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Article

Results on Hankel Determinants for the Inverse of Certain Analytic Functions Subordinated to the Exponential Function

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Abstract: In the present paper, we aimed to discuss certain coefficient-related problems for the inverse functions associated with a bounded turning functions class subordinated with the exponential function. We calculated the bounds of some initial coefficients, the Fekete–Szegö-type inequality, and the estimation of Hankel determinants of second and third order. All of these bounds were proven to be sharp.

Keywords: univalent function; inverse function; coefficient bounds; Hankel determinant

MSC: 30C45 30C80

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1. Introduction and Definitions

Let $\mathcal{A}$ and $\mathcal{S}$ be represented here as the classes of normalized analytic and univalent functions, respectively. These classes are defined in the form of

$$\mathcal{A} =: \left\{ f \in \mathcal{H}(\mathbb{D}) : f(0) = f'(0) - 1 = 0, \quad z \in \mathbb{D} \right\} \tag{1}$$

and

$$\mathcal{S} =: \{ f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D} \}, \tag{2}$$

where $\mathcal{H}(\mathbb{D})$ stands for the set of analytic or holomorphic functions in the region

$$\mathbb{D} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \tag{3}$$

If $f \in \mathcal{A}$, then it can be expressed in the series expansion of the form

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l, \quad (z \in \mathbb{D}). \tag{4}$$

In 1985, De Branges [1] solved the renowned Bieberbach conjecture by establishing that, if $f \in \mathcal{S}$, then $|a_n| \leq n$ for $n \geq 2$, where the equality holds if f is a Köebe function or its rotation, where the Köebe function is given by

$$K(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n. \tag{5}$$

Before the Bieberbach conjecture was settled, many interesting subclasses of $\mathcal{S}$ linked to different image domains were studied by different scholars. The most fundamental subfamilies are the starlike $\mathcal{S}^*$ and convex $\mathcal{K}$ functions, which are defined as

$$\begin{aligned} \mathcal{S}^* &=: \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > 0, \quad (z \in \mathbb{D}) \right\}, \\ \mathcal{K} &=: \left\{ f \in \mathcal{A} : \Re \frac{(zf'(z))'}{f'(z)} > 0, \quad (z \in \mathbb{D}) \right\}. \end{aligned}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, $0 \leq \alpha < 1$, if, for $z \in \mathbb{D}$,

$$\Re \frac{zf'(z)}{f(z)} > \alpha. \tag{6}$$

It is known that $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*$ for $0 \leq \alpha < 1$. Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta \leq 1$,

$$\mathcal{SS}^*(\beta) =: \left\{ f \in \mathcal{A} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, \quad (z \in \mathbb{D}) \right\}. \tag{7}$$

Using subordination, Ma and Minda [2] introduced the class $\mathcal{S}^*(\phi)$ given by

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \mathbb{D}) \right\}, \tag{8}$$

where ϕ is a regular function with a positive real part, $\phi(0) = 1$ and $\phi'(0) > 0$. In addition, the function ϕ maps $\mathbb{D}$ onto a star-shaped region with respect to $\phi(0) = 1$, and is symmetric with the real axis.

Another interesting subclass of univalent functions is the close-to-convex function $\mathcal{KC}$, which satisfies the condition

$$\Re \frac{zf'(z)}{g(z)} > 0, \quad (z \in \mathbb{D}), \tag{9}$$

where g is a starlike function.

If we choose $g(z) = z$, then we obtain the subclass $\mathcal{BT}$ of bounded turning functions defined by

$$\mathcal{BT} =: \{ f \in \mathcal{A} : \Re f'(z) > 0, \quad (z \in \mathbb{D}) \}. \tag{10}$$

For each univalent functions f defined in $\mathbb{D}$, the famous $1/4$-theorem of Köebe ensures that its inverse f^{-1} exists at least on a disc of radius $1/4$ with the Taylor's series of the form representation

$$f^{-1}(w) := w + \sum_{n=2}^{\infty} B_n w^n, \quad (|w| < 1/4). \tag{11}$$

Utilizing the representation $f\left(f^{-1}(w)\right) = w$, we obtain

$$B_2 = -a_2, \tag{12}$$
$$B_3 = -a_3 + 2a_2^2, \tag{13}$$
$$B_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{14}$$
$$B_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \tag{15}$$

In recent years, researchers have shown a great deal of interest in understanding the geometric behavior of the inverse function. For example, Krzyz et al. [3] determined the upper bounds of the initial coefficient contained in the inverse function f^{-1} when $f \in \mathcal{S}^*(\alpha)$ with $0 \leq \alpha < 1$. These findings were improved later by Kapoor and Mishra in [4]. In addition, for the class $\mathcal{SS}^*(\xi)$ $(0 < \xi \leq 1)$ of a strongly starlike function, Ali [5] investigated the sharp bounds of the first four initial coefficients along with the sharp estimate of the Fekete–Szegö coefficient functional of the inverse function. For more contributions in this direction, see Juneja and Rajasekaran [6], Libera et al. [7], Ponnusamy et al. [8], Silverman [9], and Sim and Thomas [10].

