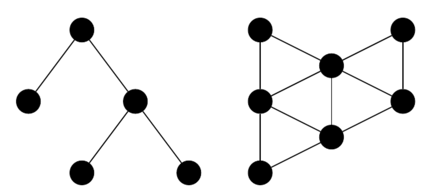

Figure 2: A 1-tree and a 2-tree

Definitions

A *paired dominating set* of a graph G is a set of vertices D such that any vertex not in D is adjacent to one that is and the subgraph of G induced by D has a perfect matching. Put simply, it's a collection of vertices on which we can place pairs of guards so that they can "see" the entire graph.

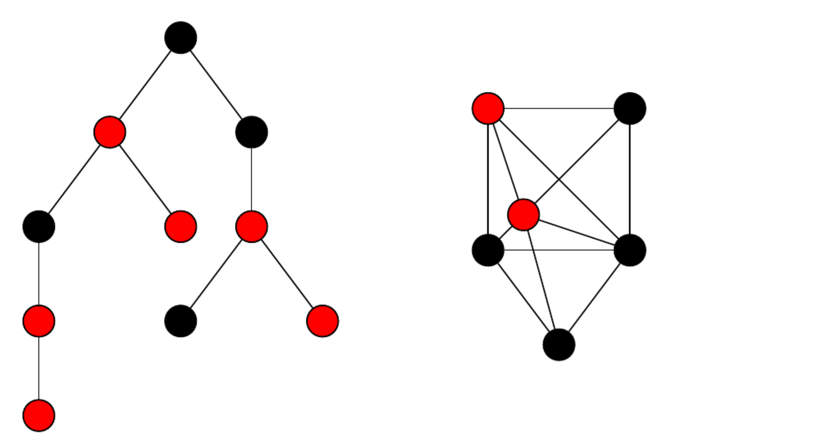

Figure 3: Examples of paired dominating sets (red vertices) on a 1-tree and a 3-tree.

A *paired* ***eternal*** *dominating set* of a graph G is exactly the above, with the added condition that the guards in this set have to be able to defend against an arbitrary sequence of attacks on vertices in G.

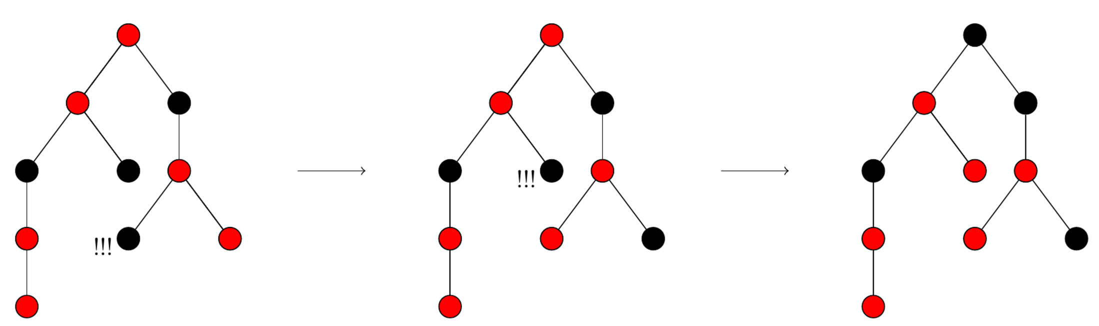

Figure 4: An example sequence of possible attacks on G.

We define the *paired eternal domination number* of G, denoted by $\gamma_{\text{pr}}^{\infty}(G)$, to be the size of the smallest possible paired eternal dominating set on G. For example, the graph above has $\gamma_{\text{pr}}^{\infty}(G) = 6$.

Previous Results

Paired eternal domination has previously been investigated in trees by Moss in [1].

