# Sarvate-Beam Group Divisible Designs and Related Multigraph Decomposition Problems

by

Joanna Niezen
M.Sc., University of Victoria, 2013
B.Math, University of Waterloo, 2010

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## ABSTRACT

A design is a set of points, $V$, together with a set of subsets of $V$ called blocks. A classic type of design is a balanced incomplete block design, where every pair of points occurs together in a block the same number of times. This 'balanced' condition can be replaced with other properties. An adesign is a design where instead every pair of points occurs a different number of times together in a block. The number of times a specified pair of points occurs together is called the pair frequency.

Here, a special type of adesign is explored, called a Sarvate-Beam design, named after its founders D.G. Sarvate and W. Beam. In such an adesign, the pair frequencies cover an interval of consecutive integers. Specifically the existence of Sarvate-Beam group divisible designs are investigated. A group divisible design, in the usual sense, is a set of points and blocks where the points are partitioned into subsets called groups. Any pair of points contained in a group have pair frequency zero and pairs of points from different groups have pair frequency one. A Sarvate-Beam group divisible design, or SBGDD, is a group divisible design where instead the frequencies of pairs from different groups form a set of distinct nonnegative consecutive integers. The SBGDD is said to be uniform when the groups are of equal size.

The main result of this dissertation is to completely settle the existence question for uniform SBGDDs with blocks of size three where the smallest pair frequency, called the starting frequency, is zero. Higher starting frequencies are also considered and settled for all positive integers except when the SBGDD is partitioned into eight groups where a few possible exceptions remain.

A relationship between these designs and graph decompositions is developed and leads to some generalizations. The use of matrices and linear programming is also explored and give rise to related results.

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# Chapter 1

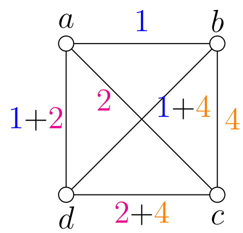
# Introduction

A *design* is a pair $(V, \mathcal{B})$ where $V$ is a set of points and $\mathcal{B}$ a set of subsets of $V$, called *blocks*. For example, a pairwise balanced design, or PBD$(v, K)$, is a design with $v$ points and blocks of size $k$ for some $k \in K$. A block of size $k$ is sometimes called a $k$-block. In a typical design, every pair of points occur in a block together a constant $\lambda \geq 1$ number of times, called the *index* of the design. This is what the term 'balanced' means in a PBD. This property is referred to as the *defining property* of designs like PBDs.

An adesign has points and blocks similarly to a pairwise balanced design. One might think of an adesign as a pairwise 'unbalanced' design since the defining property of an adesign is that every pair of points is covered a different number of times in its block set. The *pair frequency* of a pair of points is the number of times the pair appear in a block together. For example, in a pairwise balanced design the frequency of any pair of points is equal to the index $\lambda$. A Sarvate-Beam design is a special type of adesign. Its defining property is that the set of all pair frequencies are distinct consecutive integers. That is, a Sarvate-Beam design with $v$ points has pair frequencies $\mu, \mu+1, \ldots, \mu+\binom{v}{2}-1$, for some nonnegative integer $\mu$, called the *starting frequency*.

For example, consider a design with four points and blocks of size three. Call the points

$a, b, c, d$ and take all possible blocks: $abc, abd, acd$, and $bcd$ with multiplicities $0, 1, 2$, and $4$ respectively. The result is a Sarvate-Beam design with starting frequency one, shown in Figure 1.1, which is used throughout the dissertation.




| Blocks | $abc$ | $abd$ | $acd$ | $bcd$ | | |
|---|---|---|---|---|---|---|
| Multiplicity | 0 | 1 | 2 | 4 | | |
| Pairs | $ab$ | $ac$ | $ad$ | $bc$ | $bd$ | $cd$ |
| Frequency | 1 | 2 | 3 | 4 | 5 | 6 |

Figure 1.1: A Sarvate-Beam design on four points

The graph in Figure 1.1 depicts the relationship between the design and a decomposition of a certain complete multigraph into edge-disjoint triangles. The pair frequencies of the design are the edge multiplicities of the underlying complete graph. The block set, taken with multiplicities, describes a triangle decomposition of this multigraph since each of the blocks have size three. An edge-decomposition of a graph is often used to depict the blocks of a design. The relationship is discussed more deeply in Section 2.3.2.

A Sarvate-Beam design with blocks of size three is called a *Sarvate-Beam triple system* or SBTS. An SBTS with $v$ points and starting frequency $\mu$ is denoted $\text{SBTS}_\mu(v)$. The Sarvate-Beam design in Figure 1.1 is in fact an $\text{SBTS}_1(4)$. Starting frequency $0$ is the focus of this dissertation and $\text{SBTS}(v)$ is used to denote $\text{SBTS}_0(v)$. Higher starting frequencies are otherwise specified.

It is natural to take $\mu = 1$ which is a common convention in the literature. However, the constructions for any $\mu > 0$ often follow from $\mu = 0$. Moreover, considering designs with $\mu = 0$ loosens the necessary conditions for such a triple system to exist. In particular, the necessary conditions of existence for an $\text{SBTS}_\mu(v)$ are [12]

- $v > 3$, and

- $v \equiv 0, 1 \pmod 3$ or $\mu \equiv 0 \pmod 3$.

Stanton [25] handled the case of $v \equiv 2 \pmod 3$ and $\mu = 1$ by allowing for a single block of size two. Note that this is equivalent to starting frequency zero with uniform block size. The following theorem was proved by Dukes and Short-Gershman [12].

**Theorem 1.1.** There exists an $\text{SBTS}_\mu(v)$ whenever the necessary conditions are satisfied, except when $(v, \mu) = (4, 0)$.

To serve as an example, the non-existence of an $\text{SBTS}_0(4)$ is shown. As there are $\binom{4}{2} = 6$ pairs of points to cover, the frequencies are $0, 1, \ldots, 5$. Let the points be $a, b, c, d$ and suppose $ab$ is the pair with frequency zero. Then $b$ only occurs in the block $bcd$, depicted with blue dashed lines in Figure 1.2. This means that the pairs $bc$ and $bd$ have frequency equal to the multiplicity of $bcd$ and are therefore equal. Consequently, the defining property of distinct pair frequencies is not possible in this case.

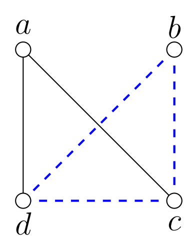



Figure 1.2: Non-existence of an $\text{SBTS}_0(4)$

A natural extension of an SBTS is a Sarvate-Beam group divisible design, abbreviated Sarvate-Beam GDD or SBGDD, which are related to group divisible designs in the same way SBTSs are related to pairwise balanced designs. A *group divisible design*, or GDD, is a design $(V, G, \mathcal{B})$, where $V$ is a set of points, $\mathcal{B}$ is a set of blocks, and $G$ is a partition of $V$ into subsets called *groups*. The defining property of a GDD is that every pair of points from distinct groups are in exactly $\lambda$ blocks together, while points from the same group have pair frequency zero. The defining property of a Sarvate-Beam GDD is that the set of pair frequencies of points from distinct groups are distinct consecutive nonnegative integers. Like a usual GDD, the pair frequencies of points in the same group of an SBGDD are zero. A GDD with $u$ groups each containing $g$ points is called a GDD of *type* $g^u$. The same notation is adapted for SBGDDs. Chapter 2 has more background on group divisible designs and their related structures.

The purpose of this dissertation is to answer the question; for what values of $g$ and $u$ does a Sarvate-Beam GDD of type $g^u$ with blocks of size three exist? Note that when $g = 1$ such an SBGDD is equivalent to an SBTS$(u)$, the same way a GDD of type $1^u$ is equivalent to a pairwise balanced design. The case when $g = 1$ is therefore taken care of by Theorem 1.1, and $g \geq 2$ is focused on. Theorem 1.2 presents a summary of the main result.

**Theorem 1.2.** A Sarvate-Beam GDD of type $g^u$ with blocks of size three and starting frequency zero exists for all $g \geq 2$ and $u \geq 3$.

Note that it is necessary to have $u \geq 3$ for block size three. The following theorem presents a partial result for starting frequency greater than zero.

**Theorem 1.3.** A Sarvate-Beam GDD of type $g^u$ with blocks of size three and starting frequency $\mu > 0$ exists for all integers $g \geq 2$ and $u \geq 3$ whenever the necessary conditions of existence are satisfied, except possibly when $u = 8, \mu \equiv 3 \pmod 6$, and $g \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$.

The proof of these theorems are a culmination of all the constructions built throughout Chapters 3 and 4. The case when $u = 3$ has a special geometric interpretation and the relevant objects are called 'Sarvate-Beam cubes'. For that reason, SBGDDs of type $g^3$ are considered separately in Chapter 3. Chapter 4 covers SBGDDs with more than three groups. The main constructions are in Sections 4.1 and 4.3. Small exceptional values are taken care of in Section 4.4. In order to access the rich theory of matrix analysis, a special type of incidence matrix is discussed in Chapter 5. Smith normal form is used to obtain results in a relaxation of the problem which allows for negative integer block multiplicities. In Chapter 6, the relationship to graph decompositions and corresponding generalizations of the problem are explored. First, some required background is given.

# Chapter 2

# Background

## 2.1 Designs

A *balanced incomplete block design*, or BIBD, is a pair $(V, \mathcal{B})$ where $V$ is a set of points and $\mathcal{B}$ is a set of $k$-subsets of $V$ called *blocks*. The defining property of a BIBD is that every pair of points in $V$ occurs in exactly $\lambda$ blocks together. The positive integer $\lambda$ is called the *index* of the design. A BIBD with $v$ points is said to have *order* $v$ and is often called a $(v, k, \lambda)$-design. The primary case of interest in the literature is when $\lambda = 1$, however, designs with higher index are useful for later constructions. A $(7, 3, 1)$-design and a $(4, 3, 2)$-design are depicted in Figures 2.1 and 2.8 respectively.

A *Steiner triple system* is a $(v, 3, 1)$-design, also denoted STS$(v)$. The only difference between an STS$(v)$ and an SBTS$(v)$ are their defining properties. The $(7, 3, 1)$-design in Figure 2.1 is also an STS$(7)$, or more commonly known as the Fano plane. Note that blocks are drawn as smooth lines through points, instead of triangles. That is, the block set is $\{abd, acg, aef, bce, bfg, cdf, deg\}$.

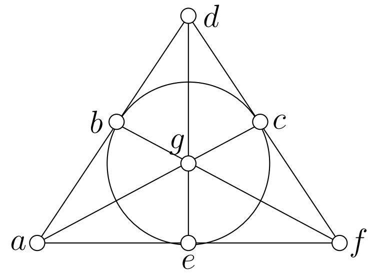


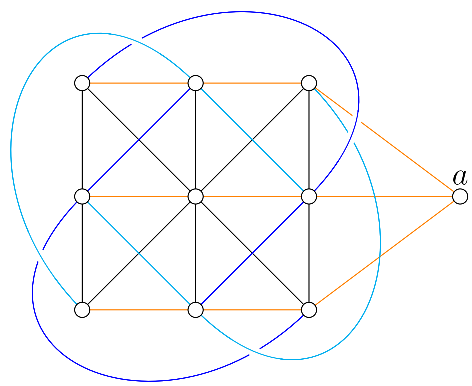

Figure 2.1: A Fano plane

### 2.1.1 Pairwise Balanced Designs

A *pairwise balanced design*, or PBD, is a system of points and blocks, $(V, \mathcal{B})$, such that every pair of points in $V$ is in exactly one block in $\mathcal{B}$. A PBD can be thought of as a BIBD where block sizes are allowed to vary. If $K$ is a set containing all block sizes in $\mathcal{B}$ then a pairwise balanced design of order $v$ is denoted PBD$(v, K)$, or sometimes $K$-PBD. If $K = \{k\}$, the PBD is sometimes called a $k$-PBD, or more commonly in this case it is denoted by its equivalent object, a $(v, k, 1)$-design. Note that not all block sizes need to occur. A PBD that necessarily has a block of size $k \in K$ is denoted by PBD$(v, K \cup \{k^*\})$.

**Example 2.1.1.** The following is a construction for a PBD$(11, \{3, 5\})$ consisting of a single block of size five, $B = \{0', 1', 2', 3', 4'\}$, and a 1-factorization of the complete graph, $K_6$. The remaining blocks consist of the matched points in a 1-factor of $K_6$ together with one point from $B$. More specifically, let the points in $K_6$ be $0, 1, 2, 3, 4, \infty$. Consider the 1-factor with edges $\{0, \infty\}$, $\{1, 4\}$, and $\{2, 3\}$. The corresponding blocks in the PBD are $\{0', 0, \infty\}, \{0', 1, 4\}$, and $\{0', 2, 3\}$. Since the 1-factor spans all points, this set of blocks cover all pairs that include $0'$ with a point outside of $B$. The remaining blocks are determined by additive shifts modulo 5 applied to each point in the block, pairing every 1-factor in $K_6$ with a different point of $B$.

Figure 2.2 depicts a PBD$(10, \{3, 4\})$. The block set is also explicitly given for later use. The blocks are listed without brackets or commas for succinctness. This is done throughout the dissertation. That is, rather than writing $\{a, b, c\}$ to be the block containing points $a, b$, and $c$, this is abbreviated to $abc$. While it is not needed here, repeated blocks occur throughout the dissertation. When $K$ is a multiset, the notation $8abc$, for example, is used to indicate that block $abc$ occurs with multiplicity eight in the design.




| | | | |
|---|---|---|---|
| $abcd$ | $beh$ | $cgi$ | $dgh$ |
| $aefg$ | $bfi$ | $cej$ | $dfj$ |
| $ahij$ | $bgj$ | $cfh$ | $dei$ |

Figure 2.2: A PBD$(10, \{3, 4\})$

### 2.1.2 Group Divisible Designs

A *group divisible design* is a triple $(V, G, \mathcal{B})$ where $V$ is a set of points, $\mathcal{B}$ is a set of blocks, and $G$ is a partition of $V$ into sets called *groups*. The defining property of a GDD is that every pair of points from distinct groups occur in $\lambda$ blocks together while points within the same group do not occur in any blocks together. A group divisible design is denoted by $(K, \lambda)$-GDD when it has block sizes in $K$ where $K$ is some subset of the positive integers. When $\lambda = 1$, which is the case typically of interest, this is abbreviated to $K$-GDD or

$k$-GDD when $K = \{k\}$.

Let $j_i$ and $g_i$ be natural numbers for each $i = 1, 2, \ldots, m$. A GDD of *type* $g_1^{j_1} g_2^{j_2} \ldots g_m^{j_m}$ has $j_i$ groups each containing $g_i$ points, for each value of $i$. A GDD with groups of all the same size is said to be *uniform*. An example of a 3-GDD of type $2^3$ is given in Figure 2.3 with blocks depicted as differently coloured triangles and groups of points enclosed in ovals. When the block size is equal to the number of groups of the GDD, as in Figure 2.3, the GDD is also called a transversal design. More specifically, a *transversal design* is a $k$-GDD of type $g^k$, denoted TD$(k, g)$.

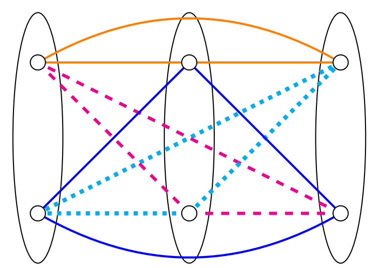


Figure 2.3: A 3-GDD of type $2^3$

Again, an SBGDD is a triple $(V, G, \mathcal{B})$ of points, blocks, and a partition on $V$. The defining property of an SBGDD is that pairs of points from different groups have distinct pair frequencies that are consecutive nonnegative integers. Like GDDs, points from the same group have pair frequency zero. For some subset $K$ of the positive integers, a $K$-SBGDD, or simply $k$-SBGDD when $K = \{k\}$, is used to denote SBGDDs with block sizes in $K$. As 3-SBGDDs are the the focus of this dissertation, 3-SBGDD is abbreviated to SBGDD and block sizes are specified otherwise. The group type notation used for GDDs is adopted for SBGDDs.

### 2.1.3 Latin Squares

A *Latin square* of side $n$ is an $n \times n$ array where each cell contains one of $n$ symbols, say $\{1, 2, \ldots, n\}$. Every symbol occurs exactly once in each row and column of the array. Two Latin squares are *orthogonal* if every possible ordered pair of elements occurs exactly once in corresponding cells of the Latin squares. A set of *mutually orthogonal Latin squares* of side $n$, denoted MOLS$(n)$, is a set of Latin squares that are pairwise orthogonal. Notice that a $k$-GDD of type $n^k$ is equivalent to $k-2$ MOLS$(n)$, where two of the groups from the GDD represent the rows and columns of the Latin squares, and the remaining $k-2$ groups represent the symbols in each respective Latin square, giving $k$ groups of size $n$ total. Points of the design are in a block together if the corresponding symbols in the Latin squares appear in the row and column of the array given by the row and column points in the block. Figure 2.4 depicts three MOLS$(4)$ and the corresponding $5$-GDD of type $4^5$. The gray cells in the Latin squares correspond to the block depicted in the GDD. Since Latin squares exist for all orders $n \geq 2$, so do the corresponding 3-GDDs of type $n^3$.

A *transversal* of a Latin square of side $n$ is a set of $n$ cells in the array where each row, column, and symbol occur exactly once. For example, in Figure 2.4, the main diagonal of the left-most blue Latin square is a transversal and the main diagonal of the right-most magenta Latin square is not. It is known that a Latin square of side $n$ has at least one orthogonal mate if and only if it has $n$ disjoint transversals. The same is true of a set of mutually orthogonal Latin squares. That is, a set of $k$ MOLS$(n)$ can be extended to a set of $k+1$ MOLS$(n)$ if and only if the $k$ Latin squares all share $n$ common transversals. For example, the first two Latin squares in Figure 2.4 share four disjoint transversals, they can be paired with the third Latin square whose distinct symbols are placed in the positions corresponding to each of the common transversals. For that reason, the removal of a Latin square from a set of MOLS means that the remaining Latin squares share $n$ disjoint transversals. Disjoint transversals in MOLS give rise to disjoint blocks in the corresponding GDD. This observation is used in Sections 4.4 and 4.5.

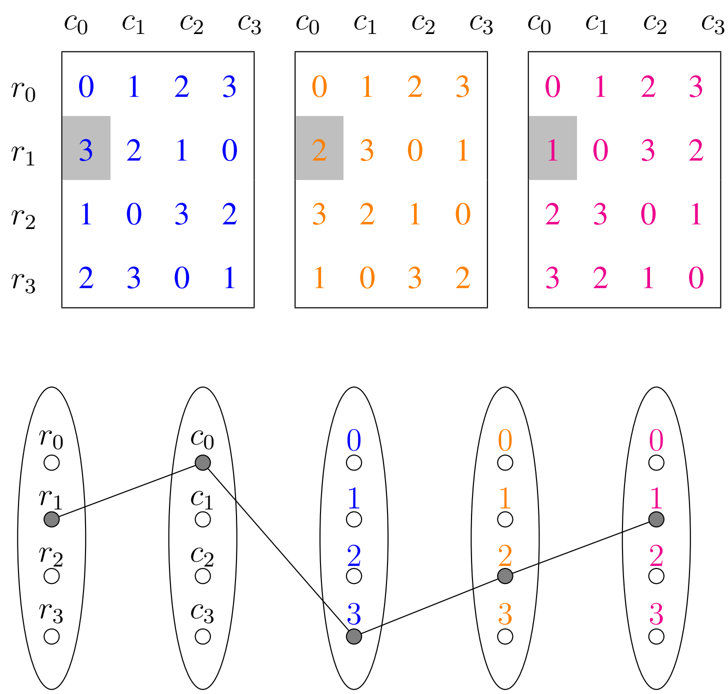



Figure 2.4: Three MOLS$(4)$ and the corresponding $5$-GDD of type $4^5$

An *incomplete Latin square* is a Latin square where every symbol occurs at most once in each row and column of the array. That is, cells of the array may be left empty. Two such Latin squares are said to be orthogonal if they have the same set of empty cells and every pair of symbols occurs at most once together. A set of *incomplete mutually orthogonal Latin squares*, or IMOLS, is a set of incomplete Latin squares that are mutually orthogonal. A set of IMOLS of side $n$ with an $m \times m$ empty subsquare is denoted IMOLS$(n, m)$.

The GDD corresponding to the IMOLS, similar to the depiction given in Figure 2.4 is called a ‘holey’ GDD. That is, a GDD where pair frequencies between points in different groups may be zero rather than the index of the design. An example is given in Figure 4.3 where IMOLS are used to construct an SBGDD. What follows is a special kind of holey GDD.

### 2.1.4 Modified Group Divisible Designs

A *modified group divisible design*, or MGDD, is a holey GDD with a special configuration. An MGDD of type $g^u$ has $gu$ points that are thought of as a grid with $g$ rows and $u$ columns. Label the points as ordered pairs, $(x_i, x_j) \in V$ with $i = 1, 2, \ldots, g$ and $j = 1, 2, \ldots, u$. The defining property of an MGDD is that every pair of points $(x_i, x_j), (y_k, y_\ell)$ with $i \neq k$ and $j \neq \ell$ are contained in a block together exactly once. If $i = k$, then $(x_i, x_j)$ and $(y_k, y_\ell)$ appear in the same group together. If $j = \ell$ then the points occur in a hole together.

Notice that an MGDD of type $g^u$ is equivalent to an MGDD of type $u^g$ as there are 'groups' of points not covered on a block in both the '$g$ direction' as well as the '$u$ direction'. For that reason, the group type of an MGDD of type $g^u$ is also denoted by $g \times u$. Figure 2.5 depicts an MGDD of type $3 \times 3$. The vertical groups are depicted by ovals and the horizontal groups correspond to the differently shaded points; white, gray, and black.

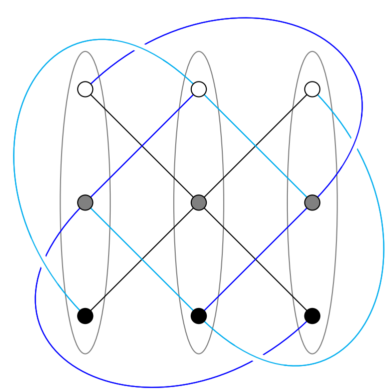


Figure 2.5: An MGDD of type $3 \times 3$

An SBMGDD of type $g \times u$ is the Sarvate-Beam equivalent of an MGDD. That is, an SBMGDD has vertical and horizontal groups defined in the same way as an MGDD, but the pair frequencies of every covered pair of points in the SBMGDD is a unique value in the interval $\{\mu, \mu+1, \ldots, \mu+\binom{u}{2}g(g-1)-1\}$.

Examples of SBMGDDs of type $3 \times 4, 3 \times 5$, and $3 \times 6$ are given in Section 3.3. What follows is a non-existence proof of an SBMGDD of type $3 \times 3$ to serve as an example.

Suppose there is an SBMGDD of type $3 \times 3$ and consider a pair of incident points, say $ab$. Both $a$ and $b$ cannot be in a block with the points in their vertical or horizontal group. That leaves a single point which is eligible to be in a block with both $a$ and $b$, call it $c$. See Figure 2.6 where the block is indicated by blue dashed lines. As the pair frequencies of $ab$ and $ac$ are therefore equal, the defining property of a Sarvate-Beam design is not satisfiable. Therefore there is no SBMGDD of type $3 \times 3$.

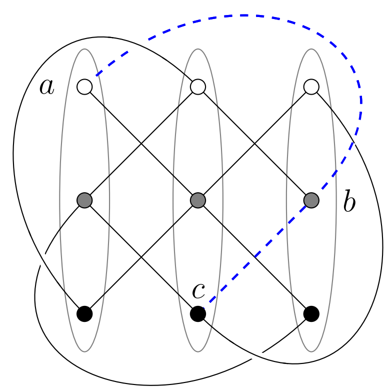



Figure 2.6: The non-existence of an SBMGDD of type $3 \times 3$

## 2.2 Conditions of Existence

Some general conditions of existence for the various designs of interested are given. The existence of designs with constant pair frequency are used to construct Sarvate-Beam designs in later chapters.

**Lemma 2.1** ([6])**.** Let $r$ be the number of occurrences of a point in a BIBD and $b$ be the total number of blocks. Then the necessary conditions of existence of a $(v, k, \lambda)$-design are

1. $vr = bk$, and

2. $r(k-1) = \lambda(v-1)$.

The conditions in Lemma 2.1 are next generalized to pairwise balanced designs.

**Lemma 2.2** ([6])**.** The necessary conditions for the existence of a PBD$(v, K)$ are

1. $v - 1 \equiv 0 \pmod{\alpha(K)}$ where $\alpha(K) = \gcd\{k-1 \mid k \in K\}$, and

2. $v(v-1) \equiv 0 \pmod{\beta(K)}$ where $\beta(K) = \gcd\{k(k-1) \mid k \in K\}$.

The reader is directed to [6] for a summary of results. For arbitrary $K$, Wilson's theory [28] says that the conditions of Lemma 2.2 are sufficient when $v$ is 'large enough'. Taking consecutive integers in $K$ often removes the congruence conditions seen in Lemma 2.2. For example, Lemma 2.3 gives a complete existence result for PBDs with block sizes $K = \{4, 5, 6\}$, which exist with $v$ points for $v$ in every congruence class.

**Lemma 2.3** ([17])**.** There exists a PBD$(v, \{4, 5, 6\})$ for all integers $v \geq 4$ excluding $v = 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23$.

Conditions similar to those in Lemma 2.1 are based on the number of pairs in a design and how many pairs are covered in each block. A similar condition is developed for Sarvate-Beam triple systems. Since each block covers three pairs in an SBTS, the sum of all the pair frequencies is equal to three times the number of blocks and is therefore divisible by three. In the case when $\mu = 0$, this sum is $0+1+2+\cdots+(\binom{v}{2}-1) = \frac{1}{2}(\binom{v}{2}-1)\binom{v}{2} = \frac{1}{8}(v-2)(v-1)v(v+1)$. Since the sum is equal to a product of four consecutive integers, it is divisible by three for any $v \geq 3$. Therefore a Sarvate-Beam triple system with starting frequency zero is admissible for any $v \geq 3$. This is not the case for all starting frequencies. The necessary conditions of existence for general starting frequency $\mu \geq 0$ are outlined in Lemma 2.4.

**Lemma 2.4.** The necessary conditions of existence for a 3-SBGDD of type $g^u$ with starting frequency $\mu \geq 0$ are

- $g \equiv 0 \pmod 3$, or
- $u \equiv 0, 1 \pmod 3$, or
- $\mu \equiv 0 \pmod 3$.