The Hankel determinant $\Lambda_{q,n}(f)$, for $q, n \in \mathbb{N} = \{1, 2, \ldots\}$, containing coefficients of the function $f \in \mathcal{S}$

$$\Lambda_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{16}$$

was examined by Pommerenke [11,12]. By varying the parameters q and n, we obtained the determinants listed as

$$\Lambda_{2,1}(f) = a_3 - a_2^2, \tag{17}$$

$$\Lambda_{2,2}(f) = a_2 a_4 - a_3^2, \tag{18}$$

$$\Lambda_{3,1}(f) = 2a_2 a_3 a_4 - a_3^3 - a_4^2 + a_3 a_5 - a_2^2 a_5. \tag{19}$$

They are referred to as first, second, and third Hankel determinants, respectively.

Recently, the problems of finding the Hankel determinants sharp bounds for a certain class of complex valued functions have attracted the interest of many specialists. For instance, Janteng et al. [13,14] estimated the sharp bounds of $|\Lambda_{2,2}(f)|$ for the families of $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{BT}$. The exact bound of the second Hankel determinant for the collection $\mathcal{S}^*(\phi)$ of starlike functions was found in [15], and further studied in [16]. This problem was also investigated for different families of bi-univalent functions in [17–19].

The task of obtaining the sharp bounds of $|\Lambda_{3,1}(f)|$ is much more difficult than calculating the bounds of $|\Lambda_{2,2}(f)|$. In [20], Babalola studied the third Hankel determinant for the $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{BT}$ families. Many scholars [21–27] calculated the upper bounds of of $|\Lambda_{3,1}(f)|$ for various subclasses of univalent functions. For the sharp bounds of the third Hankel determinant, Kowalczyk et al. [28] and Lecko et al. [29], in 2018, obtained that

$$|\Lambda_{3,1}(f)| \leq \begin{cases} \frac{4}{135}, & f \in \mathcal{K}, \\ \frac{1}{9}, & f \in \mathcal{S}^*\left(\frac{1}{2}\right). \end{cases} \tag{20}$$

Recently, more results have been found in this direction. For details, see [30–34].

The exponential function $\varphi(z) = e^z$ has a positive real part in $\mathbb{D}$, $\varphi(\mathbb{D}) = \{w \in \mathbb{C} : |\log w| < 1\}$ is symmetric with respect to the real axis and starlike with respect to 1, and $\varphi'(0) > 0$. Using the exponential function, Mendiratta et al. [35] introduced a subclass of starlike function defined by

$$\mathcal{S}^*_{\exp} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z, \quad z \in \mathbb{D} \right\}. \tag{21}$$

This class was later studied in [36] and generalized by Srivastava et al. [37], in which, the authors determined the upper bound of the Hankel determinant.

Motivated by the above works, we introduced a class of bounded turning functions $\mathcal{BT}_{\exp}$ defined as

$$\mathcal{BT}_{\exp} =: \{ f \in \mathcal{A} : \quad f'(z) \prec e^z \quad (z \in \mathbb{D}) \}. \tag{22}$$

The goal of this paper is to compute the sharp bounds of some initial coefficient results, Fekete–Szegö-type problems, and Hankel determinants for the inverse functions of this class.

2. A Set of Lemmas

Before stating the results that are applied in the main contributions, we defined the class $\mathcal{P}$ in terms of a set-builder notation:

$$\mathcal{P} = \left\{ q \in \mathcal{A} : \quad q(z) \prec \frac{1+z}{1-z}, \quad (z \in \mathbb{D}) \right\}, \tag{23}$$

where the function q has a series expansion of the form

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \mathbb{D}). \tag{24}$$

To prove our main results, we need the following Lemmas.

Lemma 1 (see [38])**.** *Let $q \in \mathcal{P}$ be given by (24). Then, for some $\rho, \sigma, x \in \overline{\mathbb{D}}$, we have*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{25}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\sigma, \tag{26}$$

$$\begin{aligned} 8c_4 = {} & c_1^4 + (4 - c_1^2)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4(4 - c_1^2)(1 - |x|^2) \\ & \left[c_1(x-1)\sigma + \overline{x}\sigma^2 - (1 - |\sigma|^2)\rho\right]. \end{aligned} \tag{27}$$

Lemma 2 (see [39])**.** *If $q \in \mathcal{P}$, and is given by (24), then, for all $\lambda \in \mathbb{R}$ and $n, k \in \mathbb{N}$,*

$$|c_{n+k} - \lambda c_n c_k| \leq 2\max(1, |2\lambda - 1|), \tag{28}$$

$$|c_n| \leq 2, \quad n \geq 1, \tag{29}$$

Lemma 3. *Let*

$$\tau(c, x) = \left(4 - c^2\right)(1 + c)x^2 + \left(c^3 + 20c^2 + 12c - 64\right)x + \frac{1}{2}c^3 - 19c^2 + 60. \tag{30}$$

Then, $\tau(c, x) > 0$ for all $(c, x) \in [0, 2) \times \left(\frac{4}{5}, 1\right)$.

