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An Upper Bound of the Third Hankel Determinant for a Subclass of *q*-Starlike Functions Associated with *k*-Fibonacci Numbers

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Abstract: In this paper, we use q-derivative operator to define a new class of q-starlike functions associated with k-Fibonacci numbers. This newly defined class is a subclass of class $\mathcal{A}$ of normalized analytic functions, where class $\mathcal{A}$ is invariant (or symmetric) under rotations. For this function class we obtain an upper bound of the third Hankel determinant.

Keywords: starlike functions; subordination; q-Differential operator; k-Fibonacci numbers

MSC: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction and Definitions

The calculus without the notion of limits is called quantum calculus; it is usually called q-calculus or q-analysis. By applying q-calculus, univalent functions theory can be extended. Moreover, the q-derivative, such as the q-calculus operators (or the q-difference) operator, are used to developed a number of subclasses of analytic functions (see, for details, the survey-cum-expository review article by Srivastava [1]; see also a recent article [2] which appeared in this journal, *Symmetry*).

Ismail et al. [3] instigated the generalization of starlike functions by defining the class of q-starilke functions. A firm footing of the usage of the q-calculus in the context of Geometric Functions Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory by Srivastava (see, for details [4]). Raghavendar and Swaminathan [5] studied certain basic concepts of close-to-convex functions. Janteng et al. [6] published a paper in which the (q) generalization of some subclasses of analytic functions have studied. Further, q-hypergeometric functions, the q-operators were studied in many recent works (see, for example, [7–9]). The q-calculus applications in operator theory could be found in [4,10]. The coefficient inequality for q-starlike and q-close-to-convex functions with respect to Janowski functions were studied by Srivastava et al. [8,11]

recently, (see also [12]). Further development on this subject could be seen in [7,9,13,14]. For a comprehensive review of the theory and applications of the q-derivative (or the q-difference) operator and related literature, we refer the reader to the above-mentioned work [1].

We denote by $\mathcal{A}$ the class of functions which are analytic and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disk $\mathbb{U}$ given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by the following conditions:

$$f(0) = 0 = f'(0) - 1.$$

The subordinate between two functions f and g in $\mathbb{U}$, given by:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if an analytic Schwarz function w exists in such way that

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

so that

$$f(z) = g\big(w(z)\big).$$

In particular, the following equivalence also holds for the univalent function g

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \implies f(0) = g(0) \ \text{ and } \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next by the $\mathcal{P}$ class of analytic functions, $p(z)$ in $\mathbb{U}$ is denoted, in which normalization conditions are given as follow:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2}$$

such that

$$\Re(p(z)) > 0 \qquad (\forall\, z \in \mathbb{U}).$$

Let k be any positive real number, then we define the k-Fibonacci number sequence $\{F_{k,n}\}_{n=0}^{\infty}$ recursively by

$$F_{k,0} = 0,\ \ F_{k,1} = 1 \ \text{ and } \ F_{k,n+1} = kF_{k,n} + F_{k,n-1} \ \text{ for } \ n \geqq 1. \tag{3}$$

The n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k - \mathcal{T}_k)^n - \mathcal{T}_k^n}{\sqrt{k^2+4}},$$

where

$$\mathcal{T}_k = \frac{k - \sqrt{k^2+4}}{2}. \tag{4}$$

If

$$\tilde{p}_k(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n,$$

then we have (see also [15])

$$\tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\mathcal{T}_k^n \qquad (n \in \mathbb{N};\ \mathbb{N} := \{1,2,3,\cdots\}). \tag{5}$$

Definition 1. *Let $q \in (0,1)$ then the q-number $[\lambda]_q$ is given by*

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}) . \end{cases}$$

Definition 2. *The q-difference (or the q-derivative) $\mathcal{D}_q$ operator of any given function f is defined, in a given subset of $\mathbb{C}$, of complex numbers by*

$$(\mathcal{D}_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

led to the existence of the derivative $f'(0)$.

From Definitions 1 and 2, we have

$$\lim_{q \to 1-} (\mathcal{D}_q f)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a differentiable function f. In addition, from (1) and (2), we observe that

$$(\mathcal{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{6}$$

In the year 1976, it was Noonan and Thomas [16] who concentrated on the function f given in (1) and gave the $\mathfrak{q}$th Hankel determinant as follows.

Let $n \geqq 0$ and $\mathfrak{q} \in \mathbb{N}$. Than the $\mathfrak{q}$th Hankel determinant is defined by

$$H_{\mathfrak{q}}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+\mathfrak{q}-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+\mathfrak{q}-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(\mathfrak{q}-1)} \end{vmatrix}$$

Several authors studied the determinant $H_{\mathfrak{q}}(n)$. In particular, sharp upper bounds on $H_2(2)$ were obtained in such earlier works as, for example, in [17,18] for various subclasses of the normalized analytic function class $\mathcal{A}$. It is well-known for the Fekete-Szegö functional $|a_3 - a_2^2|$ that

$$\left|a_3 - a_2^2\right| = H_2(1).$$

Its worth mentioning that, for a parameter μ which is real or complex, the generalization the functional $|a_3 - \mu a_2^2|$ is given in aspects. In particular, Babalola [19] studied the Hankel determinant $H_3(1)$ for some subclasses of $\mathcal{A}$.

In 2017, Güney et al. [20] explored the third Hankel determinant in some subclasses of $\mathcal{A}$ connected with the above-defined k-Fibonacci numbers. A derivation of the sharp coefficient bound for the third Hankel determinant and the conjecture for the sharp upper bound of the second Hankel determinant is also derived by them, which is employed to solve the related problems to the third Hankel determinant and to present an upper bound for this determinant.

Motivated and inspired by the above-mentioned work and also by the recent works of Güney et al. [20] and Uçar [12], we will now define a new subclass $\mathcal{SL}(k,q)$ of starlike functions associated with the k-Fibonacci numbers. We will then find the Hankel determinant $H_3(1)$ for the newly-defined functions class $\mathcal{SL}(k,q)$.

Definition 3. *Let $\mathcal{P}(\beta)$ $(0 \leqq \beta < 1)$ denote the class of analytic functions p in $\mathbb{U}$ with*

$$p(0) = 1 \quad \text{and} \quad \Re\big(p(z)\big) > \beta.$$

Definition 4. *Let the function p be said to belong to the class k-$\tilde{\mathcal{P}}_q(z)$ and let k be any positive real number if*

$$p(z) \prec \frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\,\tilde{p}_k(z)}, \tag{7}$$

where $\tilde{p}_k(z)$ is given by

$$\tilde{p}_k(z) = \frac{1+\mathcal{T}_k^2 z^2}{1-k\mathcal{T}_k z-\mathcal{T}_k^2 z^2}, \tag{8}$$

and $\mathcal{T}_k$ is given in (4).