*Proof.* In a Sarvate-Beam GDD of type $g^u$, there are $\binom{u}{2}g^2$ pairs with frequencies $\mu, \mu + 1, \ldots, \mu + \binom{u}{2}g^2 - 1$. The sum of the frequencies is divisible by three since each block covers three pairs. Therefore three divides

$$\begin{aligned} &\mu + (\mu+1) + \cdots + (\mu + g^2\binom{u}{2} - 1) \\ =&g^2\binom{u}{2}\mu + (1 + 2 + 3 + \cdots + (g^2\binom{u}{2} - 1)) \\ =&g^2\binom{u}{2}\mu + \frac{1}{2}(g^2\binom{u}{2} - 1)g^2\binom{u}{2} \\ =&g^2\binom{u}{2}(\mu + \frac{1}{2}(g^2\binom{u}{2} - 1)) \end{aligned}$$

That is $g^2\binom{u}{2}$ or $\mu + \frac{1}{2}(g^2\binom{u}{2} - 1)$ is divisible by three.

Notice that if $g \equiv 0 \pmod 3$ or $u \equiv 0, 1 \pmod 3$, then three divides $g^2\binom{u}{2}$. So assume this is not the case. Then $g^2 \equiv 1 \pmod 3$ and $u \equiv 2 \pmod 3$ and $\mu + \frac{1}{2}(g^2\binom{u}{2} - 1) \equiv 0 \pmod 3$. Thus $\mu \equiv 0 \pmod 3$ as desired.

□

Note that the conditions in Lemma 2.4 are not sufficient. For example, in Chapter 1 it was shown that there is no Sarvate-Beam triple system on four points with starting frequency zero. As previously described, an SBTS$(4)$ is equivalent to an SBGDD of type $1^4$.

The conditions for existence of a 3-GDD of type $g^u$ are given for their use in creating SBGDDs with starting frequency $\mu > 0$ from those with $\mu = 0$ throughout the dissertation.

**Lemma 2.5** ([31])**.** A $(3, \lambda)$-GDD of type $g^u$ exists if and only if

- $u \geq 3$,
- $\lambda(u-1)g \equiv 0 \pmod 2$, and
- $\lambda u(u-1)g^2 \equiv 0 \pmod 6$.

The following lemmas give similar conditions of existence for MGDDs that are also used in later constructions.

**Lemma 2.6** ([27])**.** There exists a 3-MGDD of type $g \times u$ if and only if

- $g, u \geq 3$,
- $(u-1)(g-1) \equiv 0 \pmod 2$, and
- $u(u-1)g(g-1) \equiv 0 \pmod 3$.

**Lemma 2.7** ([13])**.** There exists a $4$-MGDD of type $g \times u$ if and only if

- $g, u \geq 4$ and
- $(u-1)(g-1) \equiv 0 \pmod 3$, and
- $\{g, u\} \neq \{4, 6\}$.

## 2.3 Tools for Construction

With the existence of known designs in hand, the tools used to construct Sarvate-Beam GDDs are given.

A *point deletion* is the removal of a point from a design. Blocks that are incident to a deleted point are also removed. For example, if deleting a point from a PBD, what is left is a GDD whose groups are the removed blocks of the PBD, less the deleted point. Figure 2.7 depicts a 3-GDD of type $2^3$ obtained by deleting a point from the Fano plane (Figure 2.1). The deleted point is coloured black and the removed edges are depicted by dashed lines. The blocks of the GDD are given as differently coloured triangles.

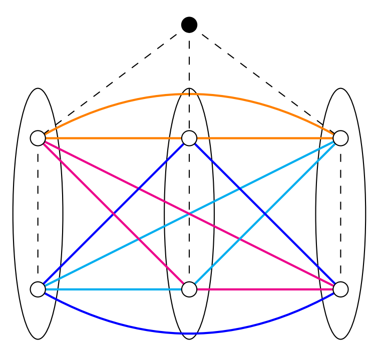


Figure 2.7: A 3-GDD of type $2^3$ obtained from a Fano plane

A *truncation* is the removal of some set of points from the design. In this case, blocks that contained a truncated point remain in the new configuration with reduced size, excluding blocks of size one. For example, truncating the point $a$ from the PBD$(10, \{3, 4\})$ given in Figure 2.2, gives a $(9, 3, 1)$-design. If instead $a$ is deleted, the blocks depicted horizontally in orange are removed and the result is a 3-GDD of type $3^3$.

A *block deletion* consists of the removal of a block from the design where points incident to the block are truncated. Blocks may also be removed without truncating points. The term *removal* is reserved for this case.

*Inflation* is the act of replacing every point in a design with a bundle of points. Typically blocks of the original design are replaced with GDDs, MGDDs, or similar designs used to cover the new pairs of points the have been created. For example, if a PBD$(v, K)$ is inflated by some positive integer $g$, the result is $gv$ points. For every $k \in K$, each original $k$-block is removed and a $k$-GDD of type $g^k$ may be put in its place. This replacement covers pairs of points from different bundles in the same way that blocks of the original design covered pairs of the original points. The pairs within each bundle of $g$ points remain uncovered. Therefore the result is a $K$-GDD of type $g^v$.

A design that is self-contained in a larger design as called a *subdesign*. The subdesign given by replacing the blocks in an inflation is sometimes referred to as an *ingredient* of the inflation, to distinguish them from the original 'global' design. In the example, the ingredients used to inflate a PBD$(v, K)$ are $k$-GDDs of type $g^k$.

Designs with higher index may be used in place of a PBD$(v, K)$. If starting with a $(v, k, \lambda)$-design, the index of the resulting GDD is also $\lambda$. An example of such an inflation follows.

**Example 2.3.1.** Start with a $(4, 3, 2)$-design, whose blocks are the set of all four possible triples. That is, taking the point set to be $\{a, b, c, d\}$, then the blocks are $abc, abd, acd,$ and $bcd$. Figure 2.8 depicts the $(4, 3, 2)$-design with blocks given in red, dotted, dashed, and solid lines. The points are lined up horizontally. This allows us to think of inflation as expanding each point vertically into a group. In this case, points are inflated by $g = 2$ and named $a_0, a_1, b_0, b_1, c_0, c_1, d_0,$ and $d_1$ so that groups are $x_0x_1$ with $x \in \{a, b, c, d\}$. The points are ordered in the groups vertically so that the points with the same subscript appear horizontally in a row. This is sometimes called the'level' of the inflated points. It becomes important to distinguish between different levels in certain circumstances, such as replacing blocks with MGDDs where horizontal groups are left uncovered. Each block $xyz$ is replaced by a 3-GDD of type $2^3$ on point set $\{x_0, x_1, y_0, y_1, z_0, z_1\}$. This replacement is shown with the block $abc$ in Figure 2.8 depicted with dashed lines and level 0 depicted below level 1. The result is a $(3, 2)$-GDD of type $2^4$. Each pair occurs twice in the GDD since each pair is covered twice in the $(4, 3, 2)$-design.

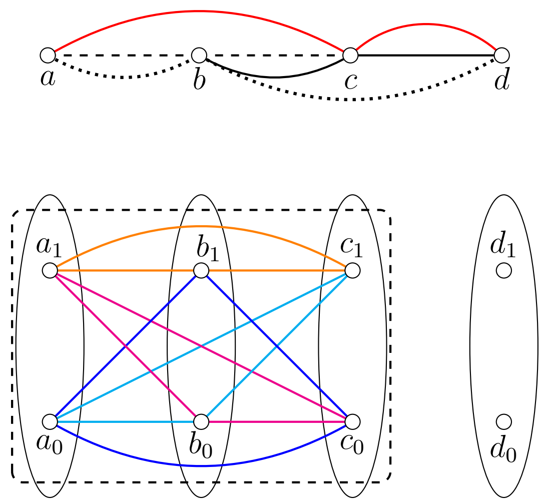



Figure 2.8: An inflated $(4, 3, 2)$-design with blocks replaced by GDDs of type $2^3$

Note that blocks need not be inflated in order to be replaced. When a design with $g$ points is placed on a group with $g$ points in a GDD, since the groups of the GDD are thought of as holes, this is often referred to as *filling in* the group. A $k$-block can also simply be replaced by a design on $k$ points. Specifically, replacing a $k$-block with an SBTS$_\mu(k)$ is an operation used throughout the dissertation, including the main constructions in Sections 3.5 and 4.1.

### 2.3.1 Lifting a Design

*Lifting* a design with $v$ points is the act of adding blocks to equally increase all pair frequencies between the $v$ points. Often this is achieved using a $(v, k, \lambda)$-design with the index $\lambda$ giving the desired increase in frequency. For example, to the $(3, 2)$-GDD of type $2^4$ given in Figure 2.8, the blocks of a 3-GDD of type $2^4$ can be added $\lambda$ times to build a

$(3, \lambda + 2)$-GDD of type $2^4$. Lifting a Sarvate-Beam design gives a Sarvate-Beam design with a higher starting frequency. For example, adding $\lambda$ copies of a $(4, 3, 2)$-design to an $\text{SBTS}_1(4)$ increases the frequency of each pair by $2\lambda$. The result is an $\text{SBTS}_{2\lambda+1}(4)$.

When replacing $k$-blocks with ingredient $\text{SBTS}_\mu(k)$, the triple systems may be lifted with care to ensure different SBTS cover back-to-back intervals of pair frequencies. An example follows to illuminate the general method.

**Example 2.3.2.** Start with a 4-GDD of type $3^4$ which is equivalent to two $\text{MOLS}(3)$. There are nine blocks, one for each position in the corresponding Latin square. Replace the first block with an $\text{SBTS}_1(4)$. Then the pairs on the first block have frequencies one through $\binom{4}{2} = 6$. Replacing the second block with an $\text{SBTS}_7(4)$ covers the pair frequencies seven through 12. Continuing in this way, the ninth block is replaced with an $\text{SBTS}_{49}(4)$ to cover pair frequencies $8(6)+1 = 49$ through 54. Since each pair is covered once in the GDD, the pair frequencies in the resulting design are exactly those given by the ingredient Sarvate-Beam triple systems. The result is a 3-GDD with distinct pair frequencies in $\{1, 2, \ldots, 54\}$. Thus, a 3-SBGDD of type $3^4$ with starting frequency one is constructed, the first SBGDD with nontrivial groups in the dissertation.

Note that the construction in Example 2.3.2 relies on the existence of the $\text{SBTS}_\mu(4)$ for specific values of $\mu$ in order to cover back-to-back intervals of points. The existence of an $\text{SBTS}_\mu(4)$ for any $\mu > 0$ is given by Theorem 1.1. As an example, a lifting construction is presented which details how an $\text{SBTS}_1(4)$ may be inflated to an $\text{SBTS}_\mu(4)$ for any $\mu > 0$. This construction works similarly to [12].

**Construction 2.1.** In order to lift the pair frequencies of an $\text{SBTS}_1(4)$ as ingredient designs, $w$ copies of a $(4, 3, 2)$-design are added to the block set. This covers the pair frequency interval $2w + 1, 2w + 1, \ldots, 2w + 6$, giving an $\text{SBTS}_{2w+1}(4)$ with any positive odd starting frequency.

To cover intervals starting with an even frequency, a few additional blocks are added to the above configuration. Suppose an $\text{SBTS}_1(4)$ has blocks with pairs $ab, ac, ad, bc, bd,$ and $cd$ whose frequencies are $1, 2, 3, 4, 5,$ and $6$ respectively, as in Figure 1.1. If the blocks $abd$ and $acd$ are added, the pair frequencies for $ab, ac, ad, bc, bd,$ and $cd$ become $2, 3, 5, 4, 6,$ and $7$. Adding $w$ copies of a $(4, 3, 2)$-design gives pair frequencies $2w+2, 2w+3, \ldots, 2w+7$, covering any positive even interval of pair frequencies.

Similarly to Construction 2.1, $\text{SBTS}_0(6)$ and $\text{SBTS}_1(6)$ are used to obtain an $\text{SBTS}_\mu(6)$ for any $\mu \geq 0$. These triple systems are used as ingredients in later constructions to replace 6-blocks in pairwise balanced designs.

A general method for lifting subdesigns to cover distinct intervals of pair frequencies is described in the following section in terms of the multigraphs corresponding to the designs.

### 2.3.2 Designs as Graph Decompositions

Edge decompositions of multigraphs are used as a tool for developing new constructions that combine small designs to make larger ones. This section starts with some basic definitions and then explains how the multigraphs are used.

The *underlying multigraph*, $G$, of a design, $D$, is a multigraph that has one vertex for each point of $D$. Two vertices $x, y \in V(G)$ are connected by an edge exactly when the pair $xy$ occur in a block together in $D$. Moreover, the edge multiplicity of $xy \in E(G)$ is equal to the pair frequency of $xy$ in $D$. That is, blocks of $D$ correspond to cliques in the underlying multigraph, occurring with multiplicity. For example, if $D$ is a $(v, k, \lambda)$-design, then in the underlying multigraph of $D$ each edge has multiplicity $\lambda$. In particular, when $\lambda = 1$, the underlying graph of $D$ is simple. For example, the underlying multigraph of a pairwise balanced design on $v$ points is $K_v$, the complete graph with $v$ vertices.

The *edge decomposition* of a multigraph $G$ is a partition of the edges of $G$ into edge-disjoint subgraphs. Notice that if $G$ is the underlying multigraph of a design, then the blocks of $D$ naturally partition the edges of $G$ into cliques.

**Example 2.3.3.** Recall from Example 2.3.1 that the block set of a $(4, 3, 2)$-design uses all four possible 3-blocks to cover each pair of points twice. As shown in Figure 2.9, the underlying graph is $K_4$ where each edge has multiplicity two. The edge decomposition is indicated by the colours and styles of the edges. The triangles in red, black, blue dashed, and dotted give a decomposition of the graph corresponding to the blocks $abc$, $abd$, $acd$, and $bcd$. Since each partition is a triangle, this decomposition is called a *triangle decomposition* of the graph.

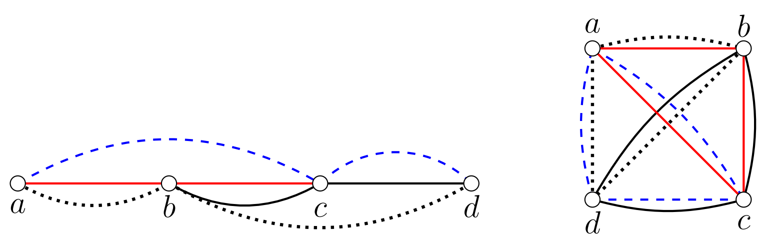


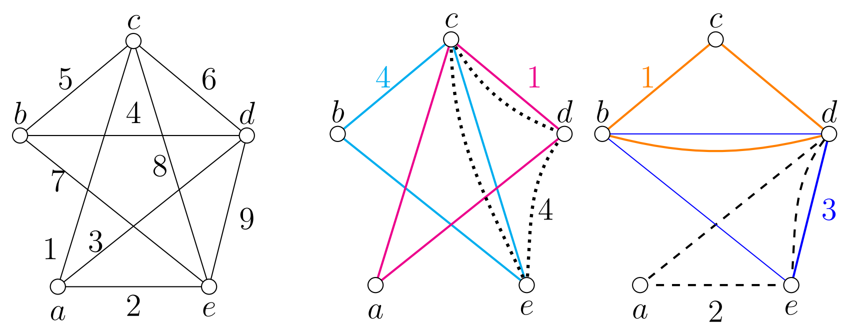

Figure 2.9: A $(4, 3, 2)$-design and its underlying graph

An example of an edge decomposition of the underlying graph of a Sarvate-Beam triple system on five points is given.

**Example 2.3.4.** Let the points be $\{a, b, c, d, e\}$ and consider the following set of blocks, $\{acd, 2ade, bcd, 4bce, 3bde, 4cde\}$. The pair frequencies are listed in Table 2.1, showing that an SBTS$(5)$ has indeed been defined. Note that Dukes and Short-Gershman [12] also gave the blocks of an SBTS$(5)$. Figure 2.10 shows the underlying multigraph depicted on the left, with edge multiplicities corresponding to the pair frequencies of the Sarvate-Beam triple system. To the right is a triangle decomposition of the underlying graph. Note that the decomposition also has five points. In the figure, the points are depicted twice to aid in distinguishing the distinct triangles.

| Pairs | $ab$ | $ac$ | $ad$ | $ae$ | $bc$ | $bd$ | $be$ | $cd$ | $ce$ | $de$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |

Table 2.1: The pair frequencies of an SBTS$(5)$




Edge Multiplicities $\{acd, 2ade, bcd, 4bce, 3bde, 4cde\}$

Figure 2.10: A triangle decomposition of the underlying graph of an SBTS$(5)$

In Figure 2.10, the two subgraphs on five vertices with edges split into triangles depicts an intermediate decomposition of the SBTS$(5)$ into two subgraphs on five points. The idea of an intermediate graph decomposition resurfaces in the next lemma which describes how to use inflation to construct new Sarvate-Beam designs using SBTS as ingredients.

**Lemma 2.8.** Suppose $\{G_1, G_2, \ldots, G_\ell\}$ are graphs that define a partition of the edges of a simple graph $G$ such that $G_i$ has a triangle decomposition for each $i = 1, 2, \ldots, \ell$. Let $M_i$ be a multigraph of $G_i$ obtained by assigning the edge multiplicities $0, 1, \ldots, |E(G_i)| - 1$ to the edges of $G_i$ so that $M_i$ also has a triangle decomposition. Then there is an assignment of the edge multiplicities $0, 1, \ldots, |E(G)| - 1$ to $G$ so that the corresponding multigraph has a triangle decomposition.

*Proof.* Let $m_i$ be the number of edges in $G_i$ and set $\mu_1 = 0$, and $\mu_i = m_1 + m_2 + \cdots + m_{i-1}$ for $i = 1, 2, \ldots, \ell$. Since $G_i$ has a triangle decomposition, add $\mu_i$ copies of each of the edges in $G_i$ to the edges of $M_i$ and call this graph $L_i$. Then $L_i$ has a triangle decomposition from a decomposition of $M_i$ taken together with the $\mu_i$ copies of triangle decomposition of $G_i$. Moreover, the edge multiplicities of $L_i$ are $\mu_i, \mu_i + 1, \ldots, \mu_i + |E(G_i)| - 1$.

Let $H$ be the multigraph given by assigning the edges of $G$ a multiplicity equal to the sum of the multiplicities of the edges in $\{L_1, L_2, \ldots, L_\ell\}$. Then $\{L_1, L_2, \ldots, L_\ell\}$ is an edge decomposition of $H$. Therefore $H$ has a triangle decomposition corresponding to the triangles in a decomposition of each $L_i$ taken altogether.

Since each edge in $G$ occurs in exactly one subgraph $G_i$, each edge in $H$ has multiplicity equal to the multiplicity of the corresponding edge in $L_i$ for some $i$. Since the edge multiplicities in each $L_i$ are $\mu_i, \mu_i + 1, \ldots, \mu_i + m_i - 1$ and $\mu_{i+1} = \mu_i + m_i$, the edge multiplicities of all the subgraphs $L_i$ taken together form the interval $0, 1, \ldots, \mu_{\ell-1} + m_\ell - 1$. Since $\mu_{\ell-1} + m_\ell - 1 = m_1 + m_2 + \cdots + m_\ell - 1 = |E(G)|, H$ has edge multiplicities $0, 1, \ldots, |E(G_i)| - 1$ and is therefore the desired multigraph of $G$. □

In the design theory setting, the subgraph $L_i$ corresponds to the underlying graph of a lifted ingredient design. Notice that Lemma 2.8 requires the underlying simple graph, $G_i$, to have a triangle decomposition in order for $L_i$ to have the desired starting frequency. These simple graphs correspond to underlying graphs of a design with index $\lambda = 1$. When dealing with ingredient designs that do not have analogous index one designs, various other approaches are used. Note that the original blocks of the design are often referred to as being lifted, rather than specifying the ingredient designs used to replace the original blocks, as these ingredients may take many different forms within the same construction.

When $k \equiv 2 \pmod 3$, an SBTS$_\mu(k)$ necessarily has starting frequency $\mu$ a multiple of three [12]. As the number of pairs covered by a $k$-block are $\binom{k}{2} \equiv 1 \pmod 3$, simply lifting the $k$-blocks one after another inevitably requires a non-existent starting frequency. Instead, the frequencies are strategically modified so that intervals between two different ingredients are interleaved. As a shorthand, this process is referred to as ‘sequencing’ the original blocks. Sequencing constructions are given in Section 4.5 and used throughout the dissertation to replace blocks of size five and eight with modified Sarvate-Beam triple systems of the same order. More generally, when replacing blocks by Sarvate-Beam designs, it is sometimes important to carry out the replacement in a certain order relative to the sub-intervals (or sub-intervals with one gap). Such cases are noted in each construction.

# Chapter 3

# Sarvate-Beam Cubes

This chapter investigates the existence of 3-SBGDDs of type $g^3$. The complete existence result is given in Section 3.5. The case where SBGDDs have three groups has an interesting geometric interpretation which is first explored.

## 3.1 Geometric Interpretation

The following geometric question is posed as it relates to Sarvate-Beam GDDs.

**Question 3.1.** For any integer $n \geq 2$, can nonnegative integers be placed into an $n \times n \times n$ cube so that the sum of a line in any direction is in the set $\{0, 1, 2, \ldots, 3n^2 - 1\}$ and all line sums are distinct?

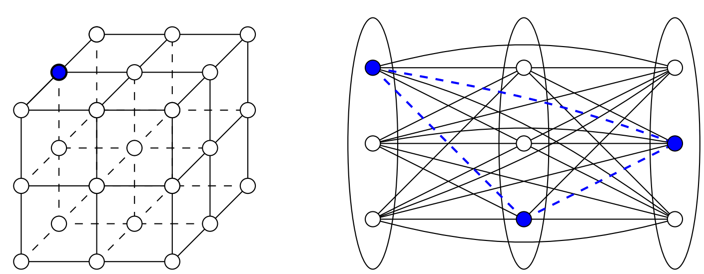


Figure 3.1: A cube of order $3$ and its corresponding SBGDD

A solution to the $n \times n \times n$ cube is equivalent to a 3-SBGDD of type $n^3$, where vertices of the cube are taken as blocks of the SBGDD. For example, the blue vertex in the cube given in Figure 3.1 corresponds to the 3-block with blue dashed edges in the SBGDD graph decomposition representation. Vertices of the cube form a straight line whenever the corresponding blocks share a pair of points.

Alternatively, starting with the cube, an SBGDD is constructed by letting points represent planes of the cube, going through a set of $n^2$ points. Planes that are parallel to faces are also included, essentially defining 'inner faces' of the cube. Points in the design are connected if they belong to the same plane. Figure 3.2 presents the same cube again, with the bottom left corner of the cube placed at the origin in three dimensions, to give it an orientation.

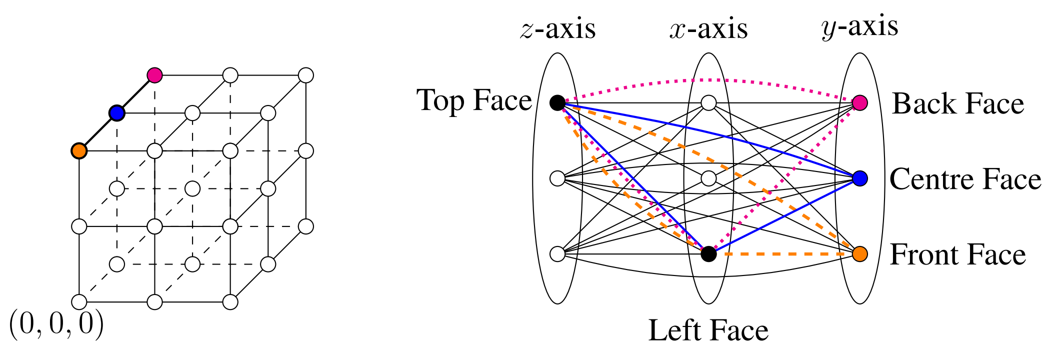



Figure 3.2: The SBGDD with points defined by the geometry of the cube

Given the orientation, the groups of the SBGDD represent the $x, y$, and $z$ directions in the cube. Each point in the SBGDD represents a position in that prescribed direction. In Figure 3.2, since the blue solid triangle and orange dashed triangles both represent points in the upper-left face of the cube, their shared points in the SBGDD define these positions. Assigning the first and second groups to be the $z$ and $x$-directions respectively, the shared point in the first group corresponds to the top face and the shared point in the second group corresponds to the left-most face. Picking one point from the final group therefore completes each triangle in the top-left line of the cube. That is, the group where the coloured triangles differ corresponds to changes in the $y$-direction.

Because of this relationship, Dukes and Short-Gershman called a 3-SBGDD of type $n^3$ a *Sarvate-Beam cube of order* $n$, or SBC$(n)$. Similarly to general SBGDDs, a Sarvate-Beam cube with starting frequency $\mu$ is denoted by SBC$_\mu(n)$. Notice that the blocks of a 3-GDD of type $n^3$, or equivalently a Latin square of side $n$, may be used to lift an SBC$(n)$. Since Latin squares of side $n$ exist for all $n \geq 2$, there is an SBC$_\mu(n)$ whenever an SBC$_0(n)$ exists.

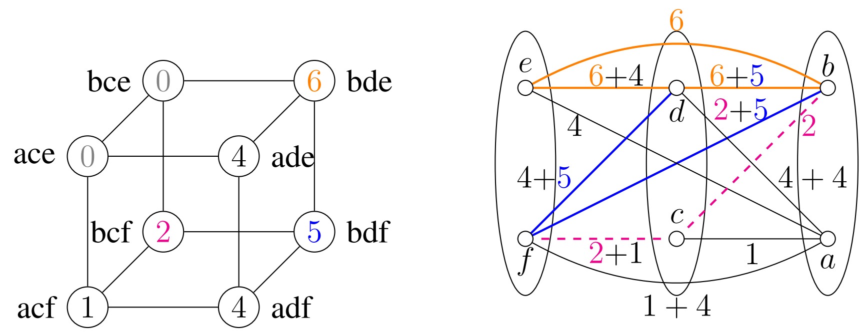
## 3.2 Small Order Cubes

Sarvate-Beam cubes of orders two and three, found by Dukes and Short-Gershman [12], are given first as they are needed to create cubes with larger orders. A cube of order five, found by computer is also given for that reason.

### 3.2.1 Order 2

Figure 3.3 depicts an SBC$(2)$ with block set $\{acf, 2bcf, 4ade, 6bde, 4adf, 5bdf\}$ and groups $\{ab, cd, ef\}$. The pair frequencies are listed as sums on each edge in the SBGDD presentation, corresponding to the triangle multiplicities listed in each point of the cube. For example, pair $bc$ occurs only once in the block $bcf$. Since this block has multiplicity two, the pair frequency of $bc$ is two. On the other hand, $bd$ occurs in blocks $bde$ and $bdf$, depicted in orange and blue in the diagram, and therefore has pair frequency $6 + 5 = 11$.