Proof. It is not difficult to observe that

$$\begin{aligned} \tau(c, x) & \geq \left(4 - c^2\right)x^2 + \left(20c^2 + 12c - 64\right)x - 19c^2 + 60 \\ & = \left(-x^2 + 20x - 19\right)c^2 + 12xc + 4x^2 - 64x + 60 := v(c, x). \end{aligned}$$

Since $\frac{4}{5} < x < 1$, we have $-x^2 + 20x - 19 > -4$ and $4x^2 - 64x + 60 > 0$. Using $x > \frac{4}{5}$, it follows that

$$v(c, x) > -4c^2 + \frac{48}{5}c = -4c(c - \frac{12}{5}) > 0.$$

This completes the proof of Lemma 3. □

Lemma 4. *Suppose that*

$$F(c, x) = \left(4 - c^2\right)\left(1 - x^2\right)\left[c^2 + 4x\left(4 - c^2\right)\right]. \tag{31}$$

Then $F(c, x) < 25$ for all $(c, x) \in [0, 2) \times \left[0, \frac{4}{5}\right]$.

Proof. Let $c^2 = s$. It is clear that $s \in [0,4)$ and

$$\begin{aligned} F(c,x) &= (4-s)\left(1-x^2\right)[s+4x(4-s)] \\ &= \left(1-x^2\right)\left[(-1+4x)s^2+4(1-8x)s+64x\right]. \end{aligned}$$

If $x \le \frac{1}{8}$, we have $-1+4x \le 0$ and $1-8x \ge 0$. Using $1-x^2 \le 1$ and $(-1+4x)s^2 \le 0$, it follows that

$$F(c,x) \le 4(1-8x)s+64x \le 16(1-8x)+64x = 16-64x \le 16. \tag{32}$$

If $\frac{1}{8} < x \le \frac{1}{4}$, we can observe that $-1+4x \le 0$ and $1-8x \le 0$. Thus, it is easily obtained that

$$F(c,x) \le \left(1-x^2\right)(64x) \le 64x \le 16. \tag{33}$$

If $\frac{1}{4} < x \le \frac{4}{5}$, we note that $-1+4x \ge 0$ and $1-8x \le 0$. Define

$$\varpi(s,x) = (-1+4x)s^2+4(1-8x)s+64x. \tag{34}$$

As

$$\frac{\partial \varpi}{\partial s} = 2(-1+4x)s+4(1-8x) \le 8(-1+4x)+4(1-8x) = -4 < 0,$$

we obtain $\varpi(s,x) \le \varpi(0,x)$. Then,

$$F(c,x) \le 64x\left(1-x^2\right) := h(x). \tag{35}$$

A basic calculation shows that $h(x)$ achieves its maximum value of approximately 24.63361 at $x \approx 0.5773503$. Therefore, we conclude that $F(c,x) < 25$ for all $(c,x) \in [0,2) \times \left[0, \frac{4}{5}\right]$. The proof of Lemma 4 is thus completed. □

3. Coefficient Bounds for the Family $\mathcal{BT}_{\exp}$

We began this part by determining the first two initial coefficients bounds for the inverse function of the function class $\mathcal{BT}_{\exp}$.

Theorem 1. *Let $f \in \mathcal{BT}_{\exp}$ be represented by (4). Then,*

$$|B_2| \le \frac{1}{2}, \quad \& \quad |B_3| \le \frac{1}{3}. \tag{36}$$

These bounds are sharp and can be obtained from the following extremal function given by

$$f_1(z) = \int_0^z e^t dt = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots. \tag{37}$$

Proof. From the definition of the class $\mathcal{BT}_{\exp}$ along with the subordination principal, there exists a Schwarz function ω such that

$$f'(z) = e^{\omega(z)}, \quad (z \in \mathbb{D}). \tag{38}$$

When writing the Schwarz function ω in terms of $p \in \mathcal{P}$, we have

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{39}$$

Or equivalently,

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \tag{40}$$

Using (4), we easily obtain

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{41}$$

By simplification and using the series expansion of $\omega(z)$, we can observe that

$$\begin{aligned} e^{\omega(z)} &= 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{8}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 + \frac{1}{48}c_1^3 - \frac{1}{4}c_1 c_2\right)z^3 \\ &\quad + \left(\frac{1}{2}c_4 - \frac{1}{8}c_2^2 + \frac{1}{384}c_1^4 + \frac{1}{16}c_1^2 c_2 - \frac{1}{4}c_1 c_3\right)z^4 + \cdots. \end{aligned} \tag{42}$$

Comparing (41) and (42), we may obtain

$$a_2 = \frac{1}{4}c_1, \tag{43}$$

$$a_3 = \frac{1}{3}\left(\frac{1}{2}c_2 - \frac{1}{8}c_1^2\right), \tag{44}$$

$$a_4 = \frac{1}{4}\left(\frac{1}{2}c_3 + \frac{1}{48}c_1^3 - \frac{1}{4}c_1 c_2\right), \tag{45}$$

$$a_5 = \frac{1}{5}\left(\frac{1}{2}c_4 - \frac{1}{8}c_2^2 + \frac{1}{384}c_1^4 + \frac{1}{16}c_1^2 c_2 - \frac{1}{4}c_1 c_3\right). \tag{46}$$

For B_2, putting (43) in (12) and then using (29), we easily obtain

$$|B_2| \le \frac{1}{2}. \tag{47}$$

For B_3, putting (43) and (44) in (13), it follows that

$$|B_3| = \frac{1}{6}\left|c_2 - c_1^2\right|. \tag{48}$$

Then, by using (28), we achieve the required bound given by

$$|B_3| \le \frac{1}{3}. \tag{49}$$

□

Now, we study the Fekete–Szegö-type problem for f^{-1} of the function $f \in \mathcal{BT}_{\exp}$.