Proposition 1. *For any tree T with a minimum star decomposition of size s, $\gamma_{\text{pr}}^{\infty}(T) = 2s$*

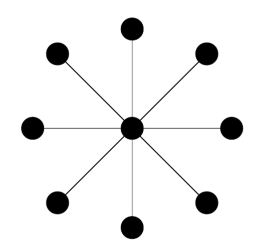

Figure 5: The star with 8 leaves, S_8

Star graphs are a subclass of trees. They consist of the trees with at most one vertex of degree greater than 1. These graphs can clearly be paired eternally dominated by only one pair of guards. A minimum star decomposition of a graph G is the smallest number of stars it takes to cover all vertices of G.

Proposition 1 therefore tells us that the optimal way to paired eternally dominate a tree is by placing a guard pair on each of the stars in a minimum star decomposition.

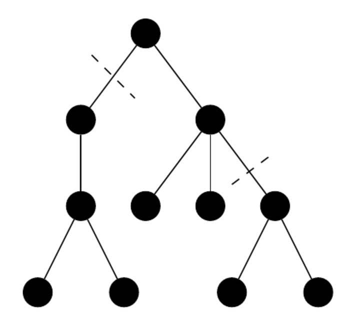

Figure 6: A tree with $s = 3$.

Corollary 1. *For a path on n vertices, P_n, $\gamma_{\text{pr}}^{\infty}(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$.*

Paths are another subclass of trees, where any vertex is adjacent to at most 2 other vertices. They are also exactly the $1 \times n$ grid graphs. According to Proposition 1, we can defend them with $2s$ guards. Clearly, we can simply divide the path into copies of S_2 in order to obtain a minimum star decomposition.

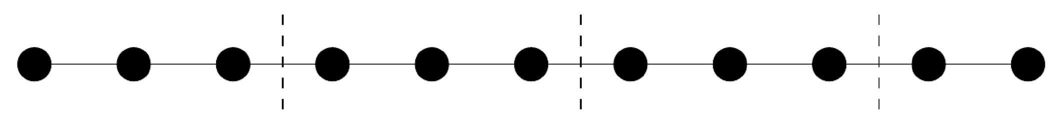

Figure 7: P_{11} can be divided into 4 stars.

Results

Grid Graphs: Bounding $\gamma_{\text{pr}}^{\infty}(G)$

In the following, we present some bounds on the paired eternal domination number for grid graphs.

Theorem 1. *For any $2 \times n$ grid graph G, $\gamma_{\text{pr}}^{\infty}(G) = 2\left\lceil \frac{n}{2} \right\rceil$.*

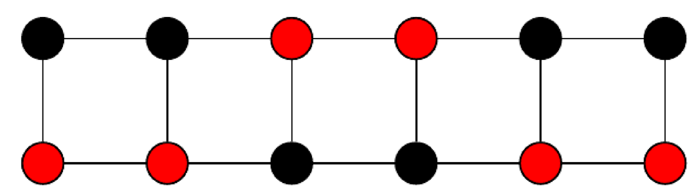

Figure 8: A paired eternal dominating set on the 2×6 grid graph

Theorem 2. *For any $3 \times n$ grid graph G, $n + 1 \leq \gamma_{\text{pr}}^{\infty}(G) \leq 6\left\lceil \frac{n}{5} \right\rceil + 6$.*

Theorem 3. *For any $4 \times n$ grid graph G, $6\left\lfloor \frac{n}{5} \right\rfloor + 2 \leq \gamma_{\text{pr}}^{\infty}(G) \leq n + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{14} \right\rfloor + 2$.*

We can use a variety of different methods to obtain bounds on $\gamma_{\text{pr}}^{\infty}(G)$:

- Comparing G to another graph H with similar structure where we know the value of $\gamma_{\text{pr}}^{\infty}(H)$.
  - Comparing grid graphs to toroidal meshes [2] gives slightly worse lower bounds than above
- Breaking the graph up into smaller pieces, where we know how to defend each of the pieces.
  - This gives us the upper bounds shown above
- Coming up with a sequence of attacks that could force at least a certain number of guards.
  - This gives us the lower bounds shown above

k-Trees: Extending the Results on Trees

We find that, subject to certain restrictions on maximum degree, we can approach the problem of paired eternal dominating sets on k-trees similarly to trees.