Note that since each group has two points, each pair appears in at most two distinct blocks. In a cube of order $n$, each pair may occur in up to $n$ distinct blocks.




| Blocks | $acf$ | $2bcf$ | $4ade$ | $6bde$ | $4adf$ | $5bdf$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pairs | $ac$ | $ad$ | $ae$ | $af$ | $bc$ | $bd$ | $be$ | $bf$ | $ce$ | $cf$ | $de$ | $df$ |
| Frequencies | 1 | 8 | 4 | 5 | 2 | 11 | 6 | 7 | 0 | 3 | 10 | 9 |

Figure 3.3: A cube of order two

### 3.2.2 Order 3

Figure 3.4 gives the entries of an SBC$(3)$ which was found via computer by Dukes and Short-Gershman [12]. Layers of the $3 \times 3 \times 3$ cube are given in separate $3 \times 3$ grids and line sums appear in bold text.

| | | | |
|---|---|---|---|
| 0 | 1 | 12 | **13** |
| 2 | 3 | 1 | **6** |
| 0 | 10 | 5 | **15** |
| **2** | **14** | **18** | |

| | | | |
|---|---|---|---|
| 0 | 14 | 8 | **22** |
| 0 | 5 | 0 | **5** |
| 3 | 7 | 0 | **10** |
| **3** | **26** | **8** | |

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | **1** |
| 7 | 11 | 3 | **21** |
| 4 | 0 | 20 | **24** |
| **11** | **12** | **23** | |

| | | |
|---|---|---|
| **0** | **16** | **20** |
| **9** | **19** | **4** |
| **7** | **17** | **25** |

Figure 3.4: A cube of order three

### 3.2.3 Order 5

Figure 3.5 gives an SBC$(5)$ in its cube presentation which was found via computer. The same layout is used as the order three cube.

| | | | | | |
|---|---|---|---|---|---|
| 16 | 9 | 6 | 9 | 1 | **41** |
| 14 | 0 | 0 | 6 | 27 | **47** |
| 4 | 13 | 5 | 14 | 0 | **36** |
| 20 | 2 | 0 | 2 | 5 | **29** |
| 7 | 8 | 41 | 0 | 4 | **60** |
| **61** | **32** | **52** | **31** | **37** | |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 10 | 2 | 0 | **12** |
| 1 | 2 | 2 | 1 | 0 | **6** |
| 0 | 0 | 0 | 2 | 0 | **2** |
| 0 | 60 | 14 | 0 | 0 | **74** |
| 0 | 7 | 8 | 49 | 3 | **67** |
| **1** | **69** | **34** | **54** | **3** | |

| | | | | | |
|---|---|---|---|---|---|
| 15 | 34 | 2 | 6 | 15 | **72** |
| 35 | 0 | 3 | 0 | 2 | **40** |
| 13 | 13 | 16 | 1 | 2 | **45** |
| 0 | 0 | 0 | 0 | 0 | **0** |
| 8 | 2 | 7 | 0 | 1 | **18** |
| **71** | **49** | **28** | **7** | **20** | |

| | | | | | |
|---|---|---|---|---|---|
| 10 | 0 | 0 | 0 | 5 | **15** |
| 1 | 6 | 5 | 5 | 0 | **17** |
| 2 | 0 | 0 | 39 | 14 | **55** |
| 5 | 2 | 32 | 12 | 0 | **51** |
| 41 | 5 | 1 | 12 | 11 | **70** |
| **59** | **13** | **38** | **68** | **30** | |

| | | | | | |
|---|---|---|---|---|---|
| 5 | 7 | 6 | 25 | 0 | **43** |
| 2 | 1 | 1 | 15 | 4 | **23** |
| 29 | 0 | 4 | 2 | 0 | **35** |
| 37 | 0 | 20 | 0 | 0 | **57** |
| 0 | 0 | 8 | 2 | 0 | **10** |
| **73** | **8** | **39** | **44** | **4** | |

| | | | | |
|---|---|---|---|---|
| **46** | **50** | **24** | **42** | **21** |
| **53** | **9** | **11** | **27** | **33** |
| **48** | **26** | **25** | **58** | **16** |
| **62** | **64** | **66** | **14** | **5** |
| **56** | **22** | **65** | **63** | **19** |

Figure 3.5: A cube of order five

## 3.3 Building Blocks for Larger Cubes

This section explores some of the required building blocks used to construct larger Sarvate-Beam cubes. The following SBMGDDs of type $3 \times 4, 3 \times 5$, and $3 \times 6$ are used in the main cube construction in Section 3.5.

### 3.3.1 An SBMGDD of type $3 \times 4$

The blocks of a Sarvate-Beam MGDD of type $3 \times 4$ with point set $\{a, b, \ldots, \ell\}$ are given in Table 3.1, found by computer. The vertical and horizontal groups are $\{aei, bfj, cgk, dhl\}$ and $\{abcd, efgh, ijkl\}$ respectively. The corresponding cube entries are given in Figure 3.6. Layers of the $4 \times 4 \times 4$ cube are given in separate $4 \times 4$ grids and line sums appear in bold text.

$afk$ $9afl$ $22agj$ $2ahj$ $5ahk$ $4bek$ $16bel$ $1bgi$ $12bgl$ $18bhi$

$26cej$ $14cfi$ $21cfl$ $9chi$ $6chj$ $8dej$ $25dek$ $2dgi$ $9dgj$

Table 3.1: The blocks of an SBGDD of type $3 \times 4$

| | | | | |
|---|---|---|---|---|
| . | . | . | . | |
| . | . | 14 | 0 | **14** |
| . | 1 | . | 2 | **3** |
| . | 18 | 9 | . | **27** |
| | **19** | **23** | **2** | |

| | | | | |
|---|---|---|---|---|
| . | . | 26 | 8 | **34** |
| . | . | . | . | |
| 22 | . | . | 9 | **31** |
| 2 | . | 6 | . | **8** |
| **24** | | **32** | **17** | |

| | | | | |
|---|---|---|---|---|
| . | 4 | . | 25 | **29** |
| 1 | . | . | 0 | **1** |
| . | . | . | . | |
| 5 | 0 | . | . | **5** |
| **6** | **4** | | **25** | |

| | | | | |
|---|---|---|---|---|
| . | 16 | 0 | . | **16** |
| 9 | . | 21 | . | **30** |
| 0 | 12 | . | . | **12** |
| . | . | . | . | |
| **9** | **28** | **21** | | |

| | | | |
|---|---|---|---|
| | **20** | **26** | **33** |
| **10** | | **35** | **0** |
| **22** | **13** | | **11** |
| **7** | **18** | **15** | |

Figure 3.6: An SBMGDD of type $3 \times 4$

### 3.3.2 An SBMGDD of type $3 \times 5$

Figure 3.7 gives the blocks of an SBMGDD of type $3 \times 5$ found by computer. Layers of the $5 \times 5 \times 5$ cube are given in separate $5 \times 5$ grids and line sums appear in bold text.

| | | | | | |
|---|---|---|---|---|---|
| . | . | . | . | . | |
| . | . | 0 | 45 | 3 | **48** |
| . | 0 | . | 10 | 1 | **11** |
| . | 3 | 0 | . | 3 | **6** |
| . | 24 | 0 | 4 | . | **28** |
| | **27** | **0** | **59** | **7** | |

| | | | | | |
|---|---|---|---|---|---|
| . | . | 49 | 9 | 0 | **58** |
| . | . | . | . | . | |
| 1 | . | . | 9 | 2 | **12** |
| 6 | . | 2 | . | 2 | **10** |
| 19 | . | 0 | 25 | . | **44** |
| **26** | | **51** | **43** | **4** | |

| | | | | | |
|---|---|---|---|---|---|
| . | 8 | . | 27 | 1 | **36** |
| 3 | . | . | 10 | 27 | **40** |
| . | . | . | . | . | |
| 3 | 12 | . | . | 19 | **34** |
| 12 | 10 | . | 0 | . | **22** |
| **18** | **30** | | **37** | **47** | |

| | | | | | |
|---|---|---|---|---|---|
| . | 9 | 1 | . | 15 | **25** |
| 42 | . | 2 | . | 2 | **46** |
| 4 | 4 | . | . | 0 | **8** |
| . | . | . | . | . | |
| 7 | 1 | 49 | . | . | **57** |
| **53** | **14** | **52** | | **17** | |

| | | | | | |
|---|---|---|---|---|---|
| . | 22 | 0 | 19 | . | **41** |
| 0 | . | 0 | 1 | . | **1** |
| 0 | 9 | . | 0 | . | **9** |
| 33 | 0 | 21 | . | . | **54** |
| . | . | . | . | . | |
| **33** | **31** | **21** | **20** | | |

| | | | | |
|---|---|---|---|---|
| | **39** | **50** | **55** | **16** |
| **45** | | **2** | **56** | **32** |
| **5** | **13** | | **19** | **3** |
| **42** | **15** | **23** | | **24** |
| **38** | **35** | **49** | **29** | |

Figure 3.7: An SBMGDD of type $3 \times 5$

### 3.3.3 An SBMGDD of type $3 \times 6$

Figure 3.8 gives the blocks of an SBMGDD of type $3 \times 6$, found by computer. Layers of the $6 \times 6 \times 6$ cube are given in separate $6 \times 6$ grids and line sums appear in bold text.

| | | | | | | |
|---|---|---|---|---|---|---|
| . | . | . | . | . | . | |
| . | . | 6 | 12 | 1 | 9 | **28** |
| . | 5 | . | 4 | 12 | 16 | **37** |
| . | 2 | 3 | . | 0 | 2 | **7** |
| . | 3 | 11 | 1 | . | 36 | **51** |
| . | 19 | 2 | 4 | 2 | . | **27** |
| | **29** | **22** | **21** | **15** | **63** | |

| | | | | | | |
|---|---|---|---|---|---|---|
| . | . | 1 | 0 | 1 | 1 | **3** |
| . | . | . | . | . | . | |
| 5 | . | . | 0 | 0 | 0 | **5** |
| 8 | . | 1 | . | 14 | 16 | **39** |
| 70 | . | 0 | 0 | . | 14 | **84** |
| 5 | . | 43 | 0 | 32 | . | **80** |
| **88** | | **45** | **0** | **47** | **31** | |

| | | | | | | |
|---|---|---|---|---|---|---|
| . | 7 | . | 1 | 24 | 16 | **48** |
| 29 | . | . | 7 | 5 | 1 | **42** |
| . | . | . | . | . | . | |
| 3 | 52 | . | . | 10 | 1 | **66** |
| 0 | 11 | . | 0 | . | 1 | **12** |
| 3 | 0 | . | 9 | 4 | . | **16** |
| **35** | **70** | | **17** | **43** | **19** | |

| | | | | | | |
|---|---|---|---|---|---|---|
| . | 3 | 0 | . | 13 | 16 | **32** |
| 2 | . | 0 | . | 43 | 36 | **81** |
| 40 | 10 | . | . | 6 | 11 | **67** |
| . | . | . | . | . | . | |
| 7 | 18 | 42 | . | . | 20 | **87** |
| 8 | 21 | 20 | . | 12 | . | **61** |
| **57** | **52** | **62** | | **74** | **83** | |

| | | | | | | |
|---|---|---|---|---|---|---|
| . | 21 | 56 | 0 | . | 1 | **78** |
| 0 | . | 0 | 9 | . | 0 | **9** |
| 0 | 12 | . | 5 | . | 9 | **26** |
| 2 | 4 | 0 | . | . | 66 | **72** |
| . | . | . | . | . | . | |
| 38 | 1 | 0 | 10 | . | . | **49** |
| **40** | **38** | **56** | **24** | | **76** | |

| | | | | | | |
|---|---|---|---|---|---|---|
| . | 38 | 7 | 0 | 37 | . | **82** |
| 48 | . | 5 | 2 | 4 | . | **59** |
| 15 | 50 | . | 1 | 2 | . | **68** |
| 1 | 0 | 4 | . | 1 | . | **6** |
| 9 | 1 | 2 | 1 | . | . | **13** |
| . | . | . | . | . | . | |
| **73** | **89** | **18** | **4** | **44** | | |

| | | | | | |
|---|---|---|---|---|---|
| | **69** | **64** | **1** | **75** | **34** |
| **79** | | **11** | **30** | **53** | **46** |
| **60** | **77** | | **10** | **20** | **36** |
| **14** | **58** | **8** | | **25** | **85** |
| **86** | **33** | **55** | **2** | | **71** |
| **54** | **41** | **65** | **23** | **50** | |

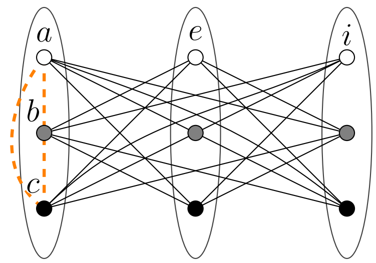
Figure 3.8: An SBMGDD of type $3 \times 6$

### 3.3.4 An SBMGDD Variant

To build cubes of order $7, 11, 13,$ and $17,$ objects similar to SBMGDDs are used which covers the groups in addition to lifting the blocks. Define $J_k$ to be an SBMGDD of type $3 \times k$ where additionally the pairs in exactly one group of size three have been covered. Unless otherwise noted, the starting frequency of $J_k$ is zero. Figure 3.9 gives an example of $J_k$ with

$k = 3$. The vertical groups are $\{abc, deg, ghi\}$ and horizontal groups are $\{aei, bfg, cdh\}$. The covered group is $abc$, depicted with an orange dashed line in Figure 3.9.

| | | |
|---|---|---|
| $3abc$ | $acg$ | $5bci$ |
| $2abd$ | $8adg$ | $bdi$ |
| $2abh$ | $2afh$ | $13beh$ |
| $16acf$ | $6bce$ | $11ceg$ |




| | | | |
|---|---|---|---|
| 3 | 6 | 5 | **14** |
| 2 | . | 1 | **3** |
| 2 | 13 | . | **15** |
| **7** | **19** | **6** | |

| | | | |
|---|---|---|---|
| 16 | . | 0 | **16** |
| . | . | . | |
| 2 | . | . | **2** |
| **18** | | **0** | |

| | | | |
|---|---|---|---|
| 1 | 11 | . | **12** |
| 8 | . | . | **8** |
| . | . | . | |
| **9** | **11** | | |

| | | |
|---|---|---|
| **20** | **17** | **5** |
| **10** | | **1** |
| **4** | **13** | |

Figure 3.9: $J_3$, an SBMGDD of type $3 \times 3$ with one group covered

The blocks of $J_4$ are given in Table 3.2, also found by computer. The vertical and horizontal groups are $\{aei, bfj, cgk, dhl\}$ and $\{abcd, efgh, ijk\ell\}$ respectively. Figure 3.10 depicts $J_4$ as a $4 \times 4 \times 4$ cube.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $afk$ | $9afl$ | $22agj$ | $2ahj$ | $5ahk$ | $4bek$ | $16bel$ | $bgi$ | $12bgl$ | $18bhi$ |
| $26cej$ | $14cfi$ | $21cfl$ | $9chi$ | $6chj$ | $8dej$ | $25dek$ | $2dgi$ | $9dgj$ | |

Table 3.2: The blocks of $J_4$

| | | | | |
|---|---|---|---|---|
| 1 | 13 | 4 | 2 | **20** |
| 1 | . | 19 | 10 | **30** |
| 25 | 7 | . | 0 | **32** |
| 6 | 1 | 1 | . | **8** |
| **33** | **21** | **24** | **12** | |

| | | | | |
|---|---|---|---|---|
| 3 | . | 0 | 1 | **4** |
| . | . | . | . | |
| 3 | . | . | 15 | **18** |
| 25 | . | 0 | . | **25** |
| **31** | | **0** | **16** | |

| | | | | |
|---|---|---|---|---|
| 1 | 4 | . | 31 | **36** |
| 22 | . | . | 4 | **26** |
| . | . | . | . | |
| 6 | 1 | . | . | **7** |
| **29** | **5** | | **35** | |

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 18 | . | **19** |
| 0 | . | 9 | . | **9** |
| 10 | 3 | . | . | **13** |
| . | . | . | . | |
| **11** | **3** | **27** | | |

| | | | |
|---|---|---|---|
| **6** | **17** | **22** | **34** |
| **23** | | **28** | **14** |
| **38** | **10** | | **15** |
| **37** | **2** | **1** | |

Figure 3.10: $J_4$, an SBMGDD of type $3 \times 4$ with one group covered

## 3.4 Larger Individual Cube Constructions

The next prime orders of interest are $7, 11, 13$, and $17$, which are described in Constructions 3.1, 3.2, 3.3, and 3.4 respectively.

### 3.4.1 Order 7

An SBC$(7)$ is constructed from a Fano plane.

**Construction 3.1.** Begin with a Fano plane, which is depicted in Figure 2.1. Inflate each of the seven points by three and label the new points $x_0, x_1, x_2$ for each point $x$. Replace each block with a copy of $J_3$ so that the underlined point $x$ in each of the following blocks takes the role of the covered group. The block set is $\{\underline{a}bd, \underline{b}ce, \underline{c}df, \underline{d}eg, \underline{e}af, \underline{f}bg, \underline{g}ac\}$, with underlined points covering the pairs in $\{x_0, x_1, x_2\}$. Since each point is underlined once, its inflated pairs are covered once. The copies of $J_3$ cover all pairs of inflated points except those corresponding to each level of inflated points. That is, pairs of the form $x_i y_i$ are left uncovered by the horizontal groups of each $J_3$. Therefore the seven points corresponding to each level of the inflated STS$(7)$ become the groups of the cube. Lifting with copies of a 3-MGDD of type $3 \times 3$ together with an additional block ensures all pair frequencies cover distinct consecutive integers. By Lemma 2.8, the result is an SBC$(7)$.

### 3.4.2 Order 11

An SBC$(11)$ is constructed by appending an additional point to a pairwise balanced design on ten points.

**Construction 3.2.** Start with the PBD$(10, \{3, 4\})$, given in Figure 2.2, appending an eleventh point to create a PBD$(11, \{2, 3, 4\})$ with the block set

$$\begin{array}{llll} abcd, & \underline{b}eh, & \underline{c}gi, & \underline{d}gh \\ aefg, & b\underline{f}i, & c\underline{e}j, & df\underline{j} \\ ahij, & b\underline{g}jk, & cf\underline{h}k, & de\underline{i}k \\ ak. & & & \end{array}$$

Inflate the points by three and label the points $x_0, x_1, x_2$ for each original point $x$ in $\{a, b, \ldots, k\}$. Blocks with an underlined point are replaced with $J_3$ or $J_4$, respective to the block size. The underlined point takes the role of the covered group in $J_k$. The remaining blocks of size four are replaced with a 3-SBMGDD of type $3 \times 4$, which avoids all pair bundles inflated from the same point. The block of size two is replaced with an SBC$(2)$ with groups $a_i k_i$ for $i = 0, 1, 2$. Similar to Construction 3.1, since $J_k$ and the SBMGDDs leave all pairs $x_i y_i$ uncovered by their horizontal groups, the groups of the Sarvate-Beam cube are given by the sets of seven points in each level $i$ of the design, for $i = 0, 1, 2$.

Apply Lemma 2.8, lifting each copy of $J_k$ with a 3-MGDD of type $3 \times k$ together with an added block. Copies of a 3-MGDD of type $3 \times 4$ may be used to lift the 3-SBGDDs of type $3 \times 4$.

### 3.4.3 Order 13

The construction of an SBC$(13)$ is similar to the SBC$(7)$ construction, starting with an affine plane with 13 points, or equivalently a PBD$(13, \{4\})$, rather than the Fano plane.

**Construction 3.3.** Start with a PBD$(13, \{4\})$, using all 13 blocks to underline each element once,

$$\begin{array}{llll} \underline{a}bdj, & \underline{b}cek, & c\underline{d}f\ell, & dh\underline{i}k, \\ a\underline{c}im, & b\underline{f}gi, & cgh\underline{j}, & eij\underline{\ell}, \\ a\underline{e}fh, & b\underline{h}\ell m, & deg\underline{m}, & fj\underline{k}m, \\ a\underline{g}k\ell. & & & \end{array}$$

Inflate each point by three, labelling points $x_0, x_1$, and $x_2$ for each point $x$ in the original PBD. Similarly to the previous constructions, each block is replaced with a copy of $J_4$, covering pairs $x_i x_j$ for the underlined point $x$ only. Pairs of the form $x_i y_i$ are the only pairs that remain uncovered by the horizontal groups of the $J_4$. Use Lemma 2.8 to lift each copy of $J_4$ with a 3-MGDD of type $3 \times 4$ plus an additional block. The result is an SBC$(13)$.

### 3.4.4 Order 17

The final individual cube construction, to obtain order 17, requires a cube of order five as an ingredient which is given in Section 3.2.3.

**Construction 3.4.** Start with a PBD$(16, \{4\})$ and append a seventeenth point, $q$, to obtain a PBD$(17, \{4, 5\})$ with blocks

$$\begin{array}{lllll} abcdq, & a\underline{e}ip, & bek\underline{o}, & ceg\underline{\ell}, & dehn, \\ efjmq, & a\underline{f}gh, & bf\ell\underline{n}, & cf\underline{i}o, & dfkp, \\ giknq, & a\underline{j}no, & b\underline{g}jp, & chj\underline{k}, & dij\ell, \\ h\ell opq, & ak\ell\underline{m}, & b\underline{h}im, & cmn\underline{p}, & dgmo. \end{array}$$

Note that only points $e, f, \ldots, p$ are underlined. As in previous constructions, $4$-blocks containing underlined points are replaced with a copy of $J_4$. Pairs inflated from the same point are covered in their underlined copy of $J_4$, excluding the points $a, b, c, d,$ and $q$ which are not underlined. Replace the block $abcdq$ with an SBC$(5)$ which covers the outstanding inflated pairs in addition to pairs between different points on different inflation levels. Replace the remaining blocks of size four and five with 3-SBMGDDs of type $3 \times 4$ and $3 \times 5$ respectively. Copies of $J_4$ and SBMGDDs are lifted using Lemma 2.8 and appropriately sized MGDDs, with an appended block in the case of $J_4$. The result is an SBC$(17)$ with groups corresponding to the three levels of points, the last of the small individual cube constructions.

## 3.5 General Construction for Cubes

The constructions in this section build the Sarvate-Beam cubes of all remaining orders from the small order cubes given in Section 3.2 and Section 3.4, as well as the building blocks from Section 3.3. Dukes and Short-Gershman [12] used small order cubes to construct cubes with larger orders in the following lemma.

**Lemma 3.1.** If there exists an SBC$(n)$, then there exists an SBC$(mn)$ for every positive integer $m$.

*Proof.* Take a Latin square of order $m$, or equivalently a 3-GDD of type $m^3$. Inflate each point by $n$. Replace the blocks of the Latin square with an SBC$(n)$. This covers all pairs of inflated points between groups and no points within each group. Lift the ingredient cubes to cover distinct intervals using copies of a 3-GDD of type $n^3$ as in Lemma 2.8. The result is a Sarvate-Beam cube with groups of size $mn$. □

If the existence of cubes of prime power order was known, Lemma 3.1 would complete the existence of cubes. However, as a construction for all prime power cubes remains unknown, the following construction is used to construct Sarvate-Beam cubes in a different way that only requires cubes of order three, four, and five as well as a few other small designs to complete the existence theory.

**Construction 3.5.** From a PBD$(n+1, \{4,5,6\})$, delete one point. A $\{4,5,6\}$-GDD with group sizes $3, 4,$ and $5$ is what remains. Inflate each point by three. Replace each block with a 3-SBMGDD of type $3 \times k$ for $k = 4, 5,$ or $6$. Fill in the groups with an SBC$(m)$ with $m = 3, 4,$ or $5$. Cubes of order three and five are given in Section 3.2. A cube of order four can be constructed by applying Lemma 3.1 to an SBC$(2)$.

Lift each cube of order $m$ using copies of a 3-GDD of type $m^3$ and Lemma 2.8. The SBMGDDs of type $3 \times k$ are lifted using copies of a 3-MGDD of type $3 \times k$. The lifted SBMGDDs leave the pairs in each inflated level uncovered by their horizontal groups. Therefore the points at each level of inflation become the groups of the Sarvate-Beam cube. The result is an SBC$(n)$.

Construction 3.5 relies on the existence of $\{4,5,6\}$-PBDs, which is given in Lemma 2.3. Together with a few individual cubes of small orders, the complete existence theory for Sarvate-Beam cubes of all orders is obtained.

**Theorem 3.2.** An SBC$(n)$ exists for all integers $n \geq 2$.

*Proof.* Note that Sarvate-Beam cubes of order two, three, and five are given in Section 3.2. Orders divisible by two, three, or five are therefore given by Lemma 3.1. From Lemma 2.3 together with Construction 3.5, the remaining possible exceptions for an SBC$(n)$ are $n =$

$7, 11, 13$, and $17$, which are integers with no divisors less than seven where no PBD($n + 1, \{4, 5, 6\}$) exists. These cubes are created in Constructions 3.1, 3.2, 3.3, and 3.4. $\square$

# Chapter 4

# Sarvate-Beam GDDs With More Groups

The focus of this chapter is the existence of $3$-SBGDDs with more than three groups. The main construction starts with a pairwise balanced design, similar to the main cube construction used in Theorem 3.2. However, unlike Sarvate-Beam cubes, the groups of the resulting SBGDDs of type $g^u$ come from the bundles of inflated points of the PBD.

## 4.1 General Construction Using Pairwise Balanced Designs

The main SBGDD construction, Construction 4.1, makes use of the existence of pairwise balanced designs with blocks of size $3, 4, 5$, and $6$. To start, the existence of the needed pairwise balanced designs is given.

**Lemma 4.1** ([15])**.** There exists a PBD$(v, \{3, 4, 5, 6\})$ for all $v \geq 3$ with $v \neq 8$.

Using the existence of these PBDs, a construction for 3-SBGDD of type $g^u$ for most values of $g$ and $u$ is accomplished.

**Construction 4.1.** To construct SBGDDs of type $g^u$, start with a PBD$(u, \{3, 4, 5, 6\})$ and inflate each point into $g$ points. Replace each original block of size three with an SBC$(g)$. Replace each block of size $k \in \{4, 5, 6\}$ of the original PBD with a $k$-GDD of type $g^k$, or equivalently $k-2$ MOLS$(g)$. All pairs of inflated points between groups are covered. Each of the $k$-blocks from the ingredient GDDs is replaced with an SBTS$(k)$. Note that when $k=4$ an SBTS$_1(4)$ is used.

The pair frequencies on each replaced $k$-block, for $k=4,5,$ or $6$ presently have overlapping intervals. In order to complete the construction, the block multiplicities of each SBC and SBTS are increased uniformly to cover back-to-back frequency intervals. Applying Lemma 2.8, the SBC$(g)$ may be lifted using 3-GDDs of type $g^3$. The SBTS$(k)$ are lifted by increasing the starting frequency, carefully using the tactics in Section 4.5.1 to avoid inadmissible values when $k=5$. Since SBTS$_\mu(5)$ are only admissible for $\mu \equiv 0 \pmod 3$, the interval starting at zero is covered with subdesigns on five points first and the rest of the ingredient designs are lifted to cover higher pair frequency intervals.

Since no SBTS$_0(4)$ exists, the subdesigns on four points cannot begin the interval of frequencies. In the case where the PBD contains only $4$-blocks, starting frequency zero may be unattainable. Note that a PBD$(u,\{4\})$ exists if and only if $u \equiv 1,4 \pmod{12}$ [6]. Moreover, a PBD$(u,\{3,5,6\})$ exists for all integers $u \geq 3$ except $u=4,8,10,12,14,20,$ or $22$. So the only 'bad' value is $u=4$ which is already excluded.