Theorem 2. *Let $f \in \mathcal{BT}_{\exp}$ be given by (4). Then, for $\gamma \in \mathbb{R}$,*

$$\left|B_3 - \gamma B_2^2\right| \le \max\left\{\frac{1}{3}, \frac{1}{3}\left|1 - \frac{3}{4}\gamma\right|\right\}. \tag{50}$$

This inequality is sharp.

Proof. From (12), (13), (43), and (44), we obtain

$$\left|B_3 - \gamma B_2^2\right| = \frac{1}{6}\left|\left(c_2 - \left(1 - \frac{3}{8}\gamma\right)c_1^2\right)\right|. \tag{51}$$

An application of (28) leads to

$$\left|B_3 - \gamma B_2^2\right| \le \max\{\frac{1}{3}, \frac{1}{3}\left|1 - \frac{3}{4}\gamma\right|\}, \tag{52}$$

and the required result follows. □

Putting $\gamma = 1$, we establish the below inequality.

Corollary 1. *If $f \in \mathcal{BT}_{\exp}$ and has the series expansion (4), then*

$$\left|B_3 - B_2^2\right| \le \frac{1}{3}. \tag{53}$$

The equality can be obtained from f defined by

$$f_2(z) = \int_0^z e^{t^2} dt = z + \frac{1}{3}z^3 + \frac{1}{10}z^5 + \cdots. \tag{54}$$

Now, we investigate bounds of $\left|\Lambda_{2,2}(f^{-1})\right|$ for the class $\mathcal{BT}_{\exp}$.

Theorem 3. *Let $f \in \mathcal{BT}_{\exp}$ be specified by (4). Then,*

$$\left|\Lambda_{2,2}\left(f^{-1}\right)\right| \le \frac{1}{9}. \tag{55}$$

The equality can be obtained from (54).

Proof. The determinant $\Lambda_{2,2}(f^{-1})$ can be reconfigured as

$$\Lambda_{2,2}\left(f^{-1}\right) = B_2B_4 - B_3^2 = a_2^4 - a_2^2a_3 + a_2a_4 - a_3^2. \tag{56}$$

From (43), (44), and (45), we have

$$\left|\Lambda_{2,2}\left(f^{-1}\right)\right| = \frac{1}{1152}\left|7c_1^4 - 14c_1^2c_2 + 36c_1c_3 - 32c_2^2\right|. \tag{57}$$

Using (25) and (26) to express c_2 and c_3 in terms of c_1, and noticing that we can put $c_1 = c$, with $0 \le c \le 2$ without affecting generality, we obtain

$$\begin{aligned}\left|\Lambda_{2,2}\left(f^{-1}\right)\right| &= \frac{1}{1152}\left|c^4 - 9c^2x^2\left(4-c^2\right) - 5c^2x\left(4-c^2\right)\right. \\ &\quad \left. +18c\left(4-c^2\right)\left(1-|x|^2\right)\sigma - 8x^2\left(4-c^2\right)^2\right|.\end{aligned}$$

Applying the triangle inequality and invoking $|\sigma| \le 1$, $|x| = b \le 1$, it follows that

$$\begin{aligned}\left|\Lambda_{2,2}\left(f^{-1}\right)\right| &\le \frac{1}{1152}\left\{c^4 + 9c^2b^2\left(4-c^2\right) + 5c^2b\left(4-c^2\right) + 18c\left(4-c^2\right)\right. \\ &\quad \left.\left(1-b^2\right) + 8b^2\left(4-c^2\right)^2\right\} := \phi(c,b).\end{aligned}$$

Differentiating about the parameter b, we have

$$\frac{\partial\phi}{\partial b} = \frac{1}{1152}\left[\left(2c^2 - 36c + 64\right)\left(4-c^2\right)b + 5c^2\left(4-c^2\right)\right]. \tag{58}$$

It is an easy task to illustrate that $\frac{\partial\phi}{\partial b} \ge 0$ for $b \in [0,1]$, that is, $\phi(c,b) \le \phi(c,1)$. Thus,

$$\left|\Lambda_{2,2}\left(f^{-1}\right)\right| \leq \frac{1}{1152}\left[c^4 + 14c^2\left(4-c^2\right) + 8\left(4-c^2\right)^2\right] := \zeta(c). \tag{59}$$

Since $\zeta'(c) < 0$, we have $\zeta(c) \leq \zeta(0)$. Therefore, we obtain that

$$\left|\Lambda_{2,2}\left(f^{-1}\right)\right| \leq \frac{1}{1152} \cdot 128 = \frac{1}{9}. \tag{60}$$

The equality is accomplished from (54). □

4. Third Hankel Determinant for the Class $\mathcal{BT}_{\exp}$

We can now study the determinant $\Lambda_{3,1}\left(f^{-1}\right)$ for $f \in \mathcal{BT}_{\exp}$.