Theorem 4. *For a k-tree G with maximum degree $\Delta \leq 2k$ and a minimum star decomposition of size s, $\gamma_{\text{pr}}^{\infty}(G) = 2s$.*

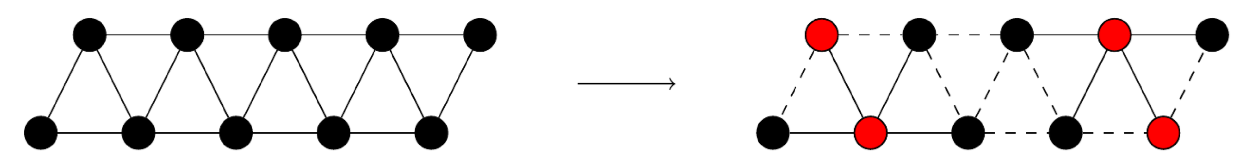

Figure 9: An example of a paired eternal dominating set on a 2-tree with maximum degree 4

With this result in hand, we can characterise the paired eternal domination number of these graphs in terms of the size of the vertex set.

Corollary 2. *For a k-tree G with maximum degree $\Delta \leq 2k$ and n vertices, $\gamma_{\text{pr}}^{\infty}(G) = 2\left\lceil \frac{n}{2k+1} \right\rceil$.*

In essence, this says that k-trees satisfying the restriction on the maximum degree allows us to choose stars that are the "largest possible", centred at vertices that attain the maximum degree and this will give us a minimum star decomposition of the graph.

Future Work

Exact values for $3 \times n$ grid graphs

Conjecture 1. *For large n, $\gamma_{\text{pr}}^{\infty}(G_{3\times n}) \sim n$.*

That is to say that the paired eternal domination number is conjectured to be closer to the proven lower bound than to the upper bound - in particular, we suspect $n + 4$ for even n. For large $3 \times n$ grids, there would be one guard in each column, with a few extra to fill in gaps.

k-Trees without restricted degree

Conjecture 2. *For any k-tree G with a minimum star decomposition of size s, $\gamma_{\text{pr}}^{\infty}(G) = 2s$.*

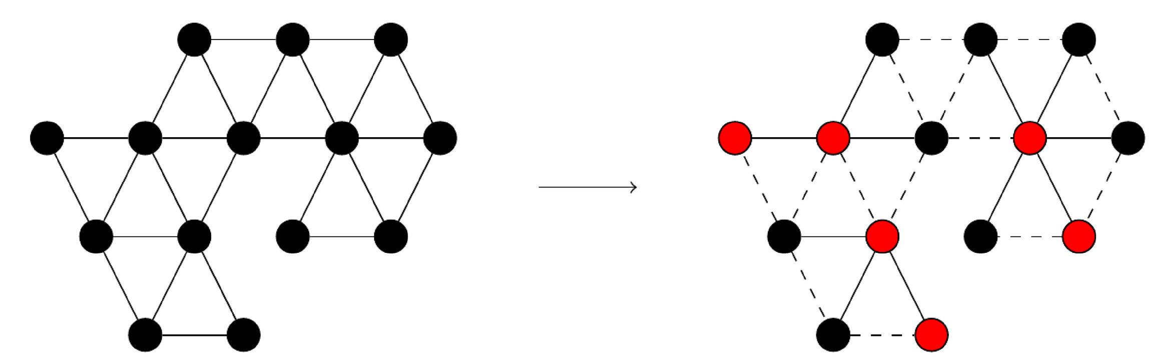

Figure 10: A minimum paired eternal dominating set on a more general 2-tree

References

[1] Elena Moss, *Paired Eternal Domination in Trees*, 2022.

[2] Fu-Tao Hu, Jun-Ming Xu, *Total and paired domination number of toroidal meshes*, Journal of Combinatorial Optimization, 2014.