While Construction 4.1 relies on the existence of Sarvate-Beam cubes, most orders can be constructed without cubes by instead starting with a PBD$(u,\{4,5,6\})$. However, in view of the exceptions listed in Lemma 2.3, the cubes are required in the construction when $u=7,9,10,11,12,14,15,18,19,$ or $23$.

Presently Construction 4.1 together with Theorem 3.2 gives the following partial result.

**Theorem 4.2.** There is a 3-SBGDD of type $g^u$ with starting frequency $\mu=0$ for all integers $u \geq 3$ and $g \geq 2$ except possibly when $u=8$ or $g \in \{2,3,4,6,10,22\}$.

The possible exceptions come from two places in Construction 4.1. In order to replace inflated blocks of size four, five, and six with GDDs, the existence of two, three, and four MOLS$(g)$ is used in the construction. It is known that four MOLS$(g)$ exist for all $g$ excluding $g = 2, 3, 4, 6, 10$, and 22 [1, 6, 26]. These possible exceptions are considered in Sections 4.3.1 and 4.4. Secondly, as there is no PBD$(8, \{3, 4, 5, 6\})$, the case of $u = 8$ is not handled in the PBD construction. An alternate construction for SBGDDs with eight groups is developed in Section 4.2.

Extending Construction 4.1 to starting frequency $\mu > 0$ is covered in Section 4.6 as it relies on the sequencing method used in Section 4.5. However, the SBGDD constructions with $\mu > 0$ corresponding to SBGDDs which are not covered by Theorem 4.2 are presented alongside their constructions with $\mu = 0$ in what follows, as the constructions are often very similar.

## 4.2 SBGDDs with Eight Groups

As there is no PBD$(8, \{3, 4, 5, 6\})$, Construction 4.1 is mimicked by instead starting with six MOLS$(g)$, when possible, which is equivalent to an 8-GDD of type $g^8$.

**Construction 4.2.** From an 8-GDD of type $g^8$, replace each of the blocks with an SBTS$(8)$. Then apply Lemma 2.8 together with the sequencing method given in Section 4.5.2 to obtain a pair frequency interval starting at zero. The result is a 3-SBGDD of type $g^8$.

Note that an 8-GDD of type $g^8$ has $g^2$ blocks of size eight, one for each position in the array in the corresponding Latin squares. For $g \equiv 0, 1$, or $2 \pmod 3$, note that $g^2 \equiv 0, 1$, or $1 \pmod 3$ so Construction 4.18 always works to produce sequenced sets of blocks to lift, provided six MOLS$(g)$ exist.

For $\mu > 0$, when it exists, a $(3, \mu)$-GDD of type $g^8$ can be used in Lemma 2.8 to lift an entire SBGDD of type $g^8$ with starting frequency zero to any desired started frequency $\mu$. By Lemma 2.4, an SBGDD of type $g^8$ has $g$ or $\mu$ are divisible by three. Therefore by Lemma 2.5, an SBGDD of type $g^8$ with starting frequency $\mu > 0$ exists whenever:

- $\mu \equiv 0 \pmod 6$, or
- $g \equiv 0 \pmod 6$, or
- $g \equiv 3 \pmod 6$ and $\mu$ even, or
- $\mu \equiv 3 \pmod 6$ and $g$ even.

The admissible SBGDDs with eight groups that remain to be constructed have starting frequency $\mu \equiv 3 \pmod 6$ and $g$ odd, or $g \equiv 3 \pmod 6$ and $\mu$ odd. As Construction 4.10 builds SBGDDs with $\mu > 0$ for any $g \equiv 0 \pmod 3$, the remaining admissible cases are therefore when $\mu \equiv 3 \pmod 6$ and $g \equiv 1, 5 \pmod 6$. As a result of Construction 4.11, the case when $g \equiv 0 \pmod 5$ is excluded from the list of exceptions. Constructions with $g \equiv 1, 5 \pmod 6, g \not\equiv 0 \pmod 5$, and $\mu \equiv 3 \pmod 6$ are presently yet to be found and left as the only exceptions in Theorem 1.3.

## 4.3 MOLS Exceptions

Individual constructions are explored for those values where the required mutually orthogonal Latin squares are not known to exist. The remaining values are left from the main PBD construction in Section 4.1 where MOLS are used as ingredients to replace inflated blocks, as well as the starting design in Section 4.2 used to construct SBGDDs with eight groups.

### 4.3.1 Avoiding Ingredient MOLS in the PBD Construction

There are several cases for which $k - 2$ MOLS$(g)$ do not exist and therefore cannot be used as ingredient designs to replace blocks of size $k = 4, 5,$ or $6$ in Construction 4.1. Table 4.1 lists all values where MOLS cannot be used and the location of the replacement construction.

| $k$ | Cases avoiding $k-2$ MOLS$(g)$ | Location Covered |
|---|---|---|
| 4 | $g=2$ | Section 4.4.1.1 |
| | $g=6$ | Construction 4.12 |
| 5 | $g=2$ | Section 4.4.2.1 |
| | $g=3$ | Section 4.4.2.2 |
| | $g=6$ | Construction 4.13 |
| | $g=10$ | Construction 4.14 |
| 6 | $g=4,6,10,22$ | Construction 4.3 |
| | $g=2$ | Section 4.4.3.2 |
| | $g=3$ | Section 4.4.3.3 |

Table 4.1: Constructions avoiding ingredient MOLS

**Construction 4.3.** When four MOLS$(g)$ do not exist, it is sometimes possible to delete a block from four MOLS$(g+1)$. After deletion, what remains is a $\{5,6\}$-GDD of type $g^6$. If the $5$-blocks can be sequenced as in Section 4.5.1, the construction continues similarly to Construction 4.1. Each of the deleted points occurs $g+1$ times, once in each row and column of the Latin squares. Once the block is deleted, there are $g$ blocks remaining that were incident to each deleted point, giving a total of $6g$ blocks of size five in what remains. As the number of $5$-blocks is a multiple of three, they can be sequenced according to Section 4.5.1. Therefore a $\{5,6\}$-GDD of type $g^6$ may be used in place of four MOLS$(g)$ in Construction 4.1.

Note that Construction 4.3 cannot be used with three MOLS$(g+1)$ unless there are disjoint transversals in the corresponding Latin squares. Otherwise, after deletion all blocks have size four and starting frequency $\mu=0$ is not possible. In the case of two MOLS$(g+1)$, after deletion individual 3-blocks are created which have no Sarvate-Beam type replacement design.

**Construction 4.4.** In place of $k-2$ MOLS$(g)$, a 4-MGDD of type $g \times k$ may be used instead, for $k = 4, 5, 6$ and $g \geq 4$. Replace 4-blocks with an SBTS$_1(4)$ and fill horizontal groups with an SBTS$(k)$.

Note that $g = k = 4$ is not admissible unless starting frequency greater than zero is desired. By Theorem 2.7, if a 4-MGDD exists then $(g-1)(k-1)$ divisible by three and $\{g, k\} \neq \{4, 6\}$. So Construction 4.4 gives an alternate construction when $(k, g) = (6, 10)$, or $(6, 22)$.

The remaining values in Table 4.1 are individually constructed in Section 4.4.

### 4.3.2 Avoiding Six MOLS$(g)$ for SBGDDs with Eight Groups

The list of exceptions when six MOLS$(g)$ are not known to exist are given in Table 4.2 [6], organized by the location of their alternate construction.

| When 6 MOLS$(g)$ cannot be used | Location |
|---|---|
| $g = 6, 10, 12, 15, 18, 22, 26, 28, 30, 35, 39,$ $42, 44, 46, 52, 54, 58, 60, 62, 66, 68, 74$ | Construction 4.5 |
| $g = 14, 21, 34, 38, 51$ | Construction 4.6 |
| $g = 2$ | Section 4.4.4.2 |
| $g = 3$ | Section 4.4.4.3 |
| $g = 4$ | Construction 4.4.4.4 |
| $g = 5$ | Construction 4.4.4.5 |
| $g = 20$ | Construction 4.8 |
| $g = 33$ | Construction 4.9 |

Table 4.2: Constructions avoiding six MOLS

Most of the exceptions are constructed using six MOLS$(g+1)$ instead of six MOLS$(g)$.

Denote $N(n)$ to be the maximum number of Latin squares in a set of MOLS of side $n$. In Table 4.3, the best lower bound known for $N(n)$ is given for values $g$ where N$(g+1) \geq 6$. That is, when $N(n)$ is not known directly, $m$ is listed such that $N(n) \geq m$, according to [6].

| $g$ | 6 | 10 | 12 | 15 | 18 | 22 | 26 | 28 | 30 | 35 | 39 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $g+1$ | 7 | 11 | 13 | 16 | 19 | 23 | 27 | 29 | 31 | 36 | 40 |
| $N(g+1)$ | 6 | 10 | 12 | 15 | 18 | 22 | 26 | 28 | 30 | 8 | 7 |
| $g$ | 42 | 44 | 46 | 52 | 54 | 58 | 60 | 62 | 66 | 68 | 74 |
| $g+1$ | 43 | 45 | 47 | 53 | 55 | 59 | 61 | 63 | 67 | 69 | 75 |
| $N(g+1)$ | 42 | 6 | 46 | 52 | 6 | 58 | 60 | 6 | 66 | 6 | 7 |

Table 4.3: The minimum number of MOLS$(g+1)$ known

**Construction 4.5.** Construction 4.3 is followed with $k = 8$. From six MOLS$(g+1)$, or an $8$-GDD of type $(g+1)^8$, delete an entire block. What is left is a $\{7, 8\}$-GDD of type $g^8$. Replacing blocks of size seven and eight with SBTS$(7)$ and SBTS$(8)$. An SBTS$(7)$ is given in [12]. Sequence the $8$-blocks according to Section 4.5.2 and lift using Lemma 2.8.

Note that before deletion, the GDD has $(g+1)^2$ blocks. Since each point is on $g+1$ blocks, each removed point reduces the size of $g$ remaining blocks. Therefore there are $(g+1)^2 - 8g - 1 = g^2 - 6g$ blocks of size eight in what remains. The different congruence classes of $g$ modulo three give $g^2 - 6g \equiv 0, 1,$ or $1 \pmod 3$ respectively. Therefore the sequencing method outlined in Section 4.5.2, specifically Construction 4.18, is sufficient to sequence the $8$-blocks for any value of $g$ whenever six MOLS$(g+1)$ exist. The SBGDDs of type $g^8$ that Construction 4.5 builds are the values of $g$ given in Table 4.3.

For $\mu > 0$, note that a $(3, \mu)$-GDD of type $g^8$ completes the construction when $g$ is divisible by six or $\mu$ is divisible by three, as discussed in Section 4.2. The remaining values are $g = 10, 15, 22, 26, 28, 30, 35, 39, 44, 46, 52, 58, 62, 68$, and $74$. In the case of $g = 15, 30$, or $39$, copies of a $(3, \mu)$-GDD may be used which exist when $\mu$ is even. Construction 4.10 also builds SBGDDs of type $g^8$ with $\mu > 0$ for any $g$ divisible by three.

In all remaining cases, as $g$ is not divisible by three and $u \equiv 2 \pmod 3$, the necessary conditions of existence in Lemma 2.4 require that $\mu \equiv 0 \pmod 3$. By Lemma 2.5, a $(3, \mu)$-GDD of type $g^8$ exists whenever $\mu \equiv 0 \pmod 3$ and $g$ is even. This covers all remaining cases except $g = 35$ which is given in Construction 4.11.

Returning to starting frequency zero, deleting two blocks from a set of six MOLS$(g + 2)$ is considered to construct SBGDDs of type $g^8$ in cases where six MOLS$(g)$ and six MOLS$(g + 1)$ are not known to exist.

**Construction 4.6.** The aim is to create an SBGDD of type $g^8$ from an $8$-GDD of type $(g + 2)^8$ by first deleting two disjoint blocks of size eight. However there is no guarantee that disjoint blocks exist in any such GDD. Consider a set of seven MOLS$(g+2)$. Take one of the Latin squares and remove it from the set. What remains is a set of six MOLS$(g+2)$, which necessarily have $g + 2$ disjoint transversals. Therefore the corresponding GDD has at least two disjoint blocks of size eight which can be deleted, leaving a $\{6, 7, 8\}$-GDD of type $g^8$. Replace blocks with SBTS$(6)$, SBTS$(7)$, and SBTS$(8)$ respective to their size.

Note that six MOLS$(g + 2)$ has $(g + 2)^2$ total blocks. In order to use Section 4.5.2 to sequence and lift $8$-blocks, the number of $8$-blocks remaining are counted by considering the untouched cells in the MOLS array. Removing two disjoint blocks removes the points corresponding to the two deleted rows and columns. This leave a $g \times g$ subsquare with row and column points intact in the Latin square array left to consider, or $g^2$ blocks. As shared rows and columns are ruled out, each point incident to a deleted block in the subsquare occurs in a separate cell. As each symbol occurs $g + 2$ times total, including once in the row/column it was deleted from and once in the row and column of the other deleted block, the subsquare contains $g + 2 - 3 = g - 1$ remaining occurrences of each deleted symbol.

Now as two symbols from different deleted blocks also occur together as a pair, there are $(8-2)(7-2) = 30$ occurrences of the deleted symbols in the same block. Thus the number of blocks of size eight remaining is $g^2 - 2 \cdot 6(g-1) + 30$. For each congruence class of $g$ modulo three, $g^2 - 12(g-1) + 30 \equiv g^2 \equiv 0, 1$, or $1 \pmod 3$ respectively. Therefore Construction 4.18 may always be used to sequence 8-blocks in this construction. Lift blocks according to Lemma 2.8 and Section 4.5.2 to obtain a 3-SBGDD of type $g^u$.

From this construction, it is clear that seven MOLS$(g+2)$ are required to delete two disjoint blocks. The maximum number of MOLS known for the remaining exceptions is given in Table 4.3.2 [6]. As there are at least seven MOLS$(g+2)$ when $g = 14, 21, 34, 38$, or $51$, a 3-SBGDD of type $g^8$ is constructed using Construction 4.6 in these cases.

| $g$ | 2 | 3 | 4 | 5 | **14** | 20 | **21** | 33 | **34** | **38** | **51** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $g+2$ | 4 | 5 | 6 | 7 | 16 | 22 | 23 | 35 | 36 | 40 | 53 |
| $N(g+2)$ | 3 | 4 | 1 | 6 | **15** | 3 | **22** | 5 | **8** | **7** | **52** |

Table 4.4: The known number of MOLS$(g+2)$

For $\mu > 0$, when $g$ is not divisible by three, the necessary conditions require that $\mu$ is divisible by three. When $\mu \equiv 0 \pmod 3$ and $g$ even, a $(3, \mu)$-GDD of type $g^8$ is used to lift the design. This takes care of values $g = 14, 34$, and $38$. The SBGDDs of type $21^8$ and $51^8$ with starting frequencies $\mu > 0$ are given in Construction 4.10.

#### 4.3.2.1 Using the Existence of SBGDDs with 3, 4, and 5 Groups

By the results given in Chapter 3 together with those in Sections 4.1, 4.3.1, and 4.4 to come, it is shown that SBGDDs of type $m^k$ exist for all $m \geq 2$ and $k = 3, 4$, or $5$. These SBGDDs are used as ingredients to construct some of the remaining possible exceptions in the following construction.

**Construction 4.7.** If there exists a $\{3, 4, 5\}$-GDD of type $h^8$ then a 3-SBGDD of type $(mh)^8$ can be constructed as follows. Inflate each point by any integer $m \geq 2$. Replace blocks with a 3-SBGDD of type $m^k$ where $k = 3, 4$, or $5$, respective to the block size. Lift blocks according to Lemma 2.8 and Section 4.5.1 and the result is a 3-SBGDD of type $(mh)^8$.

Note that if a $\{4, 5\}$-GDD of type $h^8$ is used in Construction 4.7, then $m = 1$ is admissible as the SBGDDs of type $1^k$ exist when $k$ is four or five. If the initial GDD has 5-blocks only, then starting frequency one is not attainable as there is no $\text{SBTS}_1(5)$. If the GDD has 4-blocks only, then starting frequency zero is not attainable, as there is no $\text{SBTS}_0(4)$.

With these restrictions in mind, Construction 4.7 is used to construct more SBGDDs with eight groups.

**Construction 4.8.** Start with a $\text{PBD}(41, \{5\})$ and delete a point to obtain a 5-GDD of type $4^{10}$ [6]. The points in two groups are truncated to obtain a GDD of type $4^8$. Since any block has at most one point per deleted group, the 5-blocks are reduced to 3-blocks at most. What is left is a $\{3, 4, 5\}$-GDD of type $4^8$. Use Construction 4.7 with $m = 5$ to build an SBGDD of type $20^8$.

Note that in the 5-GDD, every point is on a block with each of 36 points in other groups, and therefore incident to nine blocks. In total there are 90 blocks of size five. Now deleting two groups reduces all the blocks incident to one of eight deleted points. As there are 16 blocks incident to two deleted points, $9 \times 8 - 16 = 56$ blocks are reduced and thus 34 blocks of size five remain. Since $34 \equiv 1 \pmod 3$, Construction 4.17 may be used to sequence the 5-blocks and complete the construction of an SBGDD of type $20^8$.

For $\mu > 0$, Construction 4.20 is followed to lift the subdesigns on 5-blocks to obtain any starting frequency. Thus an SBGDD of type $20^8$ is constructed for any $\mu \geq 0$.

The following two constructions both build 3-SBGDDs of type $(3m)^8$. The first construction can be altered to obtain any starting frequency $\mu \geq 0$, but requires $m \geq 2$, while the second construction allows for any $m \geq 1$ but requires $\mu \geq 1$. Both constructions are therefore presented.

**Construction 4.9.** Truncate all the points from one group of a 4-GDD of type $3^9$ [6]. This leaves a $\{3, 4\}$-GDD of type $3^8$. Then Construction 4.7 gives an SBGDD of type $(3m)^8$ for any $m \geq 2$.

Take $m = 10$ in Construction 4.9 to create an SBGDD of type $30^8$. Note that using $m = 7$ or $17$ gives alternate constructions for $g = 21$ or $51$ respectively, first built in Construction 4.6. To obtain SBGDDs with starting frequency $\mu > 0$, as there are no 5-blocks to replace, cubes and $\text{SBTS}_1(4)$ may be used to lift the design according to Lemma 2.8. What remains is the case when $m = 1$, constructed next.

**Construction 4.10.** Start with a $\text{PBD}(25, \{4\})$ and delete a single point to get a 4-GDD of type $3^8$ [6]. Use Construction 4.7 to build an SBGDD of type $(3m)^8$. As there are no 3-blocks present, this construction works for any $m \geq 1$. However, as there are only 4-blocks, starting frequency $\mu = 0$ is not attainable in this way.

Taking $m = 1, 5, 7, 10, 11, 12$, or $17$ in Construction 4.10 creates SBGDDs with groups of size $g = 3, 15, 21, 30, 33, 39$, or $51$ respectively. This construction is also used in Section 4.4.4.

The following construction gives 3-SBGDDs of type $(5m)^8$ for any $m \geq 1$ and starting frequency $\mu$ divisible by three.

**Construction 4.11.** Start with a 5-GDD of type $5^9$ [6]. Truncate all the points in a group to give a $\{4, 5\}$-GDD of type $5^8$. Once again, as there are no 3-blocks, use Construction 4.7 to build an SBGDD of type $(5m)^8$ for any positive integer $m$. To construct $\mu \equiv 0 \pmod 3$ for $\mu > 0$, lift 5-blocks according to Construction 4.20 and Lemma 2.8.

Note that Constructions 4.4.4.5 and 4.5 also build SBGDDs of type $5^8$ and $35^8$ with starting frequency zero.

## 4.4 Remaining Small SBGDDs

Direct constructions for 3-SBGDDs of type $g^u$ are given for the outstanding cases from Theorem 4.2, as well as those small SBGDDs used in other constructions.

### 4.4.1 Small Orders with Four Groups

Note that an $\text{SBTS}_1(4)$ is given in Figure 1.1. Starting frequency $\mu > 1$ in is established in Section 2.3.1. The Sarvate-Beam triple systems on four points of all starting frequencies were first constructed by Dukes and Short-Gershman [12].

#### 4.4.1.1 SBGDD of Type $2^4$

Dukes and Short-Gershman first constructed an SBGDD of type $2^4$ [12]. It is reproduced here with groups $\{ab, cd, ef, gh\}$ for use in Construction4.4.1.2.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $ade$ | $adf$ | $2adg$ | $2adh$ | $3aeg$ | $4aeh$ | $4afg$ | $5afh$ |
| $bch$ | $bdg$ | $bdh$ | $beg$ | $2beh$ | $2bfg$ | $3bfh$ | $6ceg$ |
| $8ceh$ | $6cfg$ | $7cfh$ | $7deg$ | $7deh$ | $10dfg$ | $8dfh$ | |

Figure 4.1: Blocks of an SBGDD of type $2^4$

An SBGDD of type $2^4$ with starting frequency $\mu > 0$ can be obtained by adding $\mu$ copies of the blocks of a 3-GDD of type $2^4$ and invoking Lemma 2.8. Note that the existence of a 3-GDD of type $2^4$ is equivalent to the unique $(9, 3, 1)$-design with a point deleted.

#### 4.4.1.2 SBGDD of Type $6^4$

While there are no two $\text{MOLS}(6)$, one can make use of two $\text{IMOLS}(6, 2)$ instead. The known pair of $\text{IMOLS}(6, 2)$ is given in Figure 4.3 overlaid into a single square [6]. The symbols from distinct squares are shown in orange and blue.

| | | | | | |
|---|---|---|---|---|---|
| 51 | 62 | 35 | 46 | 13 | 24 |
| 26 | 15 | 61 | 52 | 34 | 43 |
| 64 | 53 | 16 | 25 | 41 | 32 |
| 45 | 36 | 54 | 63 | 22 | 11 |
| 12 | 44 | 23 | 31 | | |
| 33 | 21 | 12 | 44 | | |

Figure 4.2: Two IMOLS$(6, 2)$

**Construction 4.12.** Start with the pair of IMOLS$(6, 2)$ given in Figure 4.2. Using the SBGDD of type $2^4$ from Section 4.4.1.1, simply patch the $2 \times 2$ hole of the incomplete Latin squares with the SBGDD. This patch covers the pairs inside the red box given in Figure 4.3 and no others. Any pair not fully contained in the red box is covered by the IMOLS. Each of the blocks of the IMOLS is replaced with an SBTS$_\mu(4)$ for the desired frequency $\mu$ in the interval of frequencies. Note that these replaced blocks cover the interval of pair frequencies starting with $\binom{4}{2}2^2 = 24$, to cover frequencies immediately following the frequencies covered by the SBGDD of type $2^4$. By Lemma 2.8, the result is an SBGDD of type $6^4$.

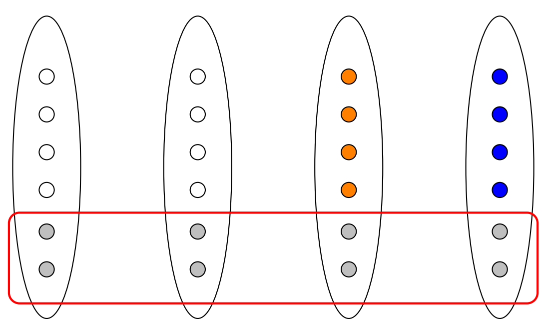


Figure 4.3: Building an SBGDD of type $6^4$ with an SBGDD of type $2^4$

Note that an SBGDD of type $6^4$ with starting frequency $\mu > 0$ can be obtained by using

Lemma 2.8 to lift the blocks in Construction 4.12 to start with an SBTS$_\mu(6)$. Alternatively, $\mu$ copies of the blocks of a 3-GDD of type $6^4$ can be used to lift the entire block set all at once.

### 4.4.2 Small Orders with Five Groups

An SBTS$(5)$ is given in Example 2.3.4. An SBTS$_\mu(5)$ exists only when $\mu \equiv 0 \pmod 3$ to satisfy the necessary conditions of Lemma 2.4. In this case, Dukes and Short-Gershman [12] showed that SBTS$_\mu(5)$ always exist.

#### 4.4.2.1 SBGDD of Type $2^5$

Start with the unique PBD$(11, \{3, 5\})$ which has a single block of size five. It is constructed in Example 2.1.1. Suppose $abcde$ is the block of size five. Delete a point not on this block to obtain a $\{3, 5\}$-GDD of type $2^5$. Put an SBTS$(5)$ on the $5$-block. By permuting the points not on the $5$-block, the frequencies in Figure 4.4 were found by hand. The blocks, excluding those in the SBTS$(5)$, are listed along side their block multiplicity. The blocks of the SBTS$(5)$ are given in Example 2.3.4. The groups are $\{ah, bj, cg, di, ef\}$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $10afg$ | $3afj$ | $1agj$ | $10aij$ | $2bfg$ | $11bfh$ | $4bfi$ | $7bgh$ | $15bgi$ |
| $8bhi$ | $9cfh$ | $13cfi$ | $6cfj$ | $3chj$ | $12cij$ | $7dfg$ | $9dfj$ | $5dgh$ |
| $7dgj$ | $13dhj$ | $6egh$ | $6egi$ | $8egj$ | $13ehi$ | $4ehj$ | $3eij$ | $4fgh$ |
| $6fgi$ | $6fgj$ | $5fhi$ | $5fhj$ | $3fij$ | $6ghi$ | $5ghj$ | $3gij$ | $7hij$ |

Figure 4.4: The SBGDD of type $2^5$

For $\mu > 0$, the necessary conditions of existence for an SBGDD of type $2^5$ are only satisfied when $\mu$ is divisible by three. Therefore copies of a 3-GDD of type $2^5$ with $\lambda = 3$, which exist by Lemma 2.5, may be added in all admissible cases to complete the existence of this group type.

#### 4.4.2.2 SBGDDs of Type $3^5$

Start with a 3-SBMGDD of type $3 \times 5$, like the one given in Figure 3.7. Fill in each horizontal group with an SBTS$(5)$. Use Section 4.5.1 to sequence the SBTS$(5)$. Lift the blocks of the SBMGDD using copies of a 3-MGDD of type $3 \times 5$, which exists by Lemma 2.6. By Lemma 2.8, the result is an SBGDD of type $3^5$.

For $\mu > 0$, instead start with a $4$-GDD of type $3^5$. Replace the first $4$-block with an SBTS$_\mu(4)$ to achieve the desired starting frequency $\mu$. Continue to replace $4$-blocks with Sarvate-Beam triple systems on four points with starting frequencies so that the intervals line up, as in Lemma 2.8. The result is an SBGDD of type $3^5$ with starting frequency $\mu > 0$.

#### 4.4.2.3 SBGDDs of Type $6^5$

Three MOLS$(7)$ are used to construct an SBGDD of type $6^5$ similarly to the constructions used to build SBGDDs with eight groups in Section 4.2.