Theorem 4. *If $f \in \mathcal{BT}_{\exp}$ with the series expansion (4), then*

$$\left|\Lambda_{3,1}\left(f^{-1}\right)\right| \leq \frac{1}{16}. \tag{61}$$

The inequality is sharp.

Proof. From the definition, we can observe that the determinant (19) is described as

$$\begin{aligned} s\Lambda_{3,1}\left(f^{-1}\right) &= 2B_2B_3B_4 - B_3^3 - B_4^2 + B_3B_5 - B_2^2B_5. \\ &= a_2^6 - 3a_2^4a_3 + 3a_2^2a_3^3 - a_2^2a_5 + 2a_2a_3a_4 - 2a_3^3 + a_3a_5 - a_4^2. \end{aligned}$$

In virtue of (43), (44), (45), and (46), along with $c_1 = c \in [0,2]$, we obtain

$$\begin{aligned} \Lambda_{3,1}\left(f^{-1}\right) &= \frac{1}{11520}\left(\frac{35}{3}c^6 - 59c^4c_2 + 15c^3c_3 + 89c^2c_2^2 - 120c^2c_4\right. \\ &\quad \left. +204cc_2c_3 - \frac{464}{3}c_2^3 + 192c_2c_4 - 180c_3^2\right). \end{aligned} \tag{62}$$

To simplify the computation, we take $t = 4 - c^2$ in (25), (26), and (27). Using (25), (26), and (27), along with straightforward algebraic computations, we have

$$\begin{aligned} \Lambda_{3,1}\left(f^{-1}\right) &= \frac{1}{11520}\left\{\frac{1}{12}c^6 + 48t^2x^3 - \frac{58}{3}t^3x^3 - 12c^2tx^2 - 3c^4tx^3 + \frac{9}{4}c^4tx^2\right. \\ &\quad -c^4tx + \frac{3}{4}c^2t^2x^4 - \frac{33}{2}c^2t^2x^3 - 45t^2\left(1-|x|^2\right)^2\sigma^2 \\ &\quad +\frac{3}{2}c^3t\left(1-|x|^2\right)\sigma + 12c^3tx\left(1-|x|^2\right)\sigma + 12c^2t\overline{x}\left(1-|x|^2\right)\sigma^2 \\ &\quad -12c^2t\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho - 3ct^2x^2\left(1-|x|^2\right)\sigma \\ &\quad -48t^2|x|^2\left(1-|x|^2\right)\sigma^2 + 12ct^2x\left(1-|x|^2\right)\sigma \\ &\quad \left. +48t^2x\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\rho\right\}. \end{aligned}$$

As $t = 4 - c^2$, it is noted that

$$\Lambda_{3,1}\left(f^{-1}\right) = \frac{1}{11520}\left(v_1(c,x) + v_2(c,x)\sigma + v_3(c,x)\sigma^2 + \Psi(c,x,\sigma)\rho\right), \tag{63}$$

where $x, \sigma, \rho \in \overline{\mathbb{D}}$, and

$$\begin{aligned} v_1(c,x) &= \frac{1}{12}c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(-\frac{88}{3}x^3 + \frac{17}{6}c^2x^3 + \frac{3}{4}c^2x^4 \right.\right. \\ &\quad \left.\left. + \frac{25}{4}c^2x^2\right) - 12c^2x^2 - 3c^4x^3 + \frac{9}{4}c^4x^2 - c^4x\right], \\ v_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(9cx - 3cx^2\right) + 12c^3x + \frac{3}{2}c^3\right], \\ v_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-3|x|^2 - 45\right) + 12c^2\overline{x}\right], \\ \Psi(c,x,\sigma) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\sigma|^2\right)\left[-12c^2 + 48x\left(4-c^2\right)\right]. \end{aligned}$$

By setting $|x| = x$, $|\sigma| = y$ and utilizing the assumption $|\rho| \leq 1$, we obtain

$$\begin{aligned} \left|\Lambda_{3,1}\left(f^{-1}\right)\right| &\leq \frac{1}{11520}\left(|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2 + |\Psi(c,x,\sigma)|\right). \\ &\leq \frac{1}{11520}Q(c,x,y), \end{aligned} \tag{64}$$

where

$$Q(c,x,y) = q_1(c,x) + q_2(c,x)y + q_3(c,x)y^2 + q_4(c,x)\left(1-y^2\right), \tag{65}$$

with

$$\begin{aligned} q_1(c,x) &= \frac{1}{12}c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(\frac{88}{3}x^3 + \frac{17}{6}c^2x^3 + \frac{3}{4}c^2x^4 \right.\right. \\ &\quad \left.\left. + \frac{25}{4}c^2x^2\right) + 12c^2x^2 + 3c^4x^3 + \frac{9}{4}c^4x^2 + c^4x\right] \\ q_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(9cx + 3cx^2\right) + 12c^3x + \frac{3}{2}c^3\right], \\ q_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(3x^2 + 45\right) + 12c^2x\right], \\ q_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[12c^2 + 48x\left(4-c^2\right)\right]. \end{aligned}$$