Remark 1. For $q = 1$, it is easily seen that

$$p(z) \prec \tilde{p}_k(z).$$

Definition 5. *Let k be any positive real number. Then the function f be in the functions class $\mathcal{SL}(k,q)$ if and only if*

$$\frac{z}{f(z)}\left(\mathcal{D}_q f\right) z \prec \frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\,\tilde{p}_k(z)}, \tag{9}$$

where $\tilde{p}_k(z)$ is given in (8).

Remark 2. For $q = 1$, we have

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z).$$

We recall that when the f belongs to the class $\mathcal{A}$ of analytic function then it is invariant (or symmetric) under rotations if and only if the function $f_\varsigma(z)$ given by

$$f_\varsigma(z) = e^{-i\varsigma} f(ze^{i\varsigma}) \qquad (\varsigma \in \mathbb{R})$$

is also in $\mathcal{A}$. A functional $\mathcal{I}(f)$ defined for functions f is in $\mathcal{A}$ is called invariant under rotations in $\mathcal{A}$ if $f_\varsigma \in \mathcal{A}$ and $\mathcal{I}(f) = \mathcal{I}(f_\varsigma)$ for all $\varsigma \in \mathbb{R}$. It can be easily checked that the functionals $|a_2a_3 - a_4|$, $|H_{2,1}|$ and $|H_{3,1}|$ considered for the class $\mathcal{SL}(k,q)$ satisfy the above definitions.

Lemma 1 (see [21])**.** *Let*

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

be in the class $\mathcal{P}$ of functions with positive real part in $\mathbb{U}$. Then

$$|c_k| \leqq 2 \qquad (k \in \mathbb{N}). \tag{10}$$

If $|c_1| = 2$, *then*

$$p(z) \cong p_1(z) \cong \frac{1+xz}{1-xz} \qquad \left(x = \frac{c_1}{2}\right).$$

Conversely, if $p(z) \cong p_1(z)$ *for some* $|x| = 1$, *then* $c_1 = 2x$ *and*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leqq 2 - \frac{|c_1^2|}{2}. \tag{11}$$

Lemma 2 (see [22])**.** *Let* $p \in \mathcal{P}$ *with its coefficients* c_k *as in Lemma 1, then*

$$\left|c_3 - 2c_1c_2 + c_1^3\right| \leqq 2. \tag{12}$$

Lemma 3 (see [23])**.** *Let* $p \in \mathcal{P}$ *with its coefficients* c_k *as in Lemma 1, then*

$$|c_1c_2 - c_3| \leqq 2. \tag{13}$$

Lemma 4 (see [20])**.** *If the function* f *given in the form* (1) *belongs to class* $\mathcal{SL}^k$, *then*

$$|a_n| \leqq |\mathcal{T}_k|^{n-1} F_{k,n}, \tag{14}$$

where $\mathcal{T}_k$ *is given in* (4)*. Equality holds true in* (14) *for the function* g *given by*

$$\begin{aligned} g_k(z) &= \frac{z}{1 - k\mathcal{T}_k z - \mathcal{T}_k^2 z} \\ &= \sum_{n=1}^{\infty} \mathcal{T}_k^{n-1} F_{k,n} z^n, \end{aligned}$$

which can be written as follows:

$$g_k(z) = z + \mathcal{T}_k z^2 + \left(k^2+1\right)(\mathcal{T}_k k + 1) z^3 + \cdots. \tag{15}$$

2. Main Results

Here, we investigate the sharp bounds for the second Hankel determinant and the third Hankel determinant. We also find sharp bounds for the Fekete-Szegö functional $|a_3 - \lambda a_2^2|$ for a real number λ. Throughout our discussion, we will assume that $q \in (0,1)$.

Theorem 1. *Let the function* $f \in \mathcal{A}$ *given in* (1) *belong to the class* $\mathcal{SL}(k,q)$*. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{1}{q^3(q+1)^2\mathcal{Q}} \left\{\mathcal{Q}(q+1)^2 + \left(|\mathcal{B}_q|k^2 + |\mathcal{C}_q|\right)16k^2\right\}\mathcal{T}_k^2, \tag{16}$$

where

$$\mathcal{Q} = \left(q + q^2 + q^3\right) \tag{17}$$

$$\mathcal{B}_q = \frac{1}{64}(q+1)^4 \left\{\frac{1}{(q+1)^2}\mathcal{Q}\left(q^2+6q-3\right) - \frac{1}{4}q^2(q-1)(2q-3)\right\} \tag{18}$$

$$\mathcal{C}_q = \frac{1}{16}(q+1)^2\left[(2q-1) - \mathcal{Q}\left(3 + \frac{1}{2}q^2(q-1)\right)(q+1)^2\right] \tag{19}$$

and $\mathcal{T}_k$ *is given in* (4).

Proof. If $f \in \mathcal{SL}(k,q)$, then it follows from the definition that

$$\frac{z\left(\mathcal{D}_q f\right)(z)}{f(z)} \prec \tilde{q}(z),$$

where

$$\tilde{q}(z) = \frac{2\check{p}_k(z)}{(1+q)+(1-q)\,\check{p}(z)}.$$

For a given $f \in \mathcal{SL}(k,q)$, we find for the function $p(z)$, where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

that

$$\frac{z\left(\mathcal{D}_q f\right)(z)}{f(z)} = p(z) := 1 + p_1 z + p_2 z^2 + \cdots,$$

where

$$p \prec \tilde{q}(z).$$

If

$$p(z) \prec \tilde{q}(z),$$

then there is an analytic function w such that

$$|w(z)| \leqq |z| \quad \text{in} \quad \mathbb{U}$$

and

$$p(z) = \tilde{q}\big(w(z)\big).$$

Therefore, the function $g(z)$, given by

$$g(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z + \cdots \qquad (\forall\, z \in \mathbb{U}), \tag{20}$$

is in the class $\mathcal{P}$. It follows that

$$w(z) = \left(\frac{c_1}{2}\right) z + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots$$

and

$$\begin{aligned}
\tilde{q}(w(z)) = 1 &+ \frac{1}{4}(q+1)\,\tilde{p}_{k,1} c_1 z \\
&+ \left[\frac{1}{4}(q+1)\,\tilde{p}_{k,1}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{c_1^2}{16}(q+1)\left[(q-1)\tilde{p}_{k,1}^2 + 2\tilde{p}_{k,2}\right]\right] z^2 \\
&+ \left[\frac{1}{4}(q+1)\,\tilde{p}_{k,1}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{1}{8}(q+1)\left\{(q-1)\tilde{p}_{k,1}^2 + 2\tilde{p}_{k,2}\right\} c_1 \right. \\
&\quad \cdot \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{64}(q+1)\left\{(q-1)^2\,\tilde{p}_{k,1}^3 \quad + 4\tilde{p}_{k,2}\tilde{p}_{k,1}(q-1) + 4\tilde{p}_{k,3}\right\} c_1^3] z^3 + \cdots \\
= p(z)&.
\end{aligned} \tag{21}$$