**Construction 4.13.** As three MOLS$(6)$ do not exist, three MOLS$(7)$ are used instead, or equivalently a $5$-GDD of type $7^5$. Delete an entire block to get a $\{4, 5\}$-GDD of type $6^5$. Note that as there is a set of six MOLS$(7)$, a set of three MOLS$(7)$ may be selected to have seven common disjoint transversals. Therefore by selecting blocks in the transversal for deletion, the remaining GDD has blocks of size five. This ensures starting frequency zero can be covered using an SBTS$(5)$.

In order to use Section 4.5.1 for sequencing, the number $5$-blocks in the $\{4, 5\}$-GDD must be a multiple of three. Since the set of three MOLS$(7)$ has 49 blocks of size five, each point occurs seven times in the array of the Latin square, including the deleted block. That means there are $5(6)$ blocks of size four, giving $49 - 30 - 1 = 18 \equiv 0 \pmod 3$ blocks of size five. Therefore Section 4.5.1 can be used to sequence the $5$-blocks.

The blocks of the $\{4,5\}$-GDD are replaced with copies of an SBTS$(5)$ or an SBTS$_1(4)$ respective to the block size. Lift according to Lemma 2.8 to give an SBGDD of type $6^5$.

For $\mu > 0$, start with a $4$-GDD of type $6^5$ [6]. Similarly to group type $3^5$, replace the first $4$-block with an SBTS$_\mu(4)$ to achieve the desired starting frequency $\mu$. Continue to replace $4$-blocks with Sarvate-Beam triple systems on four points with increasing starting frequencies so the ingredient SBTSs cover back-to-back intervals, as in Lemma 2.8.

#### 4.4.2.4 SBGDDs of Type $10^5$

A construction similar to Construction 4.12 allows us to build an SBGDD of type $10^5$ from the incomplete MOLS$(10)$ discovered by Brouwer [5].

**Construction 4.14.** Start with three IMOLS$(10,2)$, patching the hole with an SBGDD of type $2^5$ from Section 4.4.2.1. The $10^2 - 2^2 = 96$ blocks of size five are replaced with copies of an SBTS$(5)$ and blocks are lifted and sequenced by Section 4.5.1. This covers pair frequencies from zero to $959$. By adding 320 copies of a $(3,\lambda)$-GDD of type $2^5$ with $\lambda = 3$, which exists by Lemma 2.5, the SBGDD of type $2^5$ is inflated to starting frequency 960. The result is an SBGDD of type $10^5$.

For starting frequency $\mu > 0$, the necessary conditions of Lemma 2.4 require $\mu \equiv 0 \pmod 3$. Therefore the entire design may be lifted with a $(3,3)$-GDD of type $10^5$, which exists by Lemma 2.4.

### 4.4.3 Small Orders with Six Groups

The outstanding values of $g$ for Sarvate-Beam GDDs of type $g^6$ are constructed in this section.

#### 4.4.3.1 SBTS(6)

Dukes gives an SBTS$(6)$ in [10]. The blocks of an SBTS$(6)$ used throughout the dissertation are $\{ace, 2adf, 5aef, 3bcf, 4bde, 9bef, 4cde, 7cdf\}$. The corresponding pair frequencies are given in Figure 4.5.

| Pairs | $ab$ | $ac$ | $ad$ | $ae$ | $af$ | $bc$ | $bd$ | $be$ | $bf$ | $cd$ | $ce$ | $cf$ | $de$ | $df$ | $ef$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Weight | 0 | 1 | 2 | 6 | 7 | 3 | 4 | 13 | 12 | 11 | 5 | 10 | 8 | 9 | 14 |

Figure 4.5: The pair frequencies of an SBTS$(6)$

The block set of an SBTS$_1(6)$ is given as it is used in Section 2.3.1. The blocks are

$$\begin{array}{ccccc} abc, & acf, & ade, & 2adf, & 3aef, \\ bcd, & bce, & 2bcf, & 3bde, & 3bdf, \\ 4bef, & 5cde, & 5cdf, & 4cef, & 4def. \end{array}$$

An SBTS$_1(6)$ was first constructed by Stanton in [24]. Dukes and Short-Gershman [12] later constructed an SBTS$_\mu(6)$ for any $\mu > 0$.

#### 4.4.3.2 SBGDD of Type $2^6$

Place an SBTS$(6)$ on points $A = \{a, b, c, d, e, f\}$. Let $M = \{g, h, i, j, k, \ell\}$ be the remaining points and let the groups be $\{ag, bh, ci, dj, ek, f\ell\}$. Make designs with all pairs in $M$ together with a point $x \in A$. For example, take $x = a$. The goal is to give these blocks multiplicities so that the five pairs $ah, \ldots, a\ell$ have pair frequencies from 15 to 19, since the SBTS$(6)$ covers the frequencies from zero to 14. Since each block covers two pairs $am_1$ and $am_2$, the sum of the frequencies must be even. However, as $15+16+17+18+19 = 85$, frequencies $15, 16, 17, 18$, and 20 are covered instead. In the next interval, with $x = b$ similarly cover the frequencies $19, 21, 22, 23$, and 25, again skipping frequency 24 so the sum is even. Continue this pattern with all $x \in A$, taking care to select pairs $m_1m_2$ a similar number of times. Frequencies $15$ through $44$ and $46$ are covered in this manner with pairs $xy$ for $x \in A$ and $y \in M$. What remains are pairs of the form $m_1m_2$, which were carefully given similar overall frequencies in the previous steps. Add copies of a $(6, 3, 2)$-design to $M$ until the pair frequencies are sufficiently high. Then, with some adjustment, altering an SBTS$(6)$ finishes off the construction. In the block set given in Figure 4.6, the pair frequencies in $A$ were built with $x = b$ first and $x = a$ last. The given block set excludes the SBTS$(6)$ on $A$ which is given in Section 4.4.3.1.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $24ahj$ | $19ahk$ | $20aik$ | $21ail$ | $21ajl$ | $7bgi$ | $8bgj$ | $9bik$ | $1bjk$ | $8bjl$ |
| $10bkl$ | $9cgk$ | $10cgl$ | $11chj$ | $11chl$ | $12cjk$ | $2cjl$ | $13dgh$ | $13dgi$ | $2dhi$ |
| $12dhk$ | $2dhl$ | $13dil$ | $12dkl$ | $16egj$ | $15egl$ | $17ehi$ | $15ehl$ | $17eij$ | $18fgh$ |
| $18fgk$ | $20fhi$ | $2fhk$ | $17fij$ | $18fjk$ | $8ghi$ | $10ghj$ | $1ghk$ | $1ghl$ | $4gij$ |
| $11gik$ | $1gil$ | $11gjl$ | $9gkl$ | $11hil$ | $9hjk$ | $12hkl$ | $10ijk$ | $11ijl$ | $3jkl$ |

Figure 4.6: The blocks of an SBGDD of type $2^6$ excluding $A$

For $\mu > 0$, note that a 3-GDD of type $2^6$ exists by Lemma 2.5. Apply Lemma 2.8 using $\mu$ copies of the GDD to lift the block set of an SBGDD of type $2^6$ with starting frequency zero to obtain starting frequency $\mu$.

#### 4.4.3.3 SBGDD of Type $3^6$

To construct an SBGDD of type $3^6$, start with a 3-SBMGDD of type $3 \times 6$. One is given in Figure 3.8. Fill each of the horizontal groups with an SBTS$(6)$. Use Lemma 2.8 to lift the SBTS$(6)$ to cover pair frequencies after those covered in the SBMGDD.

Note that an SBGDD of type $3^6$ is obtained for any starting frequency $\mu > 0$ as follows. Use Lemma 2.8 to lift the horizontal groups with an SBTS$_x(6)$ with various starting frequencies $x$, to cover the pair frequencies in $\{\mu, \mu + 1, \ldots, \mu + 44\}$. Since a 3-MGDD of type $3 \times 6$ exists, by Lemma 2.6, add $\mu + 45$ copies of a 3-MGDD of type $3 \times 6$ to the above SBGDD of type $3 \times 6$ to complete the construction.

### 4.4.4 Small Orders with Eight Groups

The final remaining exceptions are constructed in this section, which complete the existence of SBGDDs of type $g^8$ with starting frequency zero. These are the possible exceptions left from the constructions given in Section 4.2 which start with an 8-GDD.

#### 4.4.4.1 SBTS$(8)$

An SBTS$(8)$ was constructed by Dukes and Short-Gershman [12]. The block set given in Figure 4.4.4.1 is used in Section 4.5.2.

By Lemma 2.4, values when $\mu > 0$ are only admissible when $\mu$ is divisible by three. Dukes and Short-Gershman [12] show that all admissible Sarvate-Beam triple systems with eight points exist.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $ach$ | $adg$ | $adh$ | $aef$ | $aeg$ | $aeh$ | $afg$ | $2afh$ | $2agh$ | $bce$ | $bcf$ |
| $2bcg$ | $2bch$ | $2bde$ | $2bdf$ | $2bdg$ | $2bdh$ | $2bef$ | $2beg$ | $2beh$ | $2bfg$ | $3bfh$ |
| $3bgh$ | $3cde$ | $3cdf$ | $3cdg$ | $4cdh$ | $4cef$ | $3ceg$ | $3ceh$ | $4cfg$ | $3cfh$ | $4cgh$ |
| $4def$ | $4deg$ | $5deh$ | $5dfg$ | $5dfh$ | $5dgh$ | $5efg$ | $8efh$ | $8egh$ | $4fgh$ | |

Figure 4.7: The SBTS$(8)$

#### 4.4.4.2 SBGDD of Type $2^8$

Two constructions that result in Sarvate-Beam GDDs of type $2^8$ are given.

**Construction 4.15.** This construction uses the SBGDD of type $2^6$ given in Section 4.4.3.2. The corresponding groups are $\{ag, bh, ci, dj, ek, fl\}$. Two extra groups, $mn$ and $op$, are appended. An SBC$(2)$ is placed on points $\{m, n, o, p, x, y\}$ for every group $xy$ in the $2^6$ design. Figure 4.8 shows the groups with two of the six cubes depicted in red. The cubes of order two are lifted using Latin squares so that the pair frequencies cover the interval from zero to $71$. Every pair of points containing one of $m, n, o,$ or $p$ is covered in these cubes. Pairs not containing one of the appended points are covered in the SBGDD of type $2^6$. To cover the appropriate interval, the pair frequencies are lifted using $72$ copies of a $(13, 3, 1)$-design with one point deleted. The result is a design with group type $2^8$ where pair frequencies cover zero through $107, 120, 141$ (covered twice), and $162$. The high frequency pairs are $mo, mp, no,$ and $np$ which are covered in every cube. In what follows this 'almost adesign' is called $M$. Note that an adesign is easily constructed from $M$ by permuting points in some of the cubes. For example, swapping points $o$ and $p$ in one SBC$(2)$ gives the following set of pair frequencies instead: $\{0, 1, 2, \ldots, 107, 124, 137, 145, 158\}$.

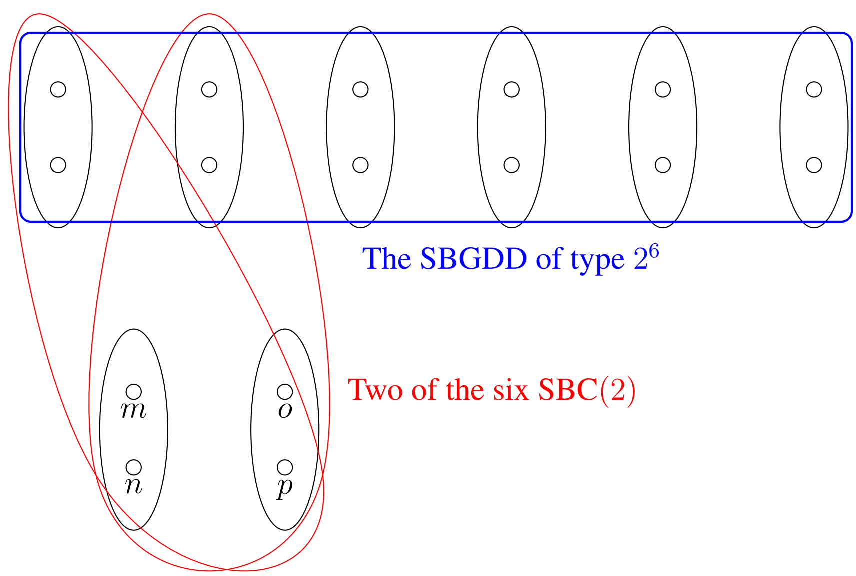



Figure 4.8: Constructing an SBGDD of type $2^8$

In order to adjust the block multiplicities of $M$ to make a proper SBGDD, the pair frequencies of these four pairs of appended points are reduced in the ingredient cubes by swapping blocks. For example, the pair frequency of $mp$ was reduced by eight in each SBC$(2)$ from the blocks $gmp, kmp, lmp, jmp$. Then multiplicity eight was added to each of the following blocks: $glm, glp, jkm, jkp$. The pairs frequencies of $gm, gp, km, kp, lm, lp, jm$, and $jp$ remain unchanged. Therefore the pair frequency of $mp$ is reduced from $141$ to $109$. The frequencies of $gl$ and $jk$ are increased by 16. Once $mo, no, np$ have been reduced similarly to $mp$ by swapping blocks, some ad hoc block swaps are performed so that the remaining displaced pairs, such as $gl$ and $jk$, find a new home in the interval of frequencies. The entire set of blocks is given in Figure 4.9. Table 4.5 lists all the block swaps performed on $M$ to achieve the true SBGDD with pair frequencies in the interval $\{0, 1, 2, \ldots, 111\}$. The blocks are listed in order of their pair frequency in $M$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $48abc$ | $ace$ | $4adf$ | $48adh$ | $46aef$ | $13aen$ | $13aep$ | $3afm$ | $afn$ | $4afo$ | $afp$ |
| $ahi$ | $24ahj$ | $19ahk$ | $ahl$ | $48aij$ | $18aik$ | $23ail$ | $2ajk$ | $22ajl$ | $49akl$ | $akn$ |
| $36amo$ | $4amp$ | $ano$ | $28anp$ | $bce$ | $2bcf$ | $2bde$ | $48bdk$ | $7bef$ | $48beg$ | $47bfi$ |
| $5bgi$ | $8bgj$ | $2bij$ | $7bik$ | $bil$ | $53bjl$ | $10bkl$ | $bmp$ | $4bno$ | $2bnp$ | $4cde$ |
| $53cdf$ | $48cel$ | $48cgh$ | $8cgk$ | $10cgl$ | $10chj$ | $10chl$ | $59cjk$ | $cjl$ | $8cmo$ | $3cmp$ |
| $1cno$ | $11cnp$ | $48dei$ | $3dem$ | $3deo$ | $12dfn$ | $12dfp$ | $11dgh$ | $13dgi$ | $47dgl$ | $2dhi$ |
| $12dhk$ | $2dhl$ | $13dil$ | $12dkl$ | $13dmo$ | $dmp$ | $4dno$ | $6dnp$ | $16egj$ | $14egl$ | $17ehi$ |
| $47ehj$ | $15ehl$ | $17eij$ | $21emo$ | $4emp$ | $eno$ | $13enp$ | $19fgh$ | $fgi$ | $48fgj$ | $19fgk$ |
| $19fhi$ | $61fhk$ | $17fij$ | $18fjk$ | $29fmo$ | $fmp$ | $4fno$ | $21fnp$ | $8ghi$ | $10ghj$ | $ghl$ |
| $4gij$ | $52gik$ | $2gil$ | $2gjl$ | $9gkl$ | $8gkn$ | $8gko$ | $8glm$ | $8glp$ | $32gmp$ | $34gno$ |
| $5gnp$ | $hij$ | $52hil$ | $2hjk$ | $11hkl$ | $2hmp$ | $5hnp$ | $9ijk$ | $11ijl$ | $8imp$ | $10ino$ |
| $5inp$ | $8jkm$ | $8jkp$ | $7jln$ | $7jlo$ | $10jmp$ | $9jno$ | $5jnp$ | $16kmp$ | $18kno$ | $5knp$ |
| $26lmp$ | $24lno$ | $5lnp$ | | | | | | | | |

Figure 4.9: An SBGDD of type $2^8$

| Pair | $M$ | SBGDD | Change | Pair | $M$ | SBGDD | Change | Pair | $M$ | SBGDD | Change |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $cn$ | 11 | 12 | 1 | $bj$ | 65 | 63 | $-2$ | $ah$ | 91 | 93 | 2 |
| $cm$ | 12 | 11 | $-1$ | $bl$ | 66 | 64 | $-2$ | $gi$ | 92 | 85 | $-7$ |
| $kn$ | 31 | 32 | 1 | $cg$ | 67 | 66 | $-1$ | $aj$ | 93 | 96 | 3 |
| $lo$ | 32 | 31 | $-1$ | $bk$ | 68 | 65 | $-3$ | $kl$ | 94 | 91 | $-3$ |
| $fo$ | 36 | 37 | 1 | $ck$ | 69 | 67 | $-2$ | $gl$ | 95 | 101 | 6 |
| $ln$ | 37 | 36 | $-1$ | $ch$ | 70 | 68 | $-2$ | $gk$ | 96 | 104 | 8 |
| $an$ | 43 | 44 | 1 | $cl$ | 71 | 69 | $-2$ | $gj$ | 97 | 88 | $-9$ |
| $am$ | 44 | 43 | $-1$ | $cj$ | 73 | 70 | $-3$ | $ik$ | 98 | 86 | $-12$ |
| $ad$ | 50 | 52 | 2 | $dg$ | 74 | 71 | $-3$ | $gh$ | 99 | 97 | $-2$ |
| $bd$ | 52 | 50 | $-2$ | $dl$ | 75 | 74 | $-1$ | $hl$ | 100 | 92 | $-8$ |
| $ce$ | 53 | 54 | 1 | $dh$ | 77 | 75 | $-2$ | $jk$ | 101 | 106 | 5 |
| $ae$ | 54 | 73 | 19 | $el$ | 78 | 77 | $-1$ | $hj$ | 102 | 94 | $-8$ |
| $af$ | 55 | 59 | 4 | $eg$ | 79 | 78 | $-1$ | $hk$ | 103 | 105 | 2 |
| $de$ | 56 | 60 | 4 | $eh$ | 80 | 79 | $-1$ | $jl$ | 104 | 103 | $-1$ |
| $df$ | 57 | 81 | 24 | $ej$ | 81 | 80 | $-1$ | $il$ | 105 | 102 | $-3$ |
| $cf$ | 58 | 55 | $-3$ | $fg$ | 84 | 87 | 3 | $hi$ | 106 | 100 | $-6$ |
| $cd$ | 59 | 57 | $-2$ | $fi$ | 85 | 84 | $-1$ | $ij$ | 107 | 109 | 2 |
| $bf$ | 60 | 56 | $-4$ | $fk$ | 86 | 98 | 12 | $mo$ | 120 | 107 | $-13$ |
| $be$ | 61 | 58 | $-3$ | $ak$ | 87 | 89 | 2 | $mp$ | 141 | 108 | $-33$ |
| $ef$ | 62 | 53 | $-9$ | $fh$ | 88 | 99 | 11 | $no$ | 141 | 110 | $-31$ |
| $bg$ | 63 | 61 | $-2$ | $ai$ | 89 | 90 | 1 | $np$ | 162 | 111 | $-51$ |
| $bi$ | 64 | 62 | $-2$ | $al$ | 90 | 95 | 5 | | | | |

Table 4.5: The blocks adjusted to create an SBGDD of type $2^8$

As the current construction relied on ad hoc methods to adjust the pair frequencies of the added points, another construction is presented, although neither construction completely avoids a few ad hoc block swaps.

**Construction 4.16.** Start with the blocks of an $(8, 4, 3)$-design [6]. Replace the $4$-blocks with copies of a $(4, 3, 2)$-design to create an $(8, 3, 6)$-design. The blocks of an $(8, 3, 6)$-design are used as a template for the blocks of the SBGDD as follows. Suppose the point set of the $(8, 3, 6)$-design is $\{a, b, \ldots, h\}$. Name the points in each group of the GDD $x_0$ and $x_1$ for $x \in \{a, b, \ldots, h\}$. For each block $xyz$ in the design, add a block $x_0y_0z_1$ to the GDD. Note the computer was used to randomly select the point $z$ to be assigned to the 'upper layer' of the GDD. The goal is to find an assignment with pair frequencies $\{0, 1, 2, 3\}$ between the pairs of type $x_iy_j$ with $x, y \in \{a, b, \ldots, h\}$ to obtain the Sarvate-Beam style of condition. For example, take the pair $ab$ which occurs six times in the $(8, 3, 6)$-design. Then using this block assignment method, the aim is to obtain the frequencies $3, 2, 1, 0$ for the pairs $a_0b_0, a_0b_1, a_1b_0, a_1b_1$ respectively. Since a single point is selected in each block to be of type $x_1$, the pairs of type $x_1y_1$ necessarily have frequency zero.

Separately, from an SBTS$(8)$, inflate each point by two and replace each block with a Latin square of order two, equivalently a 3-GDD of type $2^3$. The frequencies of the SBTS$(8)$ are repeated four times in the inflated system. That is, currently what has been constructed is a GDD-like design of type $2^8$ with frequencies zero through 27 with each frequency occurring four times. Place the blocks of the altered $(8, 3, 6)$-design onto these points.

Note that in the first part of the process, frequencies $\{0, 1, 2, 3\}$ were not obtained on all pairs $xy$. One assignment had two pairs $xz$ and $yz$ with frequencies $\{1, 1, 4, 0\}$ and $\{2, 2, 2, 0\}$ instead of the desired $\{3, 2, 1, 0\}$. Blocks of the form $x_0y_0z_0, x_0y_1z_0, x_1y_0z_0$, and $x_1y_1z_1$ were added in this case. This gives pair frequencies $\{3, 2, 4, 1\}, \{4, 2, 3, 1\}$, and $\{3, 4, 2, 1\}$ on $xz, yz$, and $xy$ (whose frequencies were already of the desired type).

To lift these points alongside the other pairs with frequencies zero through three, carefully line up points in the SBTS$(8)$ with $x, y$, and $z$ so that there is at least one block $xyz$ in the SBTS$(8)$. Remove one copy of $xyz$ before inflation. This allows us to reduce the pair frequencies by one in the altered SBTS$(8)$, making up for the extra pair frequency from the $(8, 3, 6)$-design construction. The result is an SBGDD of type $2^8$.

For $\mu > 0$, since $g = 2$ and $u = 8$, it is necessary that $\mu$ is divisible by three, by Lemma 2.4. Since $g$ is even, a $(3, \mu)$-GDD of type $2^8$ exists by Lemma 2.5. Therefore any admissible SBGDD of type $2^8$ can be constructed by lifting with the appropriate GDD of type $2^8$.

#### 4.4.4.3 SBGDD of Type $3^8$

Construction 4.10 with $m = 1$ builds a 3-SBGDD of type $3^8$ for any starting frequency $\mu > 0$.

In order to obtain starting frequency $\mu = 0$, an SBGDD of type $3^8$ with $\mu = 1$ is adjusted. The intervals of each lifted SBTS$_1(4)$ may be shifted down by one by removing two blocks as follows. Suppose the points on the original 4-block are $a, b, c, d$. When replaced with an SBTS$_1(4)$, the blocks are $abd, 2acd$, and $4bcd$ as in Figure 1.1. The pair frequencies are $\{1, 2, 3, 4, 5, 6\}$ in lexicographical order. Removing blocks $abc$ and $bcd$ gives pair frequencies $\{0, 1, 3, 2, 4, 5\}$. The result is an SBTS$_0(4)$ with the block set $\{-abc, abd, 2acd, 3bcd\}$, if negative blocks were allowed.

Except for the 4-block whose pairs cover the frequencies 1 through 6 in the SBGDD, the other 4-blocks are lifted using at lest one copy of a $(4, 3, 2)$-design in Construction 4.10. Since every possible block occurs twice in the $(4, 3, 2)$-design, after this removal action, no blocks have negative multiplicity. Currently the design has exactly one negative block. Let $N$ be this 'illegal' SBGDD of type $3^8$ with one block of multiplicity -1 and starting frequency zero.

To remove the negative block, some ad hoc block swaps are performed to interchange pair frequencies to cover the same interval. The point set used is $\{a, b, c, \ldots, x\}$. For brevity, the list of 168 distinct blocks with various multiplicities are excluded. Table 4.6 shows the pairs with altered frequency between $N$ and the final valid SBGDD of type $3^8$ with starting frequency zero. Table 4.7 lists the blocks changed from $N$ to obtain the SBGDD of type $3^8$, grouped by the $\text{SBTS}_1(4)$ used in the construction with $\mu = 1$.

| Pairs | $ad$ | $am$ | $dm$ | $at$ | $dt$ | $mt$ | $ab$ | $ak$ | $bk$ | $ap$ | $bp$ | $kp$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 12 | 13 | 14 | 15 | 16 | 17 |
| SBGDD | 0 | 1 | 2 | 4 | 5 | 3 | 12 | 14 | 13 | 15 | 17 | 16 |

| Pairs | $cd$ | $ck$ | $dk$ | $co$ | $do$ | $ko$ | $cf$ | $cn$ | $fn$ | $cr$ | $fr$ | $nr$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |
| SBGDD | 81 | 79 | 86 | 78 | 82 | 80 | 83 | 84 | 85 | 87 | 88 | 89 |

| Pairs | $gk$ | $gq$ | $kq$ | $gt$ | $kt$ | $qt$ | $hi$ | $hj$ | $ij$ | $hu$ | $iu$ | $ju$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | 204 | 205 | 206 | 207 | 208 | 209 | 210 | 211 | 212 | 213 | 214 | 215 |
| SBGDD | 205 | 204 | 206 | 207 | 215 | 208 | 210 | 209 | 212 | 211 | 214 | 213 |

Table 4.6: The altered pair frequencies, sorted by their frequency in $N$

| Block | Inflation | $\text{SBTS}_0(4)$ | Total in $N$ | SBGDD Total | Change |
|---|---|---|---|---|---|
| $akt$ | | | 0 | 3 | 3 |
| $dkt$ | | | 0 | 3 | 3 |
| $adm$ | | $-1$ | $-1$ | 0 | 1 |
| $adt$ | | 1 | 1 | 0 | $-1$ |
| $amt$ | | 2 | 2 | 1 | $-1$ |
| $dmt$ | | 3 | 3 | 2 | $-1$ |
| $abk$ | 6 | $-1$ | 5 | 4 | $-1$ |
| $abp$ | 6 | 1 | 7 | 8 | 1 |
| $akp$ | 6 | 2 | 8 | 7 | $-1$ |
| $bkp$ | 6 | 3 | 9 | 9 | |
| $cdk$ | 39 | $-1$ | 38 | 41 | 3 |
| $cdo$ | 39 | 1 | 40 | 40 | |
| $cko$ | 39 | 2 | 41 | 38 | $-3$ |
| $dko$ | 39 | 3 | 42 | 42 | |
| $cfn$ | 42 | $-1$ | 41 | 40 | $-1$ |
| $cfr$ | 42 | 1 | 43 | 43 | |
| $cnr$ | 42 | 2 | 44 | 44 | |
| $fnr$ | 42 | 3 | 45 | 45 | |
| $gkq$ | 102 | $-1$ | 101 | 101 | |
| $gkt$ | 102 | 1 | 103 | 104 | 1 |
| $gqt$ | 102 | 2 | 104 | 103 | $-1$ |
| $kqt$ | 102 | 3 | 105 | 105 | |
| $hij$ | 105 | $-1$ | 104 | 104 | |
| $hiu$ | 105 | 1 | 106 | 106 | |
| $hju$ | 105 | 2 | 107 | 105 | $-2$ |
| $iju$ | 105 | 3 | 108 | 108 | |

Table 4.7: Removing the negative block from an SBGDD of type $3^8$

#### 4.4.4.4 SBGDD of Type $4^8$

To construct an SBGDD of type $4^8$, start with a 4-MGDD of type $4 \times 8$. Fill in horizontal groups with an SBTS$(8)$. Replace blocks of size four with SBTS$_1(4)$. Sequence the 8-blocks according to Section 4.5.2. Since there are four horizontal groups, Construction 4.18 is used. Lift by Lemma 2.8, using the triple systems of order eight at the bottom of the frequency interval to cover frequency zero. The result is an SBGDD of type $4^8$.