Now, for finding the upper bound of $\left|\Lambda_{3,1}(f^{-1})\right|$, we have to maximize $Q(c,x,y)$ in the closed cuboid $\Omega : [0,2] \times [0,1] \times [0,1]$. By noting that

$$Q(0,0,1) = 720, \tag{66}$$

we know

$$\max_{(c,x,y)\in\Omega}\{Q(c,x,y)\} \geq 720. \tag{67}$$

In the following, we aim to prove that

$$\max_{(c,x,y)\in\Omega}\{Q(c,x,y)\} = 720. \tag{68}$$

It is easy to calculate that

$$Q(2,x,y) \equiv \frac{16}{3} < 720, \quad x,y \in [0,1]. \tag{69}$$

If we put $x = 1$, then $Q(c,x,y)$ becomes

$$\varsigma(c) = \frac{1}{3}\left(11c^6 - 109c^4 - 88c^2 + 1408\right). \tag{70}$$

Then, differentiating $\varsigma(c)$ with respect to c, we have

$$\frac{\partial \varsigma}{\partial c} = 22c^5 - \frac{436}{3}c^3 - \frac{176}{3}c. \tag{71}$$

Solving $\frac{\partial \varsigma}{\partial c} = 0$ for $c \in [0,2]$, we obtain $c = 0$. Thus, ς obtains a maximum valve at $c = 0$, which is $\frac{1408}{3} < 720$. Hence, the global maximum value of Q is impossible to achieve on the face $c = 2$ and $x = 1$ of Ω. Thus, we assume that $c \in [0,2)$ and $x \in [0,1)$ in the following discussions.

We first show that the global maxima of Q can only be obtained on the face $y = 1$ of Ω. Let $(c,x,y) \in [0,2) \times [0,1) \times (0,1)$. By differentiating partially (65) about y, we have

$$\begin{aligned} \frac{\partial Q}{\partial y} &= \frac{3}{2}\left(4-c^2\right)(1-x^2)\left\{4\left[\left(4-c^2\right)(x-15)+4c^2\right](x-1)y \right. \\ &\quad \left. +c\left[2x\left(4-c^2\right)(3+x)+c^2(8x+1)\right]\right\}. \end{aligned}$$

By setting $\frac{\partial Q}{\partial y} = 0$, we obtain

$$y = \frac{c\left[2x(4-c^2)(3+x)+c^2(8x+1)\right]}{4(x-1)[(4-c^2)(15-x)-4c^2]} = y_0. \tag{72}$$

If y_0 is a critical point within Ω, then $y_0 \in (0,1)$, which is only achievable if

$$c^3(8x+1) + 2cx\left(4-c^2\right)(3+x) + 4(1-x)\left(4-c^2\right)(15-x) < 16c^2(1-x), \tag{73}$$

and

$$c^2 > \frac{4(15-x)}{19-x}. \tag{74}$$

Now, we must find solutions that meet both inequality (73) and (74) for critical points to exist.

Let $q(x) = \frac{4(15-x)}{19-x}$. Then, $q'(x) < 0$ in $(0,1)$. Thus, $q(x)$ is decreasing over $(0,1)$. Hence, $c^2 > \frac{28}{9}$. Thus, if there exists a critical point (c_0,x_0,y_0) satisfying $y_0 \in (0,1)$, we can observe that $\frac{2\sqrt{7}}{3} < c_0 < 2$ and $0 < x_0 < 1$. Then, we find that

$$q_1(c_0,x_0) \le q_1(c_0,1) := \vartheta_1(c_0) \tag{75}$$

and

$$\begin{aligned} q_2(c_0,x_0) &\le \left(4-c_0^2\right)\left[12c_0\left(4-c_0^2\right)+12c_0^3+\frac{3}{2}c_0^3\right] := \vartheta_2(c_0), \\ q_3(c_0,x_0) &\le \left(4-c_0^2\right)\left[48\left(4-c_0^2\right)+12c_0^2\right] := \vartheta_3(c_0), \\ q_4(c_0,x_0) &\le \left(4-c_0^2\right)\left[12c_0^2+48\left(4-c_0^2\right)\right] := \vartheta_4(c_0). \end{aligned}$$

Thus, it yields

$$Q(c_0,x_0,y_0) \le \vartheta_1(c_0) + \vartheta_2(c_0)y + \vartheta_3(c_0)y^2 + \vartheta_4(c_0)\left(1-y^2\right). \tag{76}$$