From (5), we find the coefficient $\tilde{p}_{k,n}$ of the function $\tilde{q}$ given by

$$\tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\mathcal{T}_k^n.$$

This shows the following relevant connection $\tilde{q}$ with the sequence of k-Fibonacci numbers:

$$\begin{aligned}\tilde{q}(w(z)) = 1 &+ \frac{1}{4}(q+1)k\mathcal{T}_k c_1 z + \left[\frac{1}{4}(q+1)k\mathcal{T}_k\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^2}{16}(q+1)\right.\\ &\left.\cdot\left((q-1)k^2 + 2\left(2+k^2\right)\right)\mathcal{T}_k^2\right]z^2 + \left[\frac{1}{4}(q+1)k\mathcal{T}_k\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\right.\\ &+ \frac{1}{8}(q+1)\left\{(q-1)k^2 + 2\left(2+k^2\right)\right\}\mathcal{T}_k^2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{64}(q+1)\\ &\left.\cdot\left\{(q-1)^2k^2 + 4\left(2+k^2\right)(q-1) + 4\left(k^2+3\right)\right\}k\mathcal{T}_k^3 c_1^3\right]z^3 + \cdots\end{aligned} \tag{22}$$

If

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

then, by (21) and (22), we find that

$$p_1 = \left(\frac{q+1}{2}\right)\frac{k\mathcal{T}_k c_1}{2} \tag{23}$$

$$p_2 = \frac{1}{4}(q+1)\left(k\mathcal{T}_k\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^4}{4}\right)\left\{(q-1)k^2 + 2(2+k^2)\right\}\mathcal{T}_k^2 \tag{24}$$

$$\begin{aligned}p_3 = (q+1)&\left[\frac{k\mathcal{T}_k}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \left\{(q-1)k^2\mathcal{T}_k^2 + 2\left(2+k^2\right)\mathcal{T}_k^2\right\}\right.\\ &\cdot\left(c_2 - \frac{c_1^2}{2}\right)\frac{c_1}{8} + \frac{c_1^3}{64}\left\{(q-1)^2k^2 + 4\left(2+k^2\right)(q-1)\right.\\ &\left.\left.+4\left(k^2+3\right)\right\}k\mathcal{T}_k^3\right].\end{aligned} \tag{25}$$

Moreover, we have

$$\begin{aligned}\frac{z\left(\mathcal{D}_q f\right)(z)}{f(z)} &= 1 + qa_2 z + q\left\{(1+q)a_3 - a_2^2\right\}z^2\\ &\quad + \left\{\mathcal{Q}a_4 - q(2+q)a_2a_3 + qa_2^3\right\}z^3 + \cdots\\ &= 1 + p_1 z + p_2 z^2 + \cdots\end{aligned}$$

and

$$\begin{aligned}a_2 &= \frac{p_1}{q}\\ a_3 &= \frac{qp_2 + p_1^2}{q^2(q+1)},\\ a_4 &= \frac{q^2(q+1)p_3 - p_1^3(q+1) + (2+q)\left(p_1p_2q + p_1^3\right)}{q^3(q+1)\mathcal{Q}}\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\left|a_2a_4-a_3^2\right| = &\left|\frac{\mathcal{T}_k^2}{q^3(q+1)\mathcal{Q}}\right|\left|\left(\frac{(q+1)c_1}{2}\right)^2\left\{\frac{\mathcal{Q}c_1^2}{16}(q+1)^2\right.\right.\\ &+\left\{\mathcal{Q}-q^2(q+1)^2\right\}\left(c_2-\frac{c_1^2}{4}\right)\left.\frac{(2+k^2)}{4}\right\}\frac{k\,\mathcal{T}_k^n}{F_{k,n}}\\ &-\left(\frac{(q+1)c_1}{2}\right)^2\left\{\frac{\mathcal{Q}c_1^2}{16}(q+1)^2\left\{\mathcal{Q}-q^2(q+1)^2\right\}\right.\\ &\cdot\left(c_2-\frac{c_1^2}{4}\right)\left.\frac{(2+k^2)}{4}\right\}x_{k,n}+\left(\frac{(q+1)k}{2}\right)^2\\ &\cdot\left\{q^2\left(\frac{q+1}{2}\right)^2c_1(c_1c_2-c_3)+\frac{\mathcal{Q}}{4}\left(c_2-\frac{c_1^2}{2}\right)^2\right\}\\ &+\left\{\frac{\mathcal{Q}}{8}(q+1)-\frac{1}{4}q^2k^2\right\}\left(\frac{q+1}{2}\right)c_1^4\\ &+\left\{\mathcal{E}_qk^3\frac{c_1^2}{2}\left(c_2-\frac{c_1^2}{2}\right)+\mathcal{B}_qk^5c_1^4+\mathcal{C}_qk^3c_1^4\right\}\frac{\mathcal{T}_k^n}{F_{k,n}}\\ &\left.-\left\{\mathcal{E}_qk^3\frac{c_1^2}{2}\left(c_2-\frac{c_1^2}{2}\right)+\mathcal{B}_qk^5c_1^4+\mathcal{C}_qk^3c_1^4\right\}x_{k,n}+\mathcal{B}_qk^4c_1^4+\mathcal{C}_qk^2c_1^4\right|,\end{aligned}$$

where

$$\mathcal{E}_q=\frac{1}{16}(q+1)^2(q-1)\left\{\mathcal{Q}-q^2(q+1)^2\right\}.$$

This can be written as follows:

$$\begin{aligned}\left|a_2a_4-a_3^2\right| = &\left|\frac{\mathcal{T}_k^2}{q^2(q+1)^2\mathcal{Q}}\right|\left|\left\{\mathcal{Q}\left(\frac{q+1}{2}\right)^4\frac{c_1^4}{4}+\left(\frac{q+1}{2}\right)^2\right.\right.\\ &\cdot\left\{\mathcal{Q}-q^2(q+1)^2\right\}c_1^2\left(c_2-\frac{c_1^2}{4}\right)\left.\frac{(2+k^2)}{4}\right\}\frac{k\mathcal{T}_k^n}{F_{k,n}}\\ &+q^2\left(\frac{q+1}{2}\right)^4k^2c_1(c_1c_2-c_3)-\frac{q^2}{64}(q+1)^4k^2c_1^4+\frac{\mathcal{Q}}{16}(q+1)^2c_1^4\\ &-\mathcal{Q}\left(\frac{q+1}{2}\right)^4\frac{c_1^4}{4}kx_{k,n}+\frac{\mathcal{Q}k^2}{4}\left(\frac{q+1}{2}\right)^2c_2\left(c_2-\frac{c_1^2}{2}\right)\\ &+\frac{3}{8}(q+1)^2\left\{\frac{1}{6}\left(k^2+2\right)\left\{\mathcal{Q}-q^2(q+1)^2\right\}kx_{k,n}-\frac{k^2}{4}\right\}\\ &\cdot c_1^2\left(c_2-\frac{c_1^2}{2}\right)+\left\{\mathcal{E}_qk^3\frac{c_1^2}{2}\left(c_2-\frac{c_1^2}{2}\right)+\mathcal{B}_qk^5c_1^4+\mathcal{C}_qk^3c_1^4\right\}\frac{\mathcal{T}_k^n}{F_{k,n}}\\ &\left.-\left\{\mathcal{E}_qk^3\frac{c_1^2}{2}\left(c_2-\frac{c_1^2}{2}\right)+\mathcal{B}_qk^5c_1^4+\mathcal{C}_qk^3c_1^4\right\}x_{k,n}+\mathcal{B}_qk^4c_1^4+\mathcal{C}_qk^2c_1^4\right|.\end{aligned} \tag{26}$$

It is known that

$$\forall\, n\in\mathbb{N},\;\; \mathcal{T}_k=\frac{\mathcal{T}_k^n}{F_{k,n}}-x_{k,n},\qquad x_{k,n}=\frac{F_{k,n-1}}{F_{k,n}},\qquad \lim_{n\to\infty}\frac{F_{k,n-1}}{F_{k,n}}=|\mathcal{T}_k|. \tag{27}$$

Applying (27) together with (11)–(13), we get

$$\left|a_2a_4-a_3^2\right| \leqq \left|\frac{\mathcal{T}_k^2}{q^2(q+1)^2\mathcal{Q}}\right|\left|\left\{\mathcal{Q}\left(\frac{q+1}{2}\right)^4\frac{c_1^4}{4}+\left(\frac{q+1}{2}\right)^2\right.\right.$$
$$\left.\cdot\{\mathcal{Q}-q^2(q+1)^2\}c_1^2\left(c_2-\frac{c_1^2}{4}\right)\frac{(2+k^2)}{4}\right\}\left|\frac{k\mathcal{T}_k^n}{F_{k,n}}\right.$$
$$+\left|q^2\left(\frac{q+1}{2}\right)^4k^2\right||c_1||c_1c_2-c_3|-\left|\frac{1}{4}q^2\left(\frac{q+1}{2}\right)k^2\right|\left|c_1^4\right|$$
$$+\frac{\mathcal{Q}}{4}\left(\frac{q+1}{2}\right)^2\left|c_1^4\right|-\mathcal{Q}\left(\frac{q+1}{2}\right)^4\frac{|c_1^4|}{4}kx_{k,n}$$
$$+\left|\frac{\mathcal{Q}k^2}{4}\left(\frac{q+1}{2}\right)^2\right||c_2|\left|c_2-\frac{c_1^2}{2}\right|+\left|\frac{3}{8}\left\{2\left(\frac{q+1}{2}\right)^2\right.\right.$$
$$\left.\left.\cdot\{\mathcal{Q}-q^2(q+1)^2\}kx_{k,n}-\left(\frac{q+1}{2}\right)^2k^2\right\}\right||c_1|^2$$
$$\cdot\left|c_2-\frac{c_1^2}{2}\right|+\left\{\mathcal{E}_qk^3\frac{c_1^2}{2}\left(c_2-\frac{c_1^2}{2}\right)+\mathcal{B}_qk^5c_1^4+\mathcal{C}_qk^3c_1^4\right\}\frac{\mathcal{T}_k^n}{F_{k,n}}$$
$$-\mathcal{E}_qk^3\frac{|c_1^2|}{2}\left|2-\frac{c_1^2}{2}\right|x_{k,n}-\mathcal{B}_qk^5|c_1|^4x_{k,n}-\mathcal{C}_qk^3|c_1|^4x_{k,n}+\mathcal{B}_qk^4|c_1^4|+\mathcal{C}_qk^2|c_1^4|.$$

From (27), we obtain

$$\left(\frac{q+1}{8}\right)\left\{\mathcal{Q}\left(\frac{q+1}{2}\right)-\mathcal{Q}\left(\frac{q+1}{2}\right)^3kx_{k,n}-q^2k^2\right\}|c_1^4|>0$$

and

$$\left(\frac{q+1}{2}\right)^2\left\{\frac{2}{3}\left(k^2+2\right)\left\{\mathcal{Q}-q^2(q+1)^2\right\}kx_{k,n}-k^2\right\}>0,$$

which, for sufficiently large n, yields

$$|c_1|=:y\in[0,2].$$

After some computations, we can find that

$$\max_{y\in[0,2]}\left\{\frac{q^2}{8}(q+1)^4k^2y+\left|-\frac{q^2}{8}(q+1)k^2+\mathcal{Q}\frac{(q+1)^2}{16}\right.\right.$$
$$\left.-\frac{1}{64}\mathcal{Q}(q+1)^4kx_{k,n}\right|y^4+\left|\mathcal{Q}k^2\frac{(q+1)^2}{8}\right|\left(2-\frac{y^2}{2}\right)$$
$$+\left|\frac{3}{8}\left\{\left(\frac{q+1}{2}\right)^2\left(\frac{2}{3}(k^2+2)\{\mathcal{Q}-q^2(q+1)^2\}kx_{k,n}-k^2\right)\right\}y^2\left(2-\frac{y^2}{2}\right)\right.$$
$$\left.-|\mathcal{E}_q|k^3\frac{y^2}{2}\left(2-\frac{y^2}{2}\right)x_{k,n}-|\mathcal{B}_q|k^5y^4x_{k,n}-|\mathcal{C}_q|k^3y^4x_{k,n}+|\mathcal{B}_q|k^4y^4+|\mathcal{C}_q|k^2y^4\right|$$
$$=4\mathcal{Q}\left(\frac{q+1}{2}\right)^2\{1-kx_{k,n}\}+\left(16|\mathcal{B}_q|k^4+16|\mathcal{C}_q|k^2\right)\{1-kx_{k,n}\}.$$