For $\mu > 0$, the necessary conditions require $\mu$ is divisible by three. As $g = 4$ is even, a $(3, \mu)$-GDD of type $4^8$ exists by Lemma 2.5 and is used for lifting the design.

#### 4.4.4.5 SBGDD of Type $5^8$

To construct an SBGDD of type $5^8$, start with a 5-MGDD of type $5 \times 8$, first found by Abel and Assaf in [2]. Lift and sequence blocks according to Section 4.5, care must be taken to ensure the blocks of size five and eight can be sequenced together. First, the number of blocks of size five is determined.

Each point in the MGDD is paired with 28 other points, all other points excluding the four and seven points in its vertical and horizontal group. Since blocks cover four pairs with a specified point, each point is in seven blocks. Therefore there are total of 56 blocks of size five in the MGDD. As there are eight groups of size five, 64 copies of an SBTS$(5)$ are used as ingredients. Note that $64 \equiv 1 \pmod 3$ so Section 4.5.1 is used to sequence the blocks at the start of the interval. The sequenced SBTS$(5)$ cover pair frequencies from zero to $639$.

Sequence and lift the five $8$-blocks to have starting frequency 640, as outlined in Section 4.5.2. Note that since starting frequency $640 \equiv 4 \pmod 6$ is required and there are $5 \equiv 2 \pmod 3$ $8$-blocks to be sequenced, Construction 4.19 may be used. The result is an SBGDD of type $5^8$.

For $\mu > 0$, to satisfy Lemma 2.4 it is necessary for $\mu \equiv 0 \pmod 3$. When $\mu$ is divisible by six, the entire SBGDD may be lifted using a $(3, \mu)$-GDD of type $5^8$, whose existence is given by Lemma 2.5. Taking $m = 1$ in Construction 4.11 gives an alternate method for finding an SBGDD of type $5^8$ for any starting frequency $\mu$ divisible by three.

## 4.5 Lifting Subdesigns with $k \equiv 2 \pmod 3$ Points

What follows are lifting variants used in place of those described in Section 2.3.1 when $k \equiv 2 \pmod 3$ and designs with every starting frequency do not exist. In particular, the cases when $k = 5$ or $8$ are explored, as block sizes five and eight are used to construct SBGDDs in Chapter 4.

### 4.5.1 Five Points

In order to lift an SBTS$(5)$, the blocks of a $(5, 3, 3)$-design may be used for starting frequencies $\mu \equiv 0 \pmod 3$.

**Example 4.5.1.** Take every subset of three points from a set of five points as the blocks of the design. That is, if the points are $\{a, b, c, d, e\}$, then the blocks are $abc$, $abd$, $abe$, $acd$, $ace$, and $ade$. One can simply check that every pair of points is covered three times. Alternatively, for any pair $xy$, notice that there are exactly three points left which can complete the block. Therefore, if every block is taken once, the pair frequency is three.

Lifting an SBTS$_0(5)$ with $w$ copies of a $(5, 3, 3)$-design gives the pair frequencies from $3w$ to $3w + 9$ for any nonnegative integer $w$. However, as there is no design on five points with index one or two, nor an SBTS$_1(5)$, intervals starting at $3w + 1$ and $3w + 2$ are not attainable in this way. Lemma 4.3 shows that only one of these congruence classes is needed.

**Lemma 4.3.** The number of blocks of size five in a PBD$(v, \{3, 4, 5, 6\})$ is congruent to zero or one modulo three.

*Proof.* Let $b_i$ be the number of blocks of size $i$ where $i \in \{3,4,5,6\}$. The total number of pairs covered in a PBD$(v, \{3,4,5,6\})$ is

$$\begin{aligned}\binom{v}{2} &= \binom{3}{2}b_3 + \binom{4}{2}b_4 + \binom{5}{2}b_5 + \binom{6}{2}b_6 \\ &= 3b_3 + 6b_4 + 10b_5 + 15b_6 \\ &\equiv b_5 \pmod 3.\end{aligned}$$

If $v(v-1)$ is not a multiple of three, then $v \equiv 2 \pmod 3$ and $\frac{1}{2}v(v-1) \equiv 1 \pmod 3$. Therefore $\binom{v}{2} \equiv 0$ or $1 \pmod 3$ and $b_5 \equiv 0$ or $1 \pmod 3$. □

Instead of covering intervals of ten pairs of points with each block of size five, larger intervals of pair frequencies are covered by adjusting the frequencies in three or four 5-blocks to have interleaving frequencies. This is called *sequencing* the blocks of size five. As a result, as long as an admissible starting frequency is used for the start of the larger interval, the need for an SBTS$_1(5)$ and an SBTS$_2(5)$ can be avoided. The following construction allows for both starting frequencies zero and one. See Figure 4.11 for an example of Construction 4.17 with three blocks of size five.

**Construction 4.17.** The first of three $5$-blocks is replaced by an SBTS$(5)$, covering frequencies zero through nine. Let the second $5$-block be $fghij$. Replace this $5$-block so that $ij$ has frequency nine in an SBTS$_0(5)$. Add the block set $H = \{fgh, fij, gij, hij\}$ which covers $ij$ three times and every other pair of points exactly once. Figure 4.10 depicts $H$ with the pair $ij$ covered three times highlighted in blue and the single block not containing $ij$ is depicted using dashed lines. Taking $H$ together with the SBTS$_0(5)$, the pair frequencies covered are one through nine, and twelve. Lift using three copies of a $(5,3,3)$-design to obtain pair frequencies ten through $18$, and $21$.

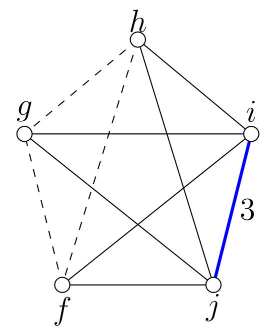



Figure 4.10: The blocks of $H$

Call the third 5-block $klmno$. Replace this block with nine copies of $H$ together with the blocks of an SBTS$(5)$, permuting the copies of $H$ so that every pair of points, except one, has frequency three in a copy of $H$. Let $km$ be the pair excluded from having frequency three in a copy of $H$. Align the SBTS$_0(5)$ so that $km$ has frequency one from the SBTS. Then $km$ has total pair frequency ten. The other pairs have frequency $3+8\cdot 1+x = 11+x$ where $x$ is its frequency in the SBTS$_0(5)$. That is, $x = 0, 2, 3, \ldots, 9$. Currently, $klmno$ covers pair frequencies $10, 11$, and $13$ through $20$. Lift with three copies of a $(5,3,3)$-design to obtain pair frequencies $19, 20$, and $22$ through $29$.

Between these three $5$-blocks, frequencies zero through $29$ have been covered. Adding $w$ copies of $(5,3,3)$-design gives all frequency intervals of the form $\{3w, 3w+1, \ldots, 3w+29\}$.

Repeat this process for all unused $5$-blocks until none are left, lifting with seven additional copies of the $(5,3,3)$-design to each subsequent set of three $5$-blocks so the intervals do not overlap.

To cover $b_5 \equiv 1 \pmod 3$ blocks of size five for $b_5 \geq 4$, lift three blocks at a time as outlined above. The final block of size five may be appended to the end of the set of three points and their corresponding pair frequencies, taking the place of the first block in the next set of three blocks and stopping there.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $B_1$ Pairs | $ab$ | $ac$ | $ad$ | $ae$ | $bc$ | $bd$ | $be$ | $cd$ | $ce$ | $de$ |
| SBTS$(5)$ | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_1$ Total | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_2$ Pairs | $fg$ | $fh$ | $fi$ | $fj$ | $gh$ | $gi$ | $gj$ | $hi$ | $hj$ | $ij$ |
| $H$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 |
| $(5,3,3)$-designs | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| SBTS$(5)$ | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_2$ Total | 10 | 11 | 13 | 12 | 15 | 14 | 17 | 16 | 18 | **21** |
| $B_3$ Pairs | $k\ell$ | $km$ | $kn$ | $ko$ | $\ell m$ | $\ell n$ | $\ell o$ | $mn$ | $mo$ | $no$ |
| Copies of $H$ | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 |
| $(5,3,3)$-designs | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| SBTS$(5)$ | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_3$ Total | **20** | **19** | 23 | 22 | 25 | 24 | 27 | 26 | 28 | 29 |

Figure 4.11: Three SBTS$(5)$ sequenced together

Since intervals with starting frequencies $\mu \equiv 2 \pmod 3$ have not been constructed, in designs with multiple block sizes the sequenced subdesigns on five points are used to cover starting frequency zero whenever possible. Moreover, placing the 5-block interval on these low pair frequencies is beneficial to avoid the need for the non-existent SBTS$_0(4)$ when present. As SBTS$_\mu(k)$ exist for any $\mu \geq 0$ when $k = 3, 6$, this decision cooperates with the restrictions on the rest of the block sizes used in most of the constructions. When block size eight is used in construction of SBGDDs of type $g^8$, these block sizes $k \equiv 2 \pmod 3$ clash, as is outlined in the next section. In this case, extra care is taken to ensure the pair frequency intervals are admissible for both block sizes.

### 4.5.2 Eight Points

This section describes how to lift Sarvate-Beam triple systems on eight points which is used in Sections 4.2 and 4.3.2. By Lemma 2.4, as $8 \equiv 2 \pmod 3$ there is no SBTS$_1(8)$. An $(8, 4, 3)$-design [6] together with Lemma 2.8 may be used to lift pair frequencies. A $(4, 3, 2)$-design can be used to split the $4$-blocks into triples, creating an $(8, 3, 6)$-design. Therefore SBTS$(8)$ can be uniformly lifted to have starting frequency $\mu = 6\ell$ for any nonnegative integer $\ell$. As there are $28$ pairs to cover between eight points, an SBTS$_\mu(8)$ covers the interval from $\mu$ to $\mu + 27$, which unfortunately requires the next SBTS$_\eta(8)$ to have starting frequency $\mu + 4 \equiv 4 \pmod 6$. As was done with $5$-blocks in Section 4.5.1, the pair frequencies are interleaved to cover larger intervals.

Consider the following design, called $L$. Every pair of points is covered twice, except one pair $ab$ which is covered six times. Similarly to Section 4.5.1, add $L$ to an SBTS$(8)$ to get starting frequency two which is used in Construction 4.18.

$$\begin{array}{cccccccccc} abc & abd & abe & abf & abg & abh & acd & aef & agh & bch \\ bde & bfg & cdf & ceg & ceh & cfg & deg & dfh & dgh & efh \end{array}$$

Figure 4.12: The blocks of $L$, with pair $ab$ of frequency six

Construction 4.18 sequences $8$-blocks three or four blocks at a time. Figure 4.13 shows the pair frequencies of three $8$-blocks described by the construction. The interleaved pair frequencies from different blocks appear in bold.

**Construction 4.18.** Replace the first $8$-block with an SBTS$(8)$ which covers pair frequencies from zero to $27$. To the next $8$-block, lift using four copies of an $(8, 3, 6)$-design so that every pair of points is covered 24 times. Replace the $8$-block itself with an SBTS$(8)$, giving starting frequency $24$. Suppose the pairs of frequency $24$ and $25$ in the SBTS$(8)$ are $ef$ and $fh$ respectively, as in Figure 4.7. Align two copies of $L$ so that $ef$ and $fh$ are the pairs of frequency six. Then $ef$ and $fh$ have frequencies $24 + 24 + 2 + 6 = 56$ and $24 + 25 + 6 + 2 = 57$. The rest of the pairs have frequencies $24 + x + 2 + 2 = 28 + x$ with $x \in \{0, 1, 2, \ldots, 23, 26, 27\}$. Pair frequencies $28$ through $51$, and$54, 55, 56$, and $57$ have therefore been covered. The next $8$-block is used to cover the frequencies $52$ and $53$.

Replace the third $8$-block with an SBTS$(8)$. Suppose the pairs of frequency zero and one are $ab$ and $ac$ respectively, as in Figure 4.7. Put 26 copies of $L$ on the eight points, two copies less than the number of pairs available, similarly to the third $5$-block in Construction 4.17. Place the copies so that every pair except for $ab$ and $ac$ take the role of the pair of frequency six in $L$. Then $ab$ and $ac$ have frequencies $0+26 \times 2 = 52$ and $1+26 \times 2 = 53$, filling in the gap of the second $8$-block. The remaining pairs have frequency $x+25\times 2+6 = 56 + x$ where $x = 2, 3, \ldots, 27$. Pair frequencies $52, 53$, and $58$ through $83$ are therefore covered.

As $84 \equiv 0 \pmod 6$, a fourth block may simply be lifted with 14 copies of a $(8, 3, 6)$-design. The same is true to sequence three more $8$-blocks in the same way the first three blocks were sequenced. If $b_8$ is the number of $8$-blocks in a design, the above method sequences the $8$-blocks as long as $b_8 \equiv 0, 1 \pmod 3$.

| Pairs | $ab$ | $ac$ | $ad$ | $ae$ | $af$ | $ag$ | $ah$ | $bc$ | $\dots$ | $dh$ | $ef$ | $eg$ | $eh$ | $fg$ | $fh$ | $gh$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SBTS$(8)$ | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 6 | $\dots$ | 22 | 24 | 23 | 27 | 21 | 25 | 26 |
| Total | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 6 | $\dots$ | 22 | 24 | 23 | 27 | 21 | 25 | 26 |
| $L$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | $\dots$ | 2 | **6** | 2 | 2 | 2 | 2 | 2 |
| $L$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | $\dots$ | 2 | 2 | 2 | 2 | 2 | **6** | 2 |
| Inflate | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | $\dots$ | 24 | 24 | 24 | 24 | 24 | 24 | 24 |
| SBTS$(8)$ | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 6 | $\dots$ | 22 | 24 | 23 | 27 | 21 | 25 | 26 |
| Total | 28 | 29 | 30 | 31 | 32 | 33 | 35 | 34 | $\dots$ | 50 | **56** | 51 | **55** | 49 | **57** | **54** |
| $26 \times L$ | **52** | **52** | 56 | 56 | 56 | 56 | 56 | 56 | $\dots$ | 56 | 56 | 56 | 56 | 56 | 56 | 56 |
| SBTS$(8)$ | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 6 | $\dots$ | 22 | 24 | 23 | 27 | 21 | 25 | 26 |
| Total | **52** | **53** | 58 | 59 | 60 | 61 | 63 | 62 | $\dots$ | 78 | 80 | 79 | 83 | 77 | 81 | 82 |

Figure 4.13: Three $8$-blocks sequenced together

Note that Construction 4.18 can be altered to any starting frequency $\mu \equiv 0 \pmod 6$ lifting with additional copies of an $(8, 3, 6)$-design on every block. For any case where the underlying design has no $5$-blocks, the $8$-blocks are sequenced first thereby ensuring the admissible starting frequency zero is available. When $5$-blocks are present, as every $5$-block covers ten pairs of points and any such design has $b_5 \equiv 0, 1 \pmod 3$ blocks of size five (see Section 4.5.1), the next uncovered frequency after $5$-blocks are sequenced is therefore $10b_5 - 1 \equiv b_5 + 2 \pmod 3 \equiv 0, 2 \pmod 3$ where $b_5$ is the number of $5$-blocks in the design. Therefore constructions to obtain starting frequencies $\mu \equiv 3 \pmod 6$ and $\mu \equiv 2 \pmod 3$ are of interest so that sequenced $8$-blocks can always be lifted to cover pair frequencies after the sequenced $5$-blocks.

Construction 4.19 alters Construction 4.18 to work with $b_8 \equiv 2 \pmod 3$.

**Construction 4.19.** In Construction 4.18, exclude the first block of size eight, whose pair frequencies are not interleaved with the subsequent $8$-blocks in the sequencing. Then the first two blocks are sequenced in the same way the second two blocks in Construction 4.18 are sequenced. Subsequent sets of three sequenced $8$-blocks are lifted to cover pair frequencies after the first two blocks. Since $b_8 \equiv 2 \pmod 3$, the remaining blocks may all be placed into sets of three sequenced blocks to be lifted in this way. Note that since 26 copies of $L$ are used to construct the interleaved sets of three $8$-blocks, the minimum pair frequency for the second $8$-block is $52$. Therefore 28 is the minimum starting frequency for this sequencing method.

Happily, only Construction 4.4.4.5 uses both $5$-blocks and $8$-blocks and requires $b_8 \equiv 2 \pmod 3$. Since the $8$-blocks in this construction require starting frequency 640, the remaining cases are left unsolved.

## 4.6 Higher Starting Frequencies

Lemma 2.8 may often be used to lift an entire SBGDD of type $g^u$ with starting frequency zero to obtain $\mu > 0$.

**Lemma 4.4.** If there is both a $(3, \lambda)$-GDD of type $g^u$ and a 3-SBGDD of type $g^u$ with starting frequency zero, then there is a 3-SBGDD of type $g^u$ with starting frequency $\mu$, for any positive integer $\mu$ that is a multiple of $\lambda$.

Construction 4.1 is altered to give SBGDDs with starting frequency $\mu > 0$ when the GDDs required to use Lemma 4.4 do not exist.

**Construction 4.20.** Start with a PBD$(u, \{3, 4, 5, 6\})$, and inflate points by $g \geq 2$. As an SBTS$_\mu(k)$ exists with any starting frequency $\mu \geq 1$ for $k = 3, 4$, or $6$, any PBD without blocks of size five may be lifted using Lemma 2.8 to give an SBGDD of type $g^u$ with starting frequency $\mu > 0$. Construction 4.17 allows us to lift and sequence the blocks for starting frequencies $\mu \equiv 0 \pmod{3}$. When this is not the case, lifting the SBTS$(5)$ requires further investigation. Once the subdesigns replacing the $5$-blocks are sequenced and lifted, remaining blocks can be lifted by applying Lemma 2.8 to complete the construction.

For starting frequency $\mu \equiv 1 \pmod{3}$, replace the $5$-blocks first as follows. Weight the points on the first and second $5$-blocks with one and nine copies of $H$, as in Construction 4.17, and $\mu - 1$ copies of a $(5, 3, 3)$-design. This covers frequencies from $\mu$ to $\mu + 19$. Then use Construction 4.17 directly to lift the remaining subdesigns on five points, three at a time, beginning with frequency $\mu + 20 \equiv 0 \pmod{3}$. As there are $b_5 - 2$ remaining 5-blocks to sequence, $b_5 - 2 \equiv 0, 1 \pmod{3}$ is required. Instead when $b_5 - 2 \equiv 2 \pmod{3}$, the proof of Lemma 4.3 tells us that $u \equiv 2 \pmod{3}$. Since $\mu \equiv 1 \pmod{3}$, by Lemma 2.4 any admissible SBGDD must have $g \equiv 0 \pmod{3}$. When this is the case, number of 5-blocks after inflation by $g$ is considered, given by the $5$-GDDs used to replace the original 5-blocks. Since there are $gb_5$ blocks of size five after the original blocks are replaced with GDDs, and since $gb_5 \equiv 0 \pmod{3}$, the same construction may be applied to sequence the 5-blocks.

For starting frequencies $\mu \equiv 2 \pmod{3}$, define a set $T$ of blocks, with one pair of frequency zero and the rest of the pair frequencies equal to two. For example, on the point set $\{a, b, c, d, e\}$, take $T = \{abd, abe, acd, ace, bcd, bce\}$, where $de$ is the pair with frequency zero. What follows describes how to interleave the SBTS$(5)$ block sets with starting frequency two using $T$ and the block set $H$ from Section 4.5.1, that has one pair of frequency three and the rest of frequency one. An example of this construction is given in Figure 4.14.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $B_1$ Pairs | $ab$ | $ac$ | $ad$ | $ae$ | $bc$ | $bd$ | $be$ | $cd$ | $ce$ | $de$ |
| $H$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | **3** |
| $H$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | **3** | 1 |
| SBTS$(5)$ | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_1$ Total | 2 | 3 | 5 | 4 | 7 | 6 | 9 | 8 | **12** | **13** |
| $B_2$ Pairs | $fg$ | $fh$ | $fi$ | $fj$ | $gh$ | $gi$ | $gj$ | $hi$ | $hj$ | $ij$ |
| $T$ | **0** | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $T$ | 2 | **0** | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $H$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | **3** |
| $H$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | **3** | 1 |
| $(5,3,3)$-designs | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| SBTS$(5)$ | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_2$ Total | **10** | **11** | 15 | 14 | 17 | 16 | 19 | 18 | **22** | **23** |
| $B_3$ Pairs | $k\ell$ | $km$ | $kn$ | $ko$ | $\ell m$ | $\ell n$ | $\ell o$ | $mn$ | $mo$ | $no$ |
| $T$ | **0** | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $T$ | 2 | **0** | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $(5,3,3)$-designs | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 18 |
| SBTS$(5)$ | 0 | 1 | 3 | 2 | 5 | 4 | 7 | 6 | 8 | 9 |
| $B_3$ Total | **20** | **21** | 25 | 24 | 27 | 26 | 29 | 28 | 30 | 31 |

Figure 4.14: Three sequenced SBTS$(5)$ with $\mu = 2$

Put two copies of $H$ so that the pairs of frequency three are given frequency eight and nine in the SBTS$(5)$. This covers frequencies two through nine, $12$, and $13$. Add two copies of $H$ to the next $5$-block similarly. Put two copies of $T$ on this block so that the pairs of frequency zero and one in the SBTS are given frequency zero in $T$. Finally lift using two copies of a $(5,3,3)$-design to cover the pair frequencies $10$ through $19, 22$, and $23$. The third block is also given two copies of $T$ so that the pairs of frequency zero have total frequency zero and one in the SBTS$(5)$. Lift using six copies of a $(5, 3, 3)$-design to cover frequencies $20, 21,$ and $24$ through $31$. Altogether the three $5$-block cover the interval of frequencies two through $31$.

The next set of three $5$-blocks is sequenced in a similar fashion, and lifted using ten additional copies of a $(5, 3, 3)$-design to obtain starting frequency 32, and so on for additional $5$-blocks sequenced three at a time. For $\mu > 2$, simply lift the above construction entirely with $(5, 3, 3)$-designs to give the desired $\mu \equiv 2 \pmod 3$.

Notice that so far this construction requires $b_5 \equiv 0 \pmod 3$. Like the previous case, if $b_5 \equiv 1 \pmod 3$, then $u \equiv 2 \pmod 3$. Since $\mu \equiv 2 \pmod 3$ also, the necessary conditions ensure that $g \equiv 0 \pmod 3$. The $gb_5$ blocks of size five are considered after replacement by the $5$-GDDs in Construction 4.1 and the sequencing construction is applied similarly.

The possible exceptions to this construction are those of group type that were possible exceptions to the original construction for $\mu = 0$. The SBGDDs of type $g^u$ with $\mu > 0$ that are not built here were constructed along side their $\mu = 0$ relatives in Sections 4.3.1 and 4.4.

# Chapter 5

# Extension Using Matrices

In the search for Sarvate-Beam Cubes, the question of interest in terms of graph decompositions is the following: Is it possible to assign multiplicities to the edges of $K_{n,n,n}$ so that the resulting multigraph has a triangle decomposition? When these edge multiplicities are distinct consecutive integers, the result is a Sarvate-Beam cube of order $n$. In the search for such assignments, the following related question naturally surfaces. Given a multigraph of $K_{n,n,n}$ with specified edge multiplicities, when can the edges be partitioned into triangles?

The fractional decomposition of $K_{n,n,n}$, where non-integral edge multiplicities are admissible, was explored by Bowditch and Dukes [4] where a bound on the minimum degree of the vertices is used to guarantee existence. This chapter addresses this question in the integral setting, with the relaxation of allowing for negative integer edge multiplicities. Some discussion of the positive integer setting follows in Section 5.4. A special kind of incidence matrix is explored in Section 5.1 which applies directly to the graph of $K_{3n}$. Adjustments are subsequently made to suit the Sarvate-Beam cube problem in Section 5.2.

## 5.1 Subset Inclusion Matrices

Consider the following special incidence matrix of a set of $v$ points, $V$. Let the rows be indexed by all the $t$-subsets of $V$ and the columns by all the $k$-subsets of $V$. Call this the *inclusion matrix* $W_{tk}(v)$. Let $w_{ij} = 1$ when the $i^{th}$ $t$-subset of $V$ is contained in the $j^{th}$ $k$-subset of $V$, and zero otherwise. Then $W_{tk}(v)$ is an $\binom{v}{t} \times \binom{v}{k}$ matrix with entries equal to zero or one. Wilson [29] proved the following theorem.

**Theorem 5.1.** Let $t \leq k \leq v - t$. A column vector $\vec{x}$ of height $\binom{v}{t}$ belongs to the column space of $W_{tk}$ over the integers if and only if

$$\frac{1}{\binom{k-i}{t-i}} W_{it}\vec{x}$$

is integral for all $i = 0, 1, \ldots, t$.

In the search for triple systems, the matrix $W_{23}(v)$ is of interest, which tracks the pairs that are covered in 3-subsets for a set of $v$ points. When $t = 2$ and $k = 3$, the properties given by Theorem 5.1 for each value of $i$ are explicitly listed below. Recall that a graph can be viewed as a design where edges are pairs and points are contained in a block together if they appear in a subgraph that is a clique. The ability to decompose $G$ into triangles is the same as decomposing a specified set of pair frequencies into 3-blocks.

From the theorem, $\frac{1}{3}W_{02}\vec{x}$, $\frac{1}{2}W_{12}\vec{x}$, and $W_{22}\vec{x}$ are required to be integral.

- When $i = 0, W_{02}$ tracks which sets of size two contain the empty set - all of them. Therefore $W_{02} = J_{1\times\binom{v}{2}}$, an all-ones row vector. As $W_{02}\vec{x}$ gives the sum over all pair frequencies, requiring $\frac{1}{3}W_{02}\vec{x}$ to be integral means that the sum is a multiple of three. This corresponds to the ability to separate the pair frequencies into blocks of size three. Equivalently, the edge multiplicities of $K_v$ sum to three so that it is possible to separate the edges into triangles.