As it is observed that $\vartheta_3(c_0) \equiv \vartheta_4(c_0)$, the above inequality leads to

$$Q(c_0,x_0,y_0) \le \vartheta_1(c_0) + \vartheta_4(c_0) + \vartheta_2(c_0)y. \tag{77}$$

Since $\vartheta_2(c_0) \ge 0$, we obtain

$$Q(c_0,x_0,y_0) \le \vartheta_1(c_0) + \vartheta_4(c_0) + \vartheta_2(c_0) := \kappa(c_0). \tag{78}$$

A basic calculation gives that

$$\begin{aligned}\kappa(c_0) &= \frac{1}{12}c_0^6 + \left(4-c_0^2\right)^2\left(\frac{59}{6}c_0^2 + 12c_0 + \frac{232}{3}\right) \\ &\quad + (4-c_0)\left(\frac{25}{4}c_0^4 + \frac{27}{2}c_0^3 + 24c_0\right). \end{aligned} \tag{79}$$

Using $c_0 < 2$, we have

$$\kappa(c_0) \le \frac{1}{12}\cdot 2^6 + \frac{422}{3}\left(4-c_0^2\right)^2 + 256(4-c_0). \tag{80}$$

Now it is easy to calculate that $\kappa(c_0)$ has an upper bound of approximately 386.6996 with the equality achieved at $c_0 \approx 1.763834$. Thus, in this situation, all critical points of $Q(c,x,y)$ in $[0,2)\times[0,1)\times(0,1)$ achieve their values at most at 386.6996. In virtue of the global maxima of Q being only possible to be obtained in critical points or at the boundary of Ω, we conclude that, for Q, it cannot gain its global optimal maximum value on any points of Ω with $y \in (0,1)$.

Making $y = 0$, $Q(c,x,0) := \Gamma(c,x)$ reduces to

$$\Gamma(c,x) = q_1(c,x) + q_4(c,x). \tag{81}$$

Let $\Delta(c,x) := Q(c,x,1)$. It is easy to observe that

$$\Delta = q_1(c,x) + q_2(c,x) + q_3(c,x). \tag{82}$$

Thus, we have

$$\begin{aligned}\Delta(c,x) - \Gamma(c,x) &= q_2(c,x) + q_3(c,x) - q_4(c,x) \\ &= 3\left(4-c^2\right)\left(1-x^2\right)\kappa(c,x),\end{aligned}$$

where

$$\kappa(c,x) = \left(4-c^2\right)(1+c)x^2 + \left(c^3 + 20c^2 + 12c - 64\right)x + \frac{1}{2}c^3 - 19c^2 + 60. \tag{83}$$

For $x > \frac{4}{5}$, when using Lemma 3, it yields to $\Delta(c,x) > \Gamma(c,x)$ for all $(c,x) \in [0,2)\times\left(\frac{4}{5},1\right)$. Hence, Q cannot achieve the global maxima with $y = 0$ and $(c,x) \in [0,2)\times\left(\frac{4}{5},1\right)$. For $x \le \frac{4}{5}$, it is easy to find that

$$q_1(c,x) \le q_1\left(c,\frac{4}{5}\right) := \chi_1(c). \tag{84}$$

From Lemma 4, it is seen that

$$q_4(c,x) \le 12\varpi(c,x) < 300. \tag{85}$$

Then, we obtain

$$k_4(c,x) \le \chi_1(c) + 300 := \mu(c). \tag{86}$$

It is not hard to calculate that $\mu(c)$ attains its maxima of approximately 540.3760 at $c \approx 0.2398224$. This implies that it is impossible for Q to gain its global maximum value with $y = 0$ and $(c,x) \in [0,2)\times\left(0,\frac{4}{5}\right]$. Thus, we conclude that $Q(c,x,0)$ has no global optimal solution in $[0,2)\times[0,1)$. Therefore, we only need to discuss Q on the face $y = 1$ of Ω to find the global optimal value.

For $y = 1$, a basic calculation shows that $Q(c, x, 1) = \Delta(c, x)$ has the form of

$$\begin{aligned}\Delta(c,x) \quad = \quad & \frac{1}{12}c^6 + \left(4-c^2\right)^2\left[\frac{3}{4}\left(c^2-4c-4\right)x^4 + \frac{1}{6}\left(17c^2-54c+176\right)x^3\right.\\ & \left.+\frac{1}{4}\left(25c^2+12c-168\right)x^2+9cx+45\right]\\ & +\left(4-c^2\right)\left[3\left(c^2-4c-4\right)c^2x^3+\frac{3}{4}\left(3c^2-2c+16\right)c^2x^2\right.\\ & \left.+\left(c^2+12c+12\right)c^2x+\frac{3}{2}c^3\right].\end{aligned}$$

By observing that $c^2 - 4c - 4 \le 0$, we have

$$\begin{aligned}\Delta(c,x) \quad \le \quad & \frac{1}{12}c^6 + \left(4-c^2\right)^2\left[\frac{1}{6}\left(17c^2-54c+176\right)x^3\right.\\ & \left.+\frac{1}{4}\left(25c^2+12c-168\right)x^2+9cx+45\right]\\ & +\left(4-c^2\right)\left[\frac{3}{4}\left(3c^2-2c+16\right)c^2x^2+\left(c^2+12c+12\right)c^2x+\frac{3}{2}c^3\right].\end{aligned}$$