As a result of the following limit formula:

$$\lim_{n\to\infty}\left[\left|\left(\frac{q+1}{2}\right)^2\frac{c_1^2}{4}\left\{\left(\frac{q+1}{2}\right)^2\mathcal{Q}c_1^2+\left\{\mathcal{Q}-q^2(q+1)^2\right\}\left(c_2-\frac{c_1^2}{4}\right)(2+k^2)\right\}\right|\right.$$
$$\left.+\left|\mathcal{E}_q k^3\frac{c_1^2}{2}\left(c_2-\frac{c_1^2}{2}\right)+\mathcal{B}_q k^5 c_1^4+\mathcal{C}_q k^3 c_1^4\right|\right]\frac{|\mathcal{T}_k^n|}{F_{k,n}}=0,$$

and by using (27), we get

$$\lim_{n\to\infty}\left[\max_{y\in[0,2]}\left\{q^2\left(\frac{q+1}{2}\right)^4 2k^2y+\left|-\frac{1}{8}q^2(q+1)k^2+\frac{\mathcal{Q}}{16}(q+1)^2\right.\right.\right.$$
$$\left.-\frac{1}{64}\mathcal{Q}(q+1)^4 kx_{k,n}\right|y^4+\mathcal{Q}k^2\frac{(q+1)^2}{8}\left(2-\frac{y^2}{2}\right)^2+\frac{3}{32}(q+1)^2$$
$$\cdot\left\{\frac{2}{3}(k^2+2)\{\mathcal{Q}-q^2(q+1)^2\}kx_{k,n}-k^2\right\}y^2\left(2-\frac{y^2}{2}\right)-|A|k^3\frac{y^2}{2}$$
$$\left.\cdot\left(2-\frac{y^2}{2}\right)x_{k,n}-|\mathcal{B}_q|k^5y^4x_{k,n}-|\mathcal{C}_q|k^3y^4x_{k,n}+|\mathcal{B}_q|k^4y^4+|\mathcal{C}_q|k^2y^4\right]$$
$$=\mathcal{Q}(q+1)^2\mathcal{T}_k^2+(|\mathcal{B}_q|k^2+|\mathcal{C}_q|)16k^2\mathcal{T}_k^2$$

We thus find that

$$\left|a_2a_4-a_3^2\right|\leqq\frac{\mathcal{T}_k^2}{q^2(q+1)^2\mathcal{Q}}\left\{\mathcal{Q}(q+1)^2+(|\mathcal{B}_q|k^2+|\mathcal{C}_q|)16k^2\right\}\mathcal{T}_k^2.$$

If, in (20), we set

$$g(z)=\frac{1+z}{1-z}=1+2z+2z^2+\cdots,$$

then, by putting $c_1=c_2=c_3=2$ in (26), we obtain

$$\left|a_2a_4-a_3^2\right|=\frac{\mathcal{T}_k^2}{q^2(q+1)^2\mathcal{Q}}\left\{\mathcal{Q}(q+1)^2+(|\mathcal{B}_q|k^2+|\mathcal{C}_q|)16k^2\right\}\mathcal{T}_k^2$$

This completes the proof of Theorem 1. □

Remark 3. In the next result, for simplicity, we take the values of $\mathcal{S}_q$, $\mathcal{L}_q$ and $\mathcal{M}_q$ as given by

$$\mathcal{S}_q=q^3(1+q)\mathcal{Q},$$

$$\mathcal{L}_q=\{q(1+q)-q(2+q)+\mathcal{Q}\}\left(\frac{q+1}{2}\right)^3\left(\frac{k^3c_1^3}{8}\right)-q^3\left(\frac{q+1}{2}\right)^2k^3c_1^3$$
$$+\frac{c_1^3k}{16}\left[\{q\mathcal{Q}-q^2(2+q)\}\left(\frac{q+1}{2}\right)^2\left\{(q-1)k^2+2\left(2+k^2\right)\right\}\right]$$
$$-\frac{c_1^3}{32}q^3\left(\frac{q+1}{2}\right)^2\left\{(q-1)k^3-(q-1)\left(2+k^2\right)k\right\}$$

and

$$\mathcal{M}_q=\left[\{q\mathcal{Q}-q^2(2+q)\}\left(\frac{q+1}{2}\right)^2-q^3\left(\frac{q+1}{2}\right)^3(q-1)k^2\right]\frac{c_1}{2}\left(c_2-\frac{c_1^2}{2}\right).$$

Theorem 2. *Let the function $f \in \mathcal{A}$ given in (1) belong to the class $\mathcal{SL}(k,q)$. Then*

$$|a_2a_3 - a_4| = \frac{2}{\mathcal{S}_q}\left(\frac{3}{4}kq^3(q+1)^2|\mathcal{T}_k^3| + \frac{1}{2}\left|\left\{\mathcal{M}_q + \left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n} + \frac{1}{2}k|\mathcal{L}_q|\right). \tag{28}$$