- When $i = 1$, the rows of $W_{12}$ each correspond to a point. When multiplied by $\vec{x}$, the result is the sum of the pair frequencies incident to that point. As a specified point covers two pairs in each 3-block, $\frac{1}{2}W_{12}\vec{x}$ being integral ensures the pairs incident to the points can be split up into 3-blocks. Equivalently, since each vertex has degree two in a triangle, this condition asserts that each vertex of the multigraph of $K_v$ has even degree.

- When $i = 2$, as $W_{22}$ is the identity matrix, this condition ensures the entries of $\vec{x}$ are integral, giving integral block/ triangle multiplicities.

Given a vector $\vec{p}$ containing the pair frequencies in some order, the goal is to solve the equation $W_{23}\vec{t} = \vec{p}$ for $\vec{t}$ which gives the multiplicities of the 3-subsets, listed in $\vec{t}$. The vector $\vec{p}$ must adhere to each of the above three properties given by Theorem 5.1.

### 5.1.1 An Example on Five Points

The inclusion matrix of $W_{23}(5)$ is constructed and analysed for a generic multigraph $G$ on five points. When the corresponding graph $G$ is decomposable into triangles is explored. A few basic necessary conditions are given below.

**Lemma 5.2.** If a multigraph on five vertices is decomposable into triangles then;

1. each vertex has even degree,

2. the sum of the edges is divisible by three, and

3. any partition of the vertices into two subsets, one of size two and one of size three, has at most two times the number of edges crossing between subgraphs as it does contained in either subgraph.

*Proof.* The first two properties follow from Theorem 5.1 and that a triangle contains three edges and each vertex has degree two. Figure 5.1 depicts what the third property looks like with edges contained in a partition coloured orange and blue and crossing edges as dashed lines. Triangles that are not contained in the partition on three points contain two crossing edges and one edge internal to a partition. Therefore, if no triangles are contained within a set in the partition, the number of crossing edges is exactly double the number of internal edges. Otherwise the number of crossing pairs is at most the number of edges internal to either subgraph. □

Figure 5.1 also gives two examples of graphs which pass conditions 1 and 2 but are clearly not triangle decomposable.

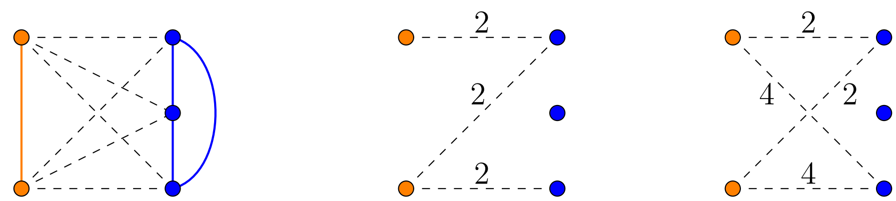



Figure 5.1: The Condition 3 partition and some defiant graphs

By investigating its pairs to triangles inclusion matrix, $W_{23}(5)$, it is shown that the conditions given in Lemma 5.2 are also sufficient for nonnegative triangle decomposability. The explicit matrix $W_{23}(5)$ and its inverse are given in Figure 5.2.

The system $W_{23}(5)\vec{t} = \vec{p}$ therefore has the following unique solution $\vec{t}$, where $\vec{p} = (p_1, p_2, \ldots, p_{10})^\top$.

$$\vec{t} = \frac{1}{6}(2p_1 + 2p_2 - p_3 - p_4 + 2p_5 - p_6 - p_7 - p_8 + 2p_9 - p_{10},$$
$$2p_1 - p_2 + 2p_3 - p_4 - p_5 + 2p_6 - p_7 - p_8 - p_9 + 2p_{10},$$
$$2p_1 - p_2 - p_3 + 2p_4 - p_5 - p_6 + 2p_7 + 2p_8 - p_9 - p_{10},$$
$$-p_1 + 2p_2 + 2p_3 - p_4 - p_5 - p_6 + 2p_7 + 2p_8 - p_9 - p_{10},$$
$$-p_1 + 2p_2 - p_3 + 2p_4 - p_5 + 2p_6 - p_7 - p_8 - p_9 + 2p_{10},$$
$$-p_1 - p_2 + 2p_3 + 2p_4 + 2p_5 - p_6 - p_7 - p_8 + 2p_9 - p_{10},$$
$$-p_1 - p_2 - p_3 + 2p_4 + 2p_5 + 2p_6 - p_7 + 2p_8 - p_9 - p_{10},$$
$$-p_1 - p_2 + 2p_3 - p_4 + 2p_5 - p_6 + 2p_7 - p_8 - p_9 + 2p_{10},$$
$$-p_1 + 2p_2 - p_3 - p_4 - p_5 + 2p_6 + 2p_7 - p_8 + 2p_9 - p_{10},$$
$$2p_1 - p_2 - p_3 - p_4 - p_5 - p_6 - p_7 + 2p_8 + 2p_9 + 2p_{10})^\top$$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&0\\ 1&0&0&1&1&0&0&0&0&0\\ 0&1&0&1&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&0\\ 1&0&0&0&0&0&1&1&0&0\\ 0&1&0&0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0&1&1&0\\ 0&0&0&1&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0&1&1\\ 0&0&0&0&1&0&0&1&0&1 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 2&2&-1&-1&2&-1&-1&-1&2&-1\\ 2&-1&2&-1&-1&2&-1&-1&-1&2\\ 2&-1&-1&2&-1&-1&2&2&-1&-1\\ -1&2&2&-1&-1&-1&2&2&-1&-1\\ -1&2&-1&2&-1&2&-1&-1&-1&2\\ -1&-1&2&2&2&-1&-1&-1&2&-1\\ -1&-1&-1&2&2&2&-1&2&-1&-1\\ -1&-1&2&-1&2&-1&2&-1&-1&2\\ -1&2&-1&-1&-1&2&2&-1&2&-1\\ 2&-1&-1&-1&-1&-1&-1&2&2&2 \end{bmatrix}$$

Figure 5.2: $W_{23}(5)$ and its inverse

The requirement that each of these entries is nonnegative corresponds to Condition 3 from Lemma 5.2, the crossing pairs condition. For example, in Figure 5.1 label the orange edge contained in the 2-set in the partition as $p_1$ and the blue edges contained in the 3-set as $p_7, p_8, p_9$. Then the crossing pairs condition with the described partition corresponds to the last entry of $\vec{t}$, that $2p_1 - p_2 - p_3 - p_4 - p_5 - p_6 - p_7 + 2p_8 + 2p_9 + 2p_{10} \geq 0$ in the solution.

Checking these constraints in $\vec{t}$ indicates when the block multiplicities, and therefore the pair frequencies in $\vec{p}$, define a multigraph which is triangle decomposable. For example, taking $\vec{p} = (0, 1, 3, 2, 5, 4, 7, 6, 9, 8)^\top$ gives the solution $\vec{t} = (0, 0, 0, 1, 0, 2, 1, 4, 3, 4)^\top$ which satisfies the solution constraints. The entries in $\vec{t}$ correspond to the block multiplicities of the SBTS$(5)$ as given in Example 2.3.4, $\{acd, 2ade, bcd, 4bce, 3bde, 4cde\}$.

Returning to the general case with any number of points, note that Conditions 1 and 2 in Lemma 5.2 are necessary for any graph to have a triangle decomposition and so the following lemma is presented.

**Lemma 5.3.** If a multigraph has a triangle decomposition then;

1. each vertex has even degree and

2. the sum of the edges is divisible by three.

## 5.2 The Inclusion Matrix of a Cube

The focus returns to the specific case of Sarvate-Beam cubes. The underlying graph for an SBC$(n)$ is $K_{n,n,n}$. The corresponding pairs to triangles inclusion matrix is a $3n^2 \times n^3$ matrix denoted $\mathcal{U}$ in this dissertation. Note that $\mathcal{U} \neq W_{23}$ as pairs within a group and triangles with pairs internal to a group are excluded from the matrix. To consider whether a list of pair frequencies $\vec{p}$ gives a Sarvate-Beam Cube, the goal is to solve the equation $\mathcal{U}\vec{t} = \vec{p}$ to find the triangle multiplicities given in $\vec{t}$. To investigate the possible solutions, the rank and nullity of $\mathcal{U}$ is determined. To do so, the Kronecker product notation is introduced.

#### 5.2.0.1 Kronecker Product

Let $A$ be an $m \times n$ matrix and $B$ be an $q \times r$ matrix. The *Kronecker product* of $A$ and $B$ is the $mq \times nr$ matrix given (in block form) by

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

Let $J_{m\times n}$ to be the $m \times n$ matrix containing all ones. Denote an $m \times n$ diagonal matrix with diagonal entries equal to one by $I_{m\times n}$. To serve as examples, the Kronecker product of $I_n$ and $J_{m\times n}$ are given with generic matrices $A$ and $B$ of any size, which is useful in what follows.

$$I_n \otimes B = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & B \end{bmatrix} \text{ and } J_{1\times n} \otimes A = \begin{bmatrix} A & A & \dots & A \end{bmatrix}.$$

### 5.2.1 Rank of the Inclusion Matrix

The rows of $\mathcal{U}$, indexed by pairs, are ordered into three row blocks according to the groups of the Sarvate-Beam cube. Each block of rows contains $n^2$ rows corresponding to all possible pairs between two groups. Splitting the row blocks into $n$ column blocks, containing $n^2$ columns, gives blocks corresponding to all triangles containing a specified point in the first group. Call these blocks of $n^2$ rows $R_1, R_2, R_3$ and denote the $n^2$ column blocks by $C_1, C_2, \ldots, C_n$. This splits $\mathcal{U}$ into $3n^2$ submatrices, each of size $n^2 \times n^2$. Let the $n^2 \times n^2$ submatrix in the row block $R_i$ and column block $C_j$ be denoted by $S_{ij}$. Let $R_{ik}$ denote the $k^{th}$ row in $R_i$, and similarly $C_{j\ell}$ denotes the $\ell^{th}$ column in block $C_j$.

Since each pair appears in $n$ triangles, every row contains $n$ nonzero entries. Ordering the columns by their incidence to the points in the first group gives the following pattern where points are labelled one through $n$ and ordered lexicographically in the rows. An example of $\mathcal{U}$ with $n = 3$ is also given in Figure 5.3.

$$\mathcal{U} = \left[\begin{array}{cccc} I_n \otimes J_{1\times n} & 0 & \ldots & 0 \\ 0 & I_n \otimes J_{1\times n} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_n \otimes J_{1\times n} \\ \hline J_{1\times n} \otimes I_n & 0 & \ldots & 0 \\ 0 & J_{1\times n} \otimes I_n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_{1\times n} \otimes I_n \\ \hline I_{n^2} & I_{n^2} & \ldots & I_{n^2} \end{array}\right] = \left[\begin{array}{c} I_{n^2} \otimes J_{1\times n} \\ \hline I_n \otimes (J_{1\times n} \otimes I_n) \\ \hline J_{1\times n} \otimes I_{n^2} \end{array}\right]$$

| | | |
|---|---|---|
| 111000000 | 000000000 | 000000000 |
| 000111000 | 000000000 | 000000000 |
| 000000111 | 000000000 | 000000000 |
| 000000000 | 111000000 | 000000000 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 000000000 | 000000000 | 000000111 |
| $I_3$ $I_3$ $I_3$ | 0 0 0 | 0 0 0 |
| 0 0 0 | $I_3$ $I_3$ $I_3$ | 0 0 0 |
| 0 0 0 | 0 0 0 | $I_3$ $I_3$ $I_3$ |
| $I_9$ | $I_9$ | $I_9$ |

Figure 5.3: The matrix $\mathcal{U}$ when $n = 3$

Row reduction can be used to show that $\text{rank}(\mathcal{U}) = 3n^2 - 3n + 1$. Therefore $\text{nullity}(\mathcal{U}) = n^3 - 3n^2 + 3n - 1 = (n-1)^3$. Given a solution $\vec{t}$ to $\mathcal{U}\vec{t} = \vec{p}$, the null space helps us understand what other solution vectors look like in the search for nonnegative integer solutions. Define a matrix $K$ to be the following $n \times n$ matrix, which appears numerous times in the process of row reducing $\mathcal{U}$.

$$K = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & \dots & & 0 & 1 \end{bmatrix}$$

**Algorithm 5.1.** To obtain the reduced row echelon form for $\mathcal{U}$, swap the rows in block $R_1$ with the rows in block $R_3$, as indicated below.

$$\mathcal{U} \sim \left[\begin{array}{cccc} I_{n^2} & I_{n^2} & \dots & I_{n^2} \\ \hline J_{1\times n}\otimes I_n & 0 & \dots & 0 \\ 0 & J_{1\times n}\otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{1\times n}\otimes I_n \\ \hline I_n\otimes J_{1\times n} & 0 & \dots & 0 \\ 0 & I_n\otimes J_{1\times n} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n\otimes J_{1\times n} \end{array}\right] = \left[\begin{array}{c} J_{1\times n}\otimes I_{n^2} \\ \hline I_n\otimes (J_{1\times n}\otimes I_n) \\ \hline I_{n^2}\otimes J_{1\times n} \end{array}\right]$$

For each column block $C_j$, perform the $n^2$ column operations which give $C_j - C_1$, for $j = 2, 3, \dots, n$.

$$\mathcal{U} \sim \left[\begin{array}{cccc} I_{n^2} & 0 & \dots & 0 \\ \hline J_{1\times n}\otimes I_n & -J_{1\times n}\otimes I_n & \dots & -J_{1\times n}\otimes I_n \\ 0 & J_{1\times n}\otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{1\times n}\otimes I_n \\ \hline I_n\otimes J_{1\times n} & -I_n\otimes J_{1\times n} & \dots & -I_n\otimes J_{1\times n} \\ 0 & I_n\otimes J_{1\times n} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n\otimes J_{1\times n} \end{array}\right] = \left[\begin{array}{c} I_{1\times n}\otimes I_{n^2} \\ \hline K\otimes (J_{1\times n}\otimes I_n) \\ \hline K\otimes J_{1\times n} \end{array}\right]$$

From the row blocks $R_2$ and $R_3$, remove the negative entries by adding rows 2 through $n^2$ to row 1, in both $R_2$ and $R_3$. That is, operations $R_{2k} - R_{21}$ and $R_{3k} - R_{31}$, for all $k = 2, \dots, n^2$.

$$\mathcal{U} \sim \left[\begin{array}{cccc} I_{n^2} & 0 & \dots & 0 \\ \hline J_{1\times n} \otimes I_n & 0 & \dots & 0 \\ 0 & J_{1\times n} \otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{1\times n} \otimes I_n \\ \hline I_n \otimes J_{1\times n} & 0 & \dots & 0 \\ 0 & I_n \otimes J_{1\times n} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n \otimes J_{1\times n} \end{array}\right] = \left[\begin{array}{c} I_{1\times n} \otimes I_{n^2} \\ \hline I_n \otimes (J_{1\times n} \otimes I_n) \\ \hline I_{n^2} \otimes J_{1\times n} \end{array}\right]$$

Then to each column block $C_j$ with $j \geq 2$, perform $C_{j\ell} - C_{j1}$ for each $\ell = 2, \dots n^2$. This zeros out the $n^2 - n$ columns in $R_2$ and leaves copies of $K \otimes J_{1\times n}$, replacing copies of $I_n \otimes J_{1\times n}$.

$$\mathcal{U} \sim \left[\begin{array}{c|ccc} I_{n^2} & 0 & \dots & 0 \\ \hline J_{1\times n} \otimes I_n & 0 & \dots & 0 \\ 0 & I_{1\times n} \otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{1\times n} \otimes I_n \\ \hline I_n \otimes J_{1\times n} & 0 & \dots & 0 \\ 0 & K \otimes J_{1\times n} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K \otimes J_{1\times n} \end{array}\right] = \left[\begin{array}{c|ccc} I_{n^2} & 0 & \dots & 0 \\ \hline J_{1\times n} \otimes I_n & 0 & \dots & 0 \\ 0 & \multicolumn{3}{c}{I_{n^2-1} \otimes (I_{1\times n} \otimes I_n)} \\ \hline I_n \otimes J_{1\times n} & \multicolumn{3}{c}{I_{n^2-1} \otimes (K \otimes J_{1\times n})} \end{array}\right]$$

As before, to return the copies of $K$ back to copies of $I_n$, perform operations $R_{3(xn+1)} - R_{3(xn+k)}$ for each $k = 2, \dots, n$, where $x = 1, \dots, n$. That is, in each $n \times n$ submatrix in $R_3$, use rows 2 through $n$ to cancel out the negative entries in the first row of that submatrix.

$$\mathcal{U} \sim \left[\begin{array}{c|cccc} I_{n^2} & 0 & \dots & 0 \\ \hline J_{1\times n} \otimes I_n & 0 & \dots & 0 \\ 0 & I_{1\times n} \otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{1\times n} \otimes I_n \\ \hline I_n \otimes J_{1\times n} & 0 & \dots & 0 \\ 0 & I_n \otimes J_{1\times n} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n \otimes J_{1\times n} \end{array}\right] = \left[\begin{array}{c|ccc} I_{n^2} & 0 & \dots & 0 \\ \hline J_{1\times n} \otimes I_n & 0 & \dots & 0 \\ 0 & \multicolumn{3}{c}{I_{n-1} \otimes (I_{1\times n} \otimes I_n)} \\ \hline I_n \otimes J_{1\times n} & 0 & \dots & 0 \\ 0 & \multicolumn{3}{c}{I_{n-1} \otimes (I_n \otimes J_{1\times n})} \end{array}\right]$$

Now the top $n$ rows in each of $R_2$ and $R_3$ may be cancelled out using the top $n^2$ rows in $R_1$ which only have nonzero entries in the first $n^2$ columns.

$$\mathcal{U} \sim \left[\begin{array}{c|ccc} I_{n^2} & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & \multicolumn{3}{c}{I_{n-1} \otimes (I_{1\times n} \otimes I_n)} \\ \hline 0 & 0 & \dots & 0 \\ 0 & \multicolumn{3}{c}{I_{n-1} \otimes (I_n \otimes J_{1\times n})} \end{array}\right]$$

Since $I_{1\times n} \otimes I_n = \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}$, cancel the rows in $R_3$ underneath a copy of $I_n$ in $R_2$. In $R_3$ this gives $n-1$ copies of the following submatrix, each containing one zero row.

$$I_n \otimes J_{1\times n} \sim \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n-1} \otimes J_{1\times n} \end{array}\right]$$

Now each column has precisely one nonzero entry, so permuting the rows gives the reduced row echelon form of $\mathcal{U}$. The $n-1$ additional zero rows from $R_3$ together with the $n$ zero rows at the top of each of $R_2$ and $R_3$ give $3n-1$ total zero rows. This means that rank$(\mathcal{U}) = 3n^2 - 3n + 1$, since $3n^2 \leq n^3$ for any $n \geq 3$. Therefore nullity$(\mathcal{U}) = n^3 - (3n^2 - 3n + 1) = n^3 - 3n^2 + 3n - 1 = (n-1)^3$.

### 5.2.2 Smith Normal Form

Since $\mathcal{U}$ is not invertible, an alternate form of the matrix is used to obtain a similar condition to Condition 3 of Lemma 5.2.

The *Smith normal form* of matrix $A$ is an integral diagonal matrix $D$ where $D = PAQ$ for some unimodular square matrices $P$ and $Q$. That is, $P$ and $Q$ are invertible matrices with determinant equal to 1 or $-1$. Moreover, $D = \text{diag}(d_1, d_2, \ldots, d_r, 0, 0, \ldots, 0)$ such that $d_i$ divides $d_{i+1}$ for all $i = 1, 2, \ldots, r-1$. Since $Q$ corresponds to the set of column operations used to obtain $D$, it has full rank. Set $Q\vec{y} = \vec{x}$ for some vector $\vec{y}$. Then the equation $A\vec{x} = \vec{b}$ is equivalent to $D\vec{y} = \vec{c}$, where $\vec{c} = P\vec{b}$. Moreover, $D\vec{y} = \vec{c}$ has an integral solution exactly when $d_1, ..., d_r$ divide the corresponding entries of $\vec{c}$, and $c_i = 0$ for $i > r$. See [20] or [23] for more details.

**Theorem 5.4.** The Smith normal form of the matrix $\mathcal{U}$ is a diagonal matrix $D$ with $3n^2 - 3n + 1$ one's on its main diagonal and rest of the entries are zero.

*Proof.* It was shown in Section 5.2.1 that rank$(\mathcal{U}) = 3n^2 - 3n + 1$ since $3n^2 \leq n^3$ for any $n \geq 3$, and therefore rank$(D) = 3n^2 - 3n + 1$. (Note that for $n = 2$, since $3n^2 > n^3$ then rank$(\mathcal{U}) = n^3 - 3n + 1 = 3$.) The nonzero entries of $D$ are all equal to one since no scaling of rows or columns of $\mathcal{U}$ was needed in Algorithm 5.1 to bring this $\{0, 1\}$-matrix into diagonal form. □

By the properties of Smith normal form, Theorem 5.4 says that any integer vector $\vec{p}$

that lies in the range of $\mathcal{U}$ admits an integer solution to the equation $\mathcal{U}\vec{t} = \vec{p}$. That is, any $\vec{p}$ where there is a solution gives a triangle decomposition. However, note that there is no guarantee that the integer triangles given by the equation are nonnegative in these solutions.

#### 5.2.2.1 An Example with $n = 3$

The constraints given by the Smith normal form of $\mathcal{U}$ are computed for $n = 3$. In this case, the following eight constraints are obtained from $c_i = 0$ where $\vec{c} = P\vec{p}$, as in Section 5.2.2.

$$0 = p_{13} + p_{14} + p_{15} - p_4 - p_5 - p_6$$

$$0 = -p_1 + p_{19} + p_{20} + p_{21} - p_4 - p_7$$

$$0 = p_1 - p_{10} - p_{11} - p_{12} + p_2 + p_3$$

$$0 = p_{16} + p_{17} + p_{18} - p_7 - p_8 - p_9$$

$$0 = -p_2 + p_{22} + p_{23} + p_{24} - p_5 - p_8$$

$$0 = -p_{10} - p_{13} - p_{16} + p_{19} + p_{22} + p_{25}$$

$$0 = -p_{11} - p_{14} - p_{17} + p_{20} + p_{23} + p_{26}$$

$$0 = p_{10} + p_{11} - p_{15} - p_{18} - p_{19} - p_{20} - p_{22} - p_{23} + p_{27} - p_3 + p_4 + p_5 + p_7 + p_8$$

These equations ensure that the frequency at every point is even and that the sum of all pairs between two groups are equal. This can been seen by summing over all the equations containing pairs in a particular group. The inclusion of the unlisted restrictions from the Smith normal form when $d_i \neq 0$ forces all pair frequencies to be integer.

Consider any of the first seven equations taken individually. These equations correspond to a different point in the design and ensure the sum of the pair frequencies from that point to one group is equal to the sum of the pair frequencies from that point to the other group. The last equation, with some rearranging, takes care of the same constraint for the final two points in the 9-point design. Thinking of the geometric interpretation of the cube, this means that in each of the nine faces of the cube ($3n$ faces for general $n$), the line sums in one direction of the face are equal to the line sums in another direction. Looking at this from the design theory perspective, notice that every block of a cube has one point in each group. Therefore the sum of all pair frequencies between any two groups must be the same.

Unfortunately, the equations given by Smith normal from do not exclude negative pair frequencies. However, this completely solves the Sarvate-Beam cube problem in the integer sense. That is, if the concept of the geometric cube is extended to allow for negative integers at its points, then these conditions are necessary and sufficient for the pair frequency vector to admit a Sarvate-Beam cube.

**Theorem 5.5.** Allowing for triangles with negative multiplicities, the equation $\mathcal{U}\vec{t} = \vec{p}$ admits an integer triangle decomposition given by the entries in $\vec{t}$ whenever $\vec{p}$ is integral and lies in the column space $\mathcal{U}$. Equivalently, $\vec{p}$ has the properties that;

1. the sum of the entries corresponding to pairs containing any specific point are even,
2. the total sum of the entries is divisible by three, and
3. the sum of the pair frequencies of one point to any group is equal to the sum of the pair frequencies on that point to the other group.

Theorem 5.5 follows from Theorem 5.4 together with the properties of the Smith normal form. Note that Conditions 2 and 3 are the necessary conditions for a triangle decomposition given in Lemma 5.3.

## 5.3 Magic Squares for Cubes

Finding pair frequency vectors $\vec{p}$ whose solutions correspond to Sarvate-Beam cubes is investigated. Some orderings of the frequencies in $\vec{p}$ correspond to a system $\mathcal{U}\vec{t} = \vec{p}$ without solutions, while others may have no nonnegative integer solutions.

From Theorem 5.5, only pair frequency vectors $\vec{p}$ where the sum of the frequencies in each third of the vector is equal to one third of the sum of all it's entries need be considered. For example, the SBC$(2)$ depicted in Figure 3.3 has groups $G = \{ab, cd, ef\}$. The blocks are $\{acf, 2bcf, 4ade, 6bde, 4adf, 5bdf\}$. The frequency between any two groups is $1 + 2 + 4 + 6 + 4 + 5 = 22 = \frac{1}{3}(0 + 1 + \cdots + 11)$.

To summarize, the goal is to assign pair frequencies so that frequencies are balanced between the groups, while having each entry in the vector distinct. These properties lead to the use of magic squares.

#### 5.3.0.1 Semi-magic Square

A *semi-magic square* is an $n \times n$ array where the sum of any row or column is the same, called the *magic constant*. A semi-magic square is called a *magic square* if both of the main diagonals also sum to the magic constant. For what comes next, the diagonal sums are of no importance. Moreover, while a semi-magic square is sufficient for the subsequent constructions, a magic square (proper) is used in the coming examples. For that reason, a semi-magic square and magic square are both simply called a magic square in what follows.

A semi-magic square is called *normal* when the numbers in the array are $\{1, 2, \ldots, n^2\}$. Semi-magic squares containing the numbers from zero to $n^2 - 1$ are of interest since the entries correspond to pair frequencies in Sarvate-Beam Cubes with starting frequency $\mu = 0$. Figure 5.4 gives an order three magic square with magic constant 12.

| | | |
|---|---|---|
| 7 | 0 | 5 |
| 2 | 4 | 6 |
| 3 | 8 | 1 |

Figure 5.4: A magic square of order 3

### 5.3.1 Constructing Semi-magic Squares

Let $A$ and $B$ be a pair of MOLS($n$). Then the array given by $M = A + n(B - J_{n\times n})$ is a semi-magic square of order $n$ [6]. To see this, consider any $m_{ij}$. Since the values $a_{ij}$ and $b_{ij}$ occur exactly once in the same position, the value $a_{ij}+nb_{ij}-n$ is unique. Moreover the sum in any row and column is $\sum a_{ij}+n\sum b_{ij}-n^2 = \frac{1}{2}(n(n+1)+n^2(n+1)-2n^2) = \frac{1}{2}n(n^2+1)$, since each row and column of a Latin square contains each symbol once.