Using $17c^2 - 54c + 176 \ge 0$, $3c^2 - 2c + 16 \ge 0$ and $0 < x < 1$, it further leads to

$$\begin{aligned}\Delta(c,x) \quad \le \quad & \frac{1}{12}c^6 + \left(4-c^2\right)^2\left[\frac{1}{6}\left(17c^2-54c+176\right)x^2\right.\\ & \left.+\frac{1}{4}\left(25c^2+12c-168\right)x^2+9cx+45\right]\\ & +\left(4-c^2\right)\left[\frac{3}{4}\left(3c^2-2c+16\right)c^2+\left(c^2+12c+12\right)c^2+\frac{3}{2}c^3\right]\\ = \quad & \frac{1}{12}c^6 + \left(4-c^2\right)^2\left[\frac{1}{12}\left(109c^2-72c-152\right)x^2+9cx+45\right]\\ & +\frac{3}{4}\left(4-c^2\right)\left(c^4+16c^3+32c^2\right)\\ \le \quad & \frac{1}{12}c^6 + \left(4-c^2\right)^2\left[\frac{1}{12}\left(109c^2-152\right)x^2+9cx+45\right]\\ & +\frac{3}{4}\left(4-c^2\right)\left(c^4+16c^3+32c^2\right) := \Xi(c,x).\end{aligned}$$

Define

$$\Theta(c,x) = \frac{1}{12}\left(109c^2-152\right)x^2+9cx+45. \tag{87}$$

If $c \ge \sqrt{\frac{152}{109}}$, it is clear that $\Theta(c,x) \le \Theta(c,1)$. For $c < \sqrt{\frac{152}{109}}$, we obviously have $\frac{1}{12}\left(109c^2-152\right) < 0$. Taking $\Theta(c,x)$ as a polynomial on x, it can be observed that the symmetric axis

$$x_0 = \frac{54c}{152-109c^2}. \tag{88}$$

Let c_0 be the only root of the equation $109c^2 + 54c - 152 = 0$. It is known that $c_0 = \frac{-27+\sqrt{17297}}{109} \approx 0.9588813$. For $c > c_0$, it is noted that $x_0 > 1$. Thus, we obtain $R(c,x) \le R(c,1)$. Thus, we can observe that

$$\Theta(c,x) \le \Theta(c,1), \quad c \in (c_0, 2). \tag{89}$$

This leads to

$$\Xi(c,x) \le \frac{1}{12}c^6 + \left(4-c^2\right)^2 R(c,1) + \frac{3}{4}\left(4-c^2\right)\left(c^4+16c^3+32c^2\right) := \eta_1(c), \tag{90}$$

where $c \in (\sqrt{c_0}, 2)$. It is calculated that $\eta_1(c)$ has a maxima of approximately 570.5751 on $c = c_0$. Now, we consider $c < c_0$. In this case, we can observe that $x_0 < 1$. Then, we have

$$\Theta(c,x) \le 45 + \frac{243c^2}{152-109c^2} \le 45 + \frac{243c^2}{152-109} = 45 + \frac{243}{43}c^2 \le 45 + 6c^2. \tag{91}$$

Hence, we obtain

$$\Xi(c,x) \le \frac{1}{12}c^6 + 3\left(4-c^2\right)^2\left(15+2c^2\right) + \frac{3}{4}\left(4-c^2\right)\left(c^4+16c^3+32c^2\right) := \eta_2(c), \tag{92}$$

where $c \in (0, c_0)$. This is a simple exercise to show that $\eta_2(c)$ attains its maximum value of 720 at $c = 0$.

Consequently, from all of the preceding situations, we established that

$$Q(c,x,y) \le 720 \text{ on } [0,2] \times [0,1] \times [0.1]. \tag{93}$$

Hence, from (64), we have

$$\left|\Lambda_{3,1}\left(f^{-1}\right)\right| \le \frac{1}{11520}Q(c,x,y) \le \frac{1}{11520} \cdot 720 = \frac{1}{16}. \tag{94}$$

Thus, the proof is completed. The extremal function for this sharp result is given by

$$f_3(z) = \int_0^z e^{t^3} dt = z + \frac{1}{4}z^4 + \frac{1}{14}z^7 + \cdots. \tag{95}$$

□

5. Conclusions

Although there is a large amount of literature on the Hankel determinants in the field of geometric function theory, it is still difficult to calculate the sharp bound on the third Hankel determinant. In the current article, we considered a family of bounded turning functions $\mathcal{BT}_{\exp}$ connected with the exponential function. For the inverse of the functions in this class, we obtained some sharp results on the coefficient-related problems. In particular, by transforming the third Hankel determinant to a real function with three variables defined on a cubioid, we found the exact bound of the third Hankel determinant with the inverse coefficient as the entry. This helps us to understand more geometric properties of this function class. By improving the present methods, we may be able to obtain more outcomes on the known various subclasses of univalent functions.

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