Proof. Let $f \in \mathcal{SL}(k,q)$ and let $p \in \mathcal{P}$ be given in (2). Then, from (23)–(25) and

$$\begin{aligned}\frac{z\mathcal{D}_q f(z)}{f(z)} &= 1 + qa_2z + \left\{\left(q+q^2\right)a_3 - qa_2^2\right\}z^2 \\ &\quad + \left\{\mathcal{Q}a_4 - \left(2q+q^2\right)a_2a_3 + qa_2^3\right\}z^3 + \cdots \\ &= 1 + p_1z + p_2z^2 + \cdots,\end{aligned}$$

we have

$$\begin{aligned}a_2a_3 - a_4 = \frac{1}{\mathcal{S}_q}&\left[\left\{q\mathcal{Q} - q^2(2+q)\right\}p_1p_2\right. \\ &\left. + \{q(1+q) - q(2+q) + \mathcal{Q}\}p_1^3 - q^3(1+q)p_3\right],\end{aligned}$$

which, together with (27), yields

$$\begin{aligned}|a_2a_3 - a_4| = \frac{2}{\mathcal{S}_q}&\left|\frac{q^3}{4}(q+1)^2\left[\frac{1}{4}\left(c_2 - \frac{c_1^3}{2}\right)c_1k^2 + \frac{1}{2}(c_1c_2 - c_3)\right.\right. \\ &- \left.\left(\frac{3k^2+4}{4}\right)c_1c_2\right]\frac{k\mathcal{T}_k}{f_{k,n}} + \frac{1}{8}q^3(q+1)^2\left(c_3 - 2c_1c_2 + c_1^3\right)kx_{k,n} \\ &+ \frac{q^3}{8}(q+1)^2\left\{\frac{1}{2}\left(4-k^2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \left(3k^2+2\right)c_2\right\}kc_1x_{k,n} \\ &+ \frac{q^3}{16}(q+1)^2\left\{\left(4-k^2\right)\left(c_2 - \frac{c_1^2}{2}\right) - 3k^2c_2\right\}c_1 - \frac{q^3}{32}(q+1)^3 \\ &\cdot(q-1)\left(c_2 - \frac{c_1^2}{2}\right)c_1k^2 + \frac{1}{2}\left\{\mathcal{M}_q + \left(1+k^2\right)\mathcal{L}_q\right\}\frac{k\mathcal{T}_k}{f_{k,n}} \\ &\left.-\frac{1}{2}\left\{\mathcal{M}_q + \left(1+k^2\right)\mathcal{L}_q\right\}kx_{k,n} + \frac{1}{2}k\mathcal{L}_q\right|.\end{aligned} \tag{29}$$

Now, applying the triangle inequality in (10)–(13), we get

$$\begin{aligned}|a_2a_3 - a_4| \leq \frac{2}{\mathcal{S}_q}&\left|\frac{q^3}{16}(q+1)^2\left(c_2 - \frac{c_1^3}{2}\right)c_1k^2 + \frac{q^3}{8}(q+1)^2(c_1c_2 - c_3)\right. \\ &\left.-\frac{q^3}{16}(q+1)^2\left(3k^2+4\right)c_1c_2\right|\frac{k\mathcal{T}_k}{f_{k,n}} + \frac{q^3}{4}(q+1)^2kx_{k,n} \\ &+ \frac{q^3}{4}(q+1)^2kx_{k,n} + \frac{q^3}{8}(q+1)^2\left|k\left(4-k^2\right)x_{k,n} - \left(4-k^2\right)\right||c_1| \\ &- \frac{q^3}{32}(q+1)^2\left|k\left(4-k^2\right)x_{k,n} - \left(4-k^2\right)\right|\left|c_1^3\right| + \frac{q^3}{4}(q+1)^2 \\ &\cdot\left|\left(3k^2+2k\right)x_{k,n} - 3k^2\right| - \frac{q^3}{32}(q+1)^3(q-1)\left(2 - \frac{|c_1|^2}{2}\right)k^2|c_1| \\ &+ \frac{1}{2}\left|\left\{\mathcal{M}_q + \left(1+k^2\right)\mathcal{L}_q\right\}\right|\frac{k\mathcal{T}_k}{f_{k,n}} - \frac{1}{2}\left|\left\{\mathcal{M}_q + \left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n} + \frac{1}{2}k|\mathcal{L}_q|.\end{aligned}$$

In addition, by using (27), we have

$$\frac{q^3}{4}(q+1)^2\left(4-k^2\right)kx_{k,n}-\left(4-k^2\right)<0 \qquad (0<k<2)$$

and

$$\frac{q^3}{4}(q+1)^2(3k+2)\,kx_{k,n}-3k^2>0$$

for $0<k\leqq 1$ and sufficiently large n. Therefore, we have got a function of the variable $|c_1|=:y\in[0,2]$ and, after some computations, we can find that

$$\begin{aligned}&\max_{y\in[0,2]}\left\{\frac{q^3}{4}(q+1)^2\left\{kx_{k,n}+\frac{1}{2}\left(k\left(4-k^2\right)x_{k,n}-\left(4-k^2\right)\right)y\right\}-\frac{q^3}{32}(q+1)^2\right.\\&\quad\left.\cdot\left(k\left(4-k^2\right)x_{k,n}-\left(4-k^2\right)\right)y^3+\left|\frac{1}{2}\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n}+\left|\frac{1}{2}k\mathcal{L}_q\right|\right\}\\&\quad=\frac{q^3}{4}(q+1)^2\left(3k^2+3k\right)x_{k,n}-3k^2+\left|\frac{1}{2}\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n}+\left|\frac{1}{2}k\mathcal{L}_q\right|.\end{aligned}$$

As a result of the following limit relation:

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{q^3}{16}(q+1)^2\left(c_2-\frac{c_1^3}{2}\right)c_1k^2+\frac{q^3}{8}(q+1)^2(c_1c_2-c_3)-\frac{q^3}{16}(q+1)^2\right.\\&\qquad\left.\cdot\left(3k^2+4\right)c_1c_2+\frac{1}{2}\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right]\frac{k\mathcal{T}_k}{f_{k,n}}=0\end{aligned}$$

and, by means of (27), we have

$$\begin{aligned}&\lim_{n\to\infty}\left[\max_{y\in[0,2]}\left\{\frac{q^3}{4}(q+1)^2\left\{kx_{k,n}+\frac{1}{2}\left(k\left(4-k^2\right)x_{k,n}-\left(4-k^2\right)\right)y\right\}-\frac{q^3}{32}(q+1)^2\right.\right.\\&\quad\left.\cdot\left(k\left(4-k^2\right)x_{k,n}-\left(4-k^2\right)\right)y^3+\frac{q^3}{4}(q+1)^2\left(3k^2+2k\right)x_{k,n}-3k^2\right\}\\&\quad\left.+\frac{1}{2}\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}kx_{k,n}+\frac{1}{2}k\mathcal{L}_q\right]\\&\quad=q^3\left(\frac{q+1}{2}\right)^2\left\{\left(3k^2+3k\right)\mathcal{T}_k-3k^2\right\}+\frac{1}{2}\left|\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n}+\frac{1}{2}k\left|\mathcal{L}_q\right|\\&\quad=q^3\left(\frac{q+1}{2}\right)^2\left\{-3k\left(\left(k^2+1\right)\mathcal{T}_k+k\right)\right\}+\frac{1}{2}\left|\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n}+\frac{1}{2}k\left|\mathcal{L}_q\right|\\&\quad=q^3\left(\frac{q+1}{2}\right)^2\left(-3k\mathcal{T}_k^3\right)+\frac{1}{2}\left|\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n}+\frac{1}{2}k\left|\mathcal{L}_q\right|\Big)\\&\quad=3q^3\left(\frac{q+1}{2}\right)^2k|\mathcal{T}_k^3|+\frac{1}{2}\left|\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|kx_{k,n}+\frac{1}{2}k\left|\mathcal{L}_q\right|\end{aligned}$$