In order for the magic squares to contain entries zero through $n^2 - 1$, instead let $M = (A-J_{n\times n})+n(B-J_{n\times n})$ which shifts the above entries down by one. Therefore the magic constant is $\frac{1}{2}(n(n^2+1) - 2n) = \frac{1}{2}n(n^2-1)$. Since two MOLS($n$) exist for every $n \geq 2$ except six, magic squares of every order $n \neq 6$ can be constructed in this way. Figure 5.5 gives a magic square of order six to complete the existence theory [6].

| | | | | | |
|---|---|---|---|---|---|
| 0 | 29 | 18 | 17 | 11 | 30 |
| 34 | 7 | 16 | 22 | 25 | 1 |
| 3 | 26 | 14 | 20 | 9 | 33 |
| 32 | 27 | 15 | 21 | 8 | 2 |
| 31 | 10 | 19 | 13 | 28 | 4 |
| 5 | 6 | 23 | 12 | 24 | 35 |

Figure 5.5: A magic square of order 6

Returning to Sarvate-Beam cubes, note that between any two groups of size $n$, there are $n^2$ pairs which need to be assigned a pair frequency. Let each row of a magic square of order $n$ represent the point in the first group and each column represent the point in the second group. Let the edge multiplicities between pair $ij$ be the $(i, j)$-entry in the magic square. Then by the properties of the square, every pair frequency is distinct and the sum of the frequencies between any two groups is equal.

The goal is to extend this property to all sets of pairs, between any two groups. A magic square of order $n$ is extended to a triple of magic squares as follows. Take three copies of the same magic square of order $n$ and multiply the entries by three. Add $0, 1,$ and $2$ to corresponding entries in the different squares. The result is the set of all values from zero to $3(n^2 - 1) + 2$ occurring once across the three distinct squares.

If the values $0, 1,$ and $2$ are added so that each row and column of the magic square is increased with equal weight, then the magic constant property is maintained. This is equivalent to finding a $3 \times n$ array with entries $0, 1,$ and $2$ where the rows have a constant sum and every column contains $0, 1,$ and $2$ exactly once. The rows of the array index the different magic squares and the columns index the values to be added to each of the $n$ diagonals of the magic squares.

For example in Figure 5.6, a magic square of order $n = 3$ is expanded into three magic squares containing distinct integers from zero to $26$. The $3 \times n$ weight array is also given. Construction 5.1 builds these weight arrays for general $n$.

| Diagonal | $j = i$ | $j = i + 1 \pmod 3$ | $j = i + 2 \pmod 3$ |
|---|---|---|---|
| $M_1$ | 0 | 2 | 1 |
| $M_2$ | 2 | 1 | 0 |
| $M_3$ | 1 | 0 | 2 |

Template

| | | |
|---|---|---|
| 7 | 0 | 5 |
| 2 | 4 | 6 |
| 3 | 8 | 1 |

$M_1$

| | | |
|---|---|---|
| 21 | 2 | 16 |
| 7 | 12 | 20 |
| 11 | 25 | 3 |

$M_2$

| | | |
|---|---|---|
| 23 | 1 | 15 |
| 6 | 14 | 19 |
| 10 | 24 | 5 |

$M_3$

| | | |
|---|---|---|
| 22 | 0 | 17 |
| 8 | 13 | 18 |
| 9 | 26 | 4 |

Figure 5.6: Magic square product construction

Lining up the values of each square in the same order as the rows of $\mathcal{U}$ gives candidate vectors for $\vec{p}$ in the equation $\mathcal{U}\vec{t} = \vec{p}$. The vector corresponding to the magic squares in Figure 5.6 is

$\vec{p}_1 = (21, 2, 16, 7, 12, 20, 11, 25, 3, 22, 8, 9, 0, 13, 26, 17, 18, 4, 23, 1, 15, 6, 14, 19, 10, 24, 5)^\top$.

### 5.3.2 Making Weight Arrays

This section describes how to make $3 \times n$ arrays, for general $n$, to weight three magic squares of order $n$ to use in the magic square construction.

**Construction 5.1.** When $n$ is even, construct the array with $\frac{n}{2}$ copies of the column $(0, 2, 1)^\top$ followed by $\frac{n}{2}$ copies of the column $(2, 0, 1)^\top$. When $n$ is odd, take $\frac{n-1}{2}$ copies of the column $(0, 2, 1)^\top$, $\frac{n-3}{2}$ copies of the column $(2, 0, 1)^\top$, and one column of each of $(2, 1, 0)^\top$ and $(1, 0, 2)^\top$. Both cases give constant row sums equal to $n$.

Note that any permutation of the columns given by Construction 5.1 also results in a valid weight array. Some of these permutations are considered for finding solutions in Section 5.3.3.

### 5.3.3 Balancing Pair Frequencies

The values in the magic square can be rearranged so that the total frequency incident to any point is the same as the other points in the design. In order to keep the row and column sums the same, the entries of each magic square are shifted along its diagonals. The diagonals in the three different magic squares are each shifted by a different value to equalize the frequencies on any particular point in the design. Figure 5.7 illustrates this with the set of magic squares from Figure 5.6, shifting the values by $0, 1$, and $2$, in that order. The points corresponding to the entry 7 in the template magic square appear in blue to depict how the diagonals are being shifted.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 21 | 2 | 16 | 23 | 1 | 15 | 22 | 0 | 17 |
| Original | 7 | 12 | 20 | 6 | 14 | 19 | 8 | 13 | 18 |
| | 11 | 25 | 3 | 10 | 24 | 5 | 9 | 26 | 4 |
| | 21 | 2 | 16 | 14 | 19 | 6 | 4 | 9 | 26 |
| Shift by 1 | 7 | 12 | 20 | 24 | 5 | 10 | 17 | 22 | 0 |
| | 11 | 25 | 3 | 1 | 15 | 23 | 18 | 8 | 13 |
| | 21 | 2 | 16 | 5 | 10 | 24 | 13 | 18 | 8 |
| Shift by 2 | 7 | 12 | 20 | 15 | 23 | 1 | 26 | 4 | 9 |
| | 11 | 25 | 3 | 19 | 6 | 14 | 0 | 17 | 22 |

Figure 5.7: Magic squares before and after balancing

The pair vectors corresponding to each set of magic squares, with entries ordered according to $\mathcal{U}$, are given by the vectors $\vec{p}_1, \vec{q}_1, \vec{r}_1$. The pair frequency vectors $\vec{p}_2, \vec{q}_2, \vec{r}_2$ correspond to the same magic squares, but with columns from the weight array reordered.

$$\vec{p}_1 = (21, 2, 16, 7, 12, 20, 11, 25, 3, 22, 8, 9, 0, 13, 26, 17, 18, 4, 23, 1, 15, 6, 14, 19, 10, 24, 5)^\top$$

$$\vec{q}_1 = (21, 2, 16, 7, 12, 20, 11, 25, 3, 4, 17, 18, 9, 22, 8, 26, 0, 13, 14, 19, 6, 24, 5, 10, 1, 15, 23)^\top$$

$$\vec{r}_1 = (21, 2, 16, 7, 12, 20, 11, 25, 3, 5, 15, 19, 10, 23, 6, 24, 1, 14, 13, 18, 8, 26, 4, 9, 0, 17, 22)^\top$$

$$\vec{p}_2 = (21, 1, 17, 8, 12, 19, 10, 26, 3, 23, 7, 9, 0, 14, 25, 16, 18, 5, 22, 2, 15, 6, 13, 20, 11, 24, 4)^\top$$

$$\vec{q}_2 = (21, 1, 17, 8, 12, 19, 10, 26, 3, 5, 16, 18, 9, 23, 7, 25, 0, 14, 13, 20, 6, 24, 4, 11, 2, 15, 22)^\top$$

$$\vec{r}_2 = (21, 1, 17, 8, 12, 19, 10, 26, 3, 4, 15, 20, 11, 22, 6, 24, 2, 13, 14, 18, 7, 25, 5, 9, 0, 16, 23)^\top$$

The corresponding solution vectors given through Sage's built in linear algebra package are given below [21]. Note that other solutions exist since nullity$(\mathcal{U}) = 8$. Whether these pair frequency vectors result in positive multiplicity triangles is investigated in Section 5.4

using integer linear programming.

$$\vec{t}_{p_1} = (66, -30, -15, -31, 14, 19, -13, 24, 5, -32, 13, 26, 12, 0, 0, 20, 0, 0, -11, 18, 4, 25, 0, 0, 3, 0, 0)^\top$$
$$\vec{t}_{q_1} = (39, -3, -15, -13, 5, 10, -22, 15, 23, -23, 22, 8, 12, 0, 0, 20, 0, 0, -2, 0, 13, 25, 0, 0, 3, 0, 0)^\top$$
$$\vec{t}_{r_1} = (39, -6, -12, -11, 4, 9, -23, 17, 22, -22, 23, 6, 12, 0, 0, 20, 0, 0, -4, 1, 14, 25, 0, 0, 3, 0, 0)^\top$$

$$\vec{t}_{p_2} = (66, -30, -15, -32, 13, 20, -11, 24, 4, -31, 14, 25, 12, 0, 0, 19, 0, 0, -13, 18, 5, 26, 0, 0, 3, 0, 0)^\top$$
$$\vec{t}_{q_2} = (39, -3, -15, -14, 4, 11, -20, 15, 22, -22, 23, 7, 12, 0, 0, 19, 0, 0, -4, 0, 14, 26, 0, 0, 3, 0, 0)^\top$$
$$\vec{t}_{r_2} = (39, -6, -12, -13, 5, 9, -22, 16, 23, -20, 22, 6, 12, 0, 0, 19, 0, 0, -5, 2, 13, 26, 0, 0, 3, 0, 0)^\top$$

## 5.4 Nonnegativity

This section explores when nonnegative integer solutions exist using integer linear programming to search for nonnegative pair frequencies. General conditions on whether a system of equations $\mathcal{U}\vec{t} = \vec{p}$ admits a nonnegative integer solution are not known. However, Farkas' Lemma gives a certificate for when a system has a nonnegative solution. See, for example, [7].

**Lemma 5.6** (Farkas' Lemma)**.** Let $A$ be an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$. The system $A\vec{x} = \vec{b}$ has a nonnegative solution if and only if there is no vector $\vec{y}$ satisfying $\vec{y}^{\top} A \geq \vec{0}$ and $\vec{y}^{\top}\vec{b} < 0$.

Recall the pair frequency vectors from Section 5.3, $\vec{p}_1, \vec{q}_1, \vec{r}_1, \vec{p}_2, \vec{q}_2, \vec{r}_2$, which respectively came from the original magic square construction, the balanced magic square constructions with different diagonal shifts, and the previous three constructions with the columns of the weight array permuted. Concerning the 'unbalanced' vectors, $\vec{p}_1$ and $\vec{p}_2$, the following certificates are given by Farkas' Lemma.

Let $\vec{y}_1 = (0,0,0,1,1,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,1)^\top$. Then $\vec{y}_1^{\top}\mathcal{U} = (0,0,0,0,0,0,0,0,1,1,1,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,1)^\top \geq \vec{0}$ and $\vec{y}_1^{\top}\vec{p}_1 = -2$ and $\vec{y}_1^{\top}\vec{p}_2 = -1$. Therefore the systems with those pair vectors have only solutions containing negative triangle multiplicities.

The particular systems of linear equations for the remaining pair vectors are explored using integer linear programming to find nonnegative solutions. The linear programming problem is given below.

$$\begin{aligned} \text{Minimize} \quad & t_1 + t_2 + \dots t_{n^3} \\ \text{subject to} \quad & \mathcal{U}\vec{t} = \vec{p} \\ & t_i \geq 0, \forall i = 1, 2, \dots, n^3. \end{aligned}$$

For example, the constraints given by $\mathcal{U}\vec{t} = \vec{r}_2$ are given in Figure 5.8. Each equation corresponds to the sum of the three triangles containing the edge with the given pair frequency. The objective function serves simply as a check since the sum of the triangles is always $117$, a third the sum of the pair frequencies. The program runs using Sage's class 'Mixed Integer Linear Program' on the default solver 'GLPK'. See [21] and [14] for more details on the integer linear programming class.

$$
\begin{array}{rrr}
t_0 + t_1 + t_2 = 21 & t_3 + t_4 + t_5 = 1 & t_6 + t_7 + t_8 = 17 \\
t_{10} + t_{11} + t_9 = 8 & t_{12} + t_{13} + t_{14} = 12 & t_{15} + t_{16} + t_{17} = 19 \\
t_{18} + t_{19} + t_{20} = 10 & t_{21} + t_{22} + t_{23} = 26 & t_{24} + t_{25} + t_{26} = 3 \\
t_0 + t_3 + t_6 = 4 & t_1 + t_4 + t_7 = 15 & t_2 + t_5 + t_8 = 20 \\
t_{12} + t_{15} + t_9 = 11 & t_{10} + t_{13} + t_{16} = 22 & t_{11} + t_{14} + t_{17} = 6 \\
t_{18} + t_{21} + t_{24} = 24 & t_{19} + t_{22} + t_{25} = 2 & t_{20} + t_{23} + t_{26} = 13 \\
t_0 + t_{18} + t_9 = 14 & t_1 + t_{10} + t_{19} = 18 & t_{11} + t_2 + t_{20} = 7 \\
t_{12} + t_{21} + t_3 = 25 & t_{13} + t_{22} + t_4 = 5 & t_{14} + t_{23} + t_5 = 9 \\
t_{15} + t_{24} + t_6 = 0 & t_{16} + t_{25} + t_7 = 16 & t_{17} + t_{26} + t_8 = 23
\end{array}
$$

Figure 5.8: Constraints used in the integer linear program with $n = 3$

The balanced vectors, $\vec{q}_1, \vec{r}_1, \vec{q}_2$, and $\vec{r}_2$, in the integer linear program each yield nonnegative integer solutions to the problem, given below.

$$
\begin{aligned}
\vec{t}_{q_1} &= (4, 17, 0, 0, 0, 2, 0, 0, 16, 5, 2, 0, 3, 5, 4, 1, 15, 4, 5, 0, 6, 21, 0, 4, 0, 0, 3)^\top \\
\vec{t}_{r_1} &= (5, 13, 3, 0, 2, 0, 0, 0, 16, 2, 5, 0, 8, 1, 3, 0, 17, 3, 6, 0, 5, 18, 1, 6, 0, 0, 3)^\top \\
\vec{t}_{q_2} &= (5, 12, 4, 0, 0, 1, 0, 4, 13, 0, 8, 0, 7, 4, 1, 2, 11, 6, 8, 0, 2, 17, 0, 9, 0, 0, 3)^\top \\
\vec{t}_{r_2} &= (4, 14, 3, 0, 1, 0, 0, 0, 17, 4, 4, 0, 7, 2, 3, 0, 16, 3, 6, 0, 4, 18, 2, 6, 0, 0, 3)^\top
\end{aligned}
$$

Note that by adding constraints, other nonnegative integer solutions were found for the same set of pair vectors. For example, the system $\mathcal{U}\vec{t} = \vec{r}_2$ has nonnegative integer solutions for each of $15 \leq t_{21} \leq 21$, but no feasible solution for other values of $t_{21}$. All admissible solutions have $t_6 = t_{15} = t_{24} = 0$ corresponding to the triangles incident to the pair of frequency zero in $\vec{r}_2$. For every other entry in the solution vector, there are feasible solutions with that entry greater than zero.

The following integer linear program is used to determine if a certificate vector $\vec{y}$ exists by Lemma 5.6, Farkas' Lemma.

$$\begin{aligned} \text{Minimize} \quad & y_1 + y_2 + \dots y_{3n^2} \\ \text{subject to} \quad & \mathcal{U}^\top \vec{y} \geq \vec{0} \\ & \vec{y}^{\,\top} \vec{p} \leq -1. \end{aligned}$$

The constraints given by $\mathcal{U}^\top \vec{y} \geq \vec{0}$ corresponds to each triangle having nonnegative multiplicity. To this, the constraint $\vec{y}^{\,\top} \vec{p} \leq -1$ is added so that the integer linear program is only feasible if the corresponding system $\mathcal{U}\vec{t} = \vec{p}$ has integer solutions that all contain at least one negative multiplicity, by Farkas' Lemma. The same objective function is used, again only as a check, since the goal is to determine when the linear program is feasible.

Running the program with $\vec{q}_1, \vec{r}_1, \vec{q}_2$, and $\vec{r}_2$ returns infeasible as expected, meaning nonnegative solutions exist. For $\vec{p}_1$, and $\vec{p}_2$ the following optimal solutions are obtained which serve as a certificate of infeasibility for the original problem.

$$\vec{y}_1 = (0,0,0,1,1,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,1)^\top$$
$$\vec{y}_2 = (0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,-1,0,0,0)^\top$$

The inadmissible configuration in the graph of $K_{3,3,3}$ corresponding to $\vec{p}_1$ and $\vec{y}_1$ is depicted in Figure 5.9. Note that $\vec{y}_1^{\top}\ \vec{p}_1 = 7 + 12 - 26 + 5 = -2$. The corresponding edges from $\vec{p}_1$ are given with solid lines. Since one edge has multiplicity five, the triangle given by the black vertices has maximum multiplicity five. If the pair frequencies have a triangle decomposition, the remaining edge multiplicity of $26$ must be split amongst the other two triangles containing that edge. However as $7 + 12 + 5 < 26$ no such nonnegative decomposition exists. This also follows from Theorem 5.4, although in this way, a graph decomposition obstruction to existence is obtained.

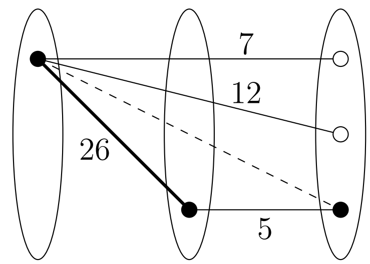



Figure 5.9: Configuration in $\vec{p}_1$ forcing no valid decomposition

In $K_{2,2,2}$, this obstruction corresponds to a path of length three where the middle edge has higher multiplicity than the sum of the outer edges. For $n > 3$, the obstruction is the tree given by a star with $n-1$ vertices in the same group, whose central vertex is connected by a path of length two with a leaf in the same group as the leaves of the star. Such a multigraph is an obstruction when the edge incident to the star and contained in the path of length two has higher multiplicity than the sum of the rest of the edges. This obstruction extends to more than three groups as well. The additional edges in the star correspond to triangles on the high multiplicity edge whose third point is contained in another group. The generalized obstructions are given in Figure 5.10 with vertices in different groups depicted in different colours.

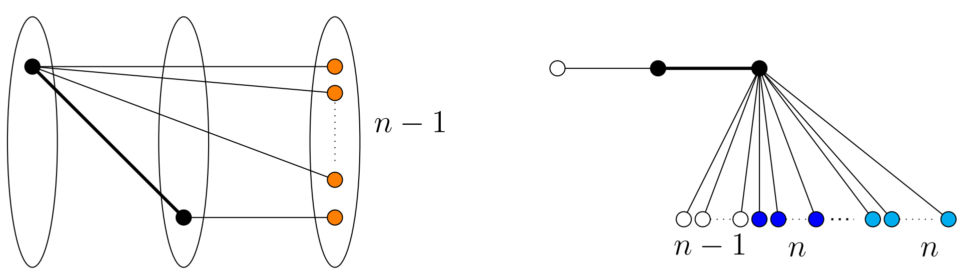



Figure 5.10: The generalized obstructions

# Chapter 6

# Wrap Up and Further Directions

The main results of this dissertation is the complete existence of 3-SBGDDs of type $g^u$ with starting frequency zero. It is nice that the main construction can be altered to cover almost any admissible $\mu > 0$. The exceptional cases in Theorem 1.3 of type $g^8$, where $\mu \equiv 3 \pmod 6$ and $g \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$ remain open. Without the existence of the required pairwise balanced designs or group divisible designs used for inflation, these values likely require a direct construction.

It is natural to extend the geometric interpretation of cubes to rectangular prisms. In order to have straight lines of the prism corresponding to blocks in the design, the Sarvate-Beam GDD would have variable group sizes. While this dissertation focused on the uniform group case, the existence of these generalized objects is of interest. One may also be interested in extending these results to general block size or considering alternate pair frequency sets, different than a set of distinct consecutive integers starting with some $\mu \geq 0$. The following questions for the existence of Sarvate-Beam designs are raised.

1. When does a 3-SBGDD exist with general group type?
2. When does a $K$-SBGDD of type $g^u$ (or $g_1^{u_1} g_2^{u_2} \ldots g_k^{u_k}$) exist? The case of three groups leads to the extension of Sarvate-Beam cubes to rectangular prisms or 'bricks'.

3. For a specified set of integer block sizes $K$, when does a Sarvate-Beam design with block sizes in $K$ exist? One might call these objects Sarvate-Beam PBDs, or SBPBDs.

Throughout the dissertation, graph decompositions are used as an aid to solve the question of existence. Extensions of the problem via graph decompositions are discussed in what follows.

## 6.1 Graph Decompositions

The setting of Sarvate-Beam designs led us to the more general study of decomposing a multigraph $G$ into copies of a subgraph $H$. In the Sarvate-Beam design setting, this dissertation investigated when a particular set of values could be assigned to the edges of a simple graph to give a multigraph that can be decomposed into triangles. Instead, given some pre-assigned edge multiplicities (i.e. the multigraph itself) under what conditions is the graph triangle decomposable? The following question which generalizes the problem from triangles to general subgraphs is of broad interest.

**Question 6.1.** For any multigraph $G$ and $H$, can the edges of $G$ be partitioned into copies of $H$?

The case when $G$ and $H$ are both simple graphs has long been studied. The generalized necessary conditions for when such a partition is possible are given in the following lemma.

**Lemma 6.1.** Let $G$ and $H$ be simple graphs. If $G$ has a partition of its edges into copies of $H$ then

1. the number of edges of $H$ divides the number of edges in $G$, and
2. $\gcd\{\deg(h_i) \mid h_i \in V(H)\}$ divides $\deg(g_i)$ for each $g_i \in V(G)$.

If $G$ and $H$ satisfy the conditions of Lemma 6.1, then $G$ is called $H$-divisible. Note that the conditions of this lemma are equivalent to Conditions 1 and 2 of Lemma 5.2 in the five vertex case, though the current setting does not allow for repeated edges. It was shown that Conditions 1 and 2 were not enough to ensure a decomposition in the multigraph setting. Counterexamples are given in Figure 5.1 on five vertices. Certainly the same is true in the simple graph setting.

Dor and Tarsi [9] showed that the problem of finding an $H$-decomposition of an input graph $G$ is NP-complete whenever $H$ contains a connected component with three or more edges. It is therefore natural to look at specific classes of graphs.

One of the first settings that is natural to explore is a decomposition into cliques. Notice that if $G$ and $H$ are both cliques, then the problem is equivalent to the existence of a $(v, k, \lambda)$-design where $V(G) = v$ and $V(H) = k$. In the case where $G$ is simple, the index, $\lambda$, equals one. For example, it is well-known that a complete graph with $v$ vertices is triangle decomposable if and only if $v \equiv 1, 3 \pmod{6}$, based on the existence theory of Steiner triple systems [6]. It is therefore no surprise that the conditions of Lemma 6.1 closely resemble the necessary conditions of the existence for designs, as given in Lemma 2.1.

The question of triangle decomposability of general simple graphs has received the majority of the attention in the field. The following is a conjecture posed by Nash-Williams on triangle decomposability based on the minimum degree in a graph.

**Conjecture** ([19])**.** Let $G$ be a $K_3$-divisible graph with $n$ vertices and minimum degree $\delta(G) \geq \frac{3}{4}n$. If $n$ is sufficiently large, then $G$ admits a $K_3$-decomposition.

While the conjecture remains open, the problem has received a lot of attention since it was posed in 1970. The following theorem by Delcourt and Postle is the best known result to date.

**Theorem 6.2** ([8])**.** Let $G$ be a $K_3$-divisible graph with $n$ vertices and minimum degree $\delta(G) \geq \left(\frac{7+\sqrt{21}}{14} + \varepsilon\right) n$. If $n$ is sufficiently large, then $G$ admits a $K_3$-decomposition for any $\varepsilon > 0$.

Theorem 6.2 is a corollary to the following result for fractional decompositions of a graph given in the same paper. A *fractional $H$-decomposition* of a simple graph $G$ is a partition of the edges of $G$ into 'fractional' copies of $H$, where the edge multiplicities may be any nonnegative real numbers.

**Theorem 6.3** ([8])**.** Any graph $G$ with minimum degree $\delta(G) \geq \left(\frac{7+\sqrt{21}}{14}\right) n$ admits a fractional $K_3$-decomposition.

In the case of simple graphs, the sum over the edge multiplicities of each copy of $H$ is equal to one. In the case where the edges have the same multiplicity, the decomposition of a multigraph is equivalent to the decomposition of the simple graph. On the other end of the spectrum is the adesign-like case, where every edge of the graph $G$ has a different multiplicity. The following question is equivalent to the existence of Sarvate-Beam Triple Systems: For the complete graph $K_v$, is there an assignment of the multiplicities $\{0, 1, \ldots, \binom{v}{2} - 1\}$ so that $K_v$ is triangle decomposable? By the result in [12], as the necessary conditions are sufficient excluding $(v, \mu) = (4, 0)$, the answer to the question is yes, whenever $v \equiv 0, 1 \pmod 3$. Relating this question back to Question 6.1, one might instead ask the following: for a given multigraph $G$ with edge multiplicities $0, 1, \ldots, \binom{v}{2} - 1$ already prescribed, when is $G$ triangle decomposable?

Considering instead SBGDDs, replace $K_v$ with general $G$ (and the corresponding reduced interval of frequencies) in the preceding question. However, as this investigation of SBGDDs was restricted to group type $g^u$, the underlying simple graph of $G$ was restricted to $K_{g,g,\ldots,g}$, the complete multipartite graph with $u$ partite sets of size $g$. While Theorem 1.2 shows that every complete multipartite graph with $g \geq 2$ and $u \geq 3$ has an assignment of the multiplicities $\{0, 1, \ldots \binom{u}{2}g^2 - 1\}$ that admit a triangle decomposition, the latter question remains open.

Returning to simple graphs, the tripartite case is further discussed. Bowditch and Dukes [4] showed that for sufficiently large $n$, every locally balanced 3-partite graph $G$ on $3n$ vertices satisfying $\frac{1}{2n}\delta(G) \geq 0.96$ admits a fractional $K_3$-decomposition. The requirement that $G$ is *locally balanced* means that every vertex of $G$ has the same multiplicity assigned to the set of neighbours in each of the other partite sets. In Section 5.3.3, a similar restriction was given by the Smith normal form of the inclusion matrix and semi-magic squares were used to balance the pair frequencies. In Section 5.4, Farkas' Lemma was used to show that a special type of tree is an obstruction to attaining positive integral edge multiplicities. However, much of the structure of edge multiplicities and how they effect the decomposition beyond the necessary conditions is not yet known. The following directions are of interest to the multigraph decomposition problem.

1. Given a list of edge multiplicities, classify the assignments of the multiplicities to the edges of a graph $G$ so that the resulting multigraph has a triangle decomposition.

2. Determine a list of obstructions which prevent a $K_3$-divisible multigraph $G$ from having a triangle decomposition.

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