If, in the formula (20) , we set

$$g(z)=\frac{1+z}{1-z}=1+2z+2z^2+\cdots,$$

then, by putting $c_1=c_2=c_3=2$ in (26), we obtain

$$|a_2a_3-a_4|=\frac{k}{2}\left\{\frac{3q^3}{\mathcal{S}_q}(q+1)^2|\mathcal{T}_k^3|+1\left|\left\{\mathcal{M}_q+\left(1+k^2\right)\mathcal{L}_q\right\}\right|x_{k,n}+|\mathcal{L}_q|\right\}$$

This completes the proof of Theorem 2. □

Theorem 3. *Let the function $f \in \mathcal{A}$ given in (1) belong to the class $\mathcal{SL}(k,q)$. Then*

$$\left|a_3 - \lambda a_2^2\right| \leqq \frac{\mathcal{T}_k^2}{q^2(q+1)} \left[\left\{ \left| \mathcal{G} \left(\frac{1+q}{2} \right)^2 \right| + \left| \frac{q}{4} \left(q^2 - 1 \right) + q \left(\frac{q+1}{2} \right) \right| \right\} k^2 + q\,(q+1) \right] \tag{30}$$

Proof. Let $f \in \mathcal{SL}(k,q)$ and let $p \in \mathcal{P}$ given in (2). Then, from (23)–(25) and

$$\begin{aligned} \frac{z\mathcal{D}_q f\,(z)}{f\,(z)} &= 1 + q a_2 z + \left\{ \left(q + q^2 \right) a_3 - q a_2^2 \right\} z^2 \\ &\quad + \left\{ \mathcal{Q} a_4 - \left(2q + q^2 \right) a_2 a_3 + q a_2^3 \right\} z^3 + \cdots \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \end{aligned}$$

we have

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{q^2\,(1+q)} \left| \left[1 + \left|\lambda^2\right| (1+q) \right] p_1^2 + q p_2 \right|.$$

Therefore, we obtain

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &= \frac{1}{q^2\,(1+q)} \left| (1 - \lambda^2\,(1+q)) \left[\left(\frac{1+q}{2} \right)^2 \left(\frac{\kappa c_1 \mathcal{T}}{2} \right)^2 \right] \right. \\ &\quad \left. + q \left\{ \frac{q+1}{4} \kappa \mathcal{T}_k (c_2 - \frac{c_1^2}{2}) c_1^2 (\frac{q+1}{16}) \right\} \left\{ (q-1)\kappa^2 + 2(2+k^2)\mathcal{T}_k^2 \right\} \right| \\ &= \frac{\mathcal{T}_k}{q^2\,(1+q)} \left| (1 + \left|\lambda^2\right| (1+q)\, (\frac{1+q}{2})^2 \frac{k^2 c_1^2}{4} \mathcal{T}_k + q \left[\frac{q+1}{4} \kappa \left(c_2 - \frac{c_1^2}{2} \right) \right. \right. \\ &\quad \left. \left. + \frac{q(q+1)c_1^2}{16} (q-1) k^2 \mathcal{T}_k + 2(2+k^2)\mathcal{T}_k^2 \right] \right|. \end{aligned}$$

Thus, by applying (27), we have

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &= \frac{\mathcal{T}_k}{q^2\,(1+q)} \left| \mathcal{G} \left(\frac{1+q}{2} \right)^2 \frac{k^2 c_1^2}{4} \mathcal{T}_k + q (\frac{q+1}{4}) \frac{c_1^2}{4} \right. \\ &\quad \cdot \left[(q-1)k^2 + 2(2+k^2) \frac{\mathcal{T}_k^n}{f_{k,n}} \right] - \left(\mathcal{G} \left(\frac{1+q}{2} \right)^2 \frac{k^2 c_1^2}{4} + q \frac{(q+1)}{4} \frac{c_1^2}{4} \right) \\ &\quad \left. \cdot \left[(q-1)k^2 + 2(2+k^2) \right] x_{k,n} + q \left(\frac{q+1}{4} \right) k \left(c_2 - \frac{c_1^2}{2} \right) \right|, \end{aligned}$$

where

$$\mathcal{G} = 1 + \left|\lambda^2\right| (1+q)\,.$$

Now, by applying the triangle inequality in (10)–(13), we have

$$
\begin{aligned}
&\left|\frac{\mathcal{T}_k}{q^2\,(1+q)}\right|\left|\mathcal{G}\left(\frac{1+q}{2}\right)^2\frac{k^2c_1^2}{4}+q\left(\frac{q+1}{4}\right)\frac{c_1^2}{4}\left[(q-1)k^2+2(2+k^2)\right]\right| \\
&\quad\cdot\frac{\mathcal{T}_k^n}{f_{k,n}}-\left[\left|\mathcal{G}\left(\frac{1+q}{2}\right)^2\frac{k^2}{4}\right||c_1|^2_{x_{k,n}}+q\left(\frac{q^2-1}{4}\right)k^2\,|c_1|^2_{x_{k,n}}+q\left(\frac{q+1}{4}\right)\right. \\
&\quad\left.\cdot\left|\frac{c_1^2}{2}(2+k^2)x_{k,n}\right|+q\left(\frac{q+1}{4}\right)k\left|c_2-\frac{c_1^2}{2}\right|\right] \\
&\quad=\left|\frac{\mathcal{T}_k}{q^2\,(1+q)}\right|\left|\frac{\mathcal{G}}{16}\,(1+q)^2\,k^2c_1^2+\frac{q}{16}\,(q+1)\,c_1^2\left[(q-1)k^2+2(2+k^2)\right]\right| \\
&\quad\cdot\frac{\mathcal{T}_k^n}{f_{k,n}}-\left[\left|\mathcal{G}\left(\frac{1+q}{2}\right)^2\frac{k^2}{4}\right||c_1|^2_{x_{k,n}}+q\left(\frac{q+1}{4}\right)(q-1)k^2\,|c_1|^2_{x_{k,n}}\right. \\
&\quad\left.+\frac{q\,(q+1)}{4}\left|\frac{c_1^2}{2}(2+k^2)x_{k,n}\right|+\left|q\left(\frac{q+1}{4}\right)k\right|\left|c_2-\frac{c_1^2}{2}\right|\right],
\end{aligned}
$$

which, after some computations, yields

$$
\begin{aligned}
&\max_{y\epsilon[0,2]}\left|\mathcal{G}\left(\frac{1+q}{2}\right)^2\frac{k^2}{4}\right|y^2x_{k,n}+\left|q\left(\frac{q+1}{4}\right)(q-1)k^2\right|x_{k,n}+\left|q\left(\frac{q+1}{4}\right)(2+k^2)\right|2x_{k,n} \\
&\quad=\left[\left|\mathcal{G}\left(\frac{1+q}{2}\right)^2\right|+\left|q\left(\frac{q^2-1}{4}\right)\right|+\left|q\left(\frac{q+1}{2}\right)\right|\right]k^2x_{k,n}+q(q+1)x_{k,n},
\end{aligned}
$$

in which we have set $y=2$. As a result of the following limit formula:

$$
\lim_{n\to\infty}\left|\mathcal{G}\left(\frac{1+q}{2}\right)^2\frac{k^2c_1^2}{4}+q\left(\frac{q+1}{2}\right)\frac{c_1^2}{4}\left\{(q-1)\,k^2+2(2+k^2)\right\}\right|\frac{\mathcal{T}_k^n}{f_{x_{k,n}}}=0,
$$

which, by applying (27), yields

$$
\begin{aligned}
\left|a_3-\lambda a_2^2\right|\leqq\frac{\mathcal{T}_k^2}{q^2(q+1)}&\left[\left\{\frac{1}{4}\left|\mathcal{G}\,(1+q)^2\right|+\left|\frac{q}{4}\left(q^2-1\right)+\frac{q}{2}\,(q+1)\right|\right\}k^2\right. \\
&\left.+q\,(q+1)\right].
\end{aligned}
$$

This completes the proof of Theorem 3. □

Theorem 4. *Let the function $f\in\mathcal{A}$ given in (1) belong to the class $\mathcal{SL}(k,q)$. Then*

$$
\begin{aligned}
|H_3(1)|\leqq&\frac{\mathcal{T}_k^6}{q^4(1+q)^3\mathcal{Q}}[(2q+(q+1)k^2]\{16\,|\mathcal{B}_q|\,k^4+16\,|\mathcal{C}_q|\,k^2\} \\
&+\frac{\mathcal{T}_k^3 2(2+q)k^3+(5q+7)k}{2q^2(1+q)\mathcal{Q}}\left|\{\mathcal{M}_q+(1+k^2)\mathcal{L}_q\}kx_{k,n}+k\mathcal{L}_q\right|.
\end{aligned}
$$

Proof. Let $f\in\mathcal{SL}(k,q)$. Then as we know that

$$
H_3(1)=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2),
$$

where $a_1=1$ so, we have

$$
|H_3(1)|\leqq|a_3|\left|a_2a_4-a_3^2\right|+|a_4|\,|a_4-a_2a_3|+|a_5|\left|a_3-a_2^2\right| \tag{31}
$$

Thus, by using Lemma 4, Theorems 1–3, as well as the formula (31), we find that

$$|H_3(1)| \leqq 2(\digamma_q k^6 + 4\Psi_q k^4 + 4\Upsilon_q k^2 + \Gamma_q)\mathcal{T}_k^6, \tag{32}$$

where

$$\digamma_q = \frac{\varkappa_{q,\lambda}}{2q^2\mathcal{Q}}\left\{1 + \frac{\chi_q(2+q)}{\mathcal{Q}q^2} - \frac{(q+3)(q+1)^2}{4q^2} + \left(\frac{q+1}{q^3}\right) + \frac{(q+1)^4}{16q^3}\right\}$$

$$\begin{aligned}\Psi_q = {} & \frac{3(2+q)}{q^3(1+q)^2\mathcal{Q}} + \frac{\varkappa_{q,\lambda}}{2q^2(\mathcal{Q}+1)}\left\{4 + \frac{(5q+7)\chi_q}{2q^2\mathcal{Q}} - \frac{(q+3)(q+1)}{2q} + \frac{4}{q^2}\right\} \\ & + \frac{(q+1)}{2q(\mathcal{Q}+1)}\left\{1 + \frac{\chi_q(2+q)}{q^2\mathcal{Q}} - \frac{(q+3)(q+1)^2}{4q^2} + \frac{q+1}{q^3} + q\left(\frac{q+1}{2q}\right)^4\right\}\end{aligned}$$

$$\begin{aligned}\Upsilon_q = {} & \frac{1}{q^2} + \frac{1}{(\mathcal{Q}+1)}\left(4 + \frac{\chi_q(5q+7)}{2q^2\mathcal{Q}}\right) - \frac{q+1}{2q}\left(\frac{(q+1)(q+3)}{2q} + \frac{4}{q^2}\right) \\ & + \frac{3(5q+7)}{2q^2\mathcal{Q}} + \frac{\varkappa_{q,\lambda}}{2q^2(\mathcal{Q}+1)}\left(2 + \frac{4}{q(1+q}\right)\end{aligned}$$

$$\begin{aligned}\Gamma_q = {} & \frac{1}{q^4(1+q)^3\mathcal{Q}}\left(2q + (q+1)k^2\right)\left(16\left|\mathcal{B}_q\right|k^4 + 16\left|\mathcal{C}_q\right|k^2\right) \\ & + \frac{2(2+q)k^3 + (5q+7)k}{2q^2(1+q)\mathcal{Q}}\left|\{\mathcal{M}_q + (1+k^2)L_q\}kx_{k,n} + kL_q\right|,\end{aligned}$$

and

$$\chi_q = q^2 + q + 2.$$

This completes the proof of Theorem 4. □

3. Conclusions

A new subclass of analytic functions associated with k-Fibonacci numbers has been introduced by means of quantum (or q-) calculus. Upper bound of the third Hankel determinant has been derived for this functions class. We have stated and proved our main results as Theorems 1–4 in this article.

Further developments based upon the the q-calculus can be motivated by several recent works which are reported in (for example) [24,25], which dealt essentially with the second and the third Hankel determinants, as well as [26–29], which studied many different aspects of the Fekete-Szegö problem.

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