MAPS PRESERVING n^{th} POWERS

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1. Introduction

Let R_1 be an arbitrary ring and R_2 be a prime ring of characteristic not 2. A classical result of Herstein states that every surjective Jordan homomorphism $\phi : R_1 \to R_2$ (that is, a surjective additive map satisfying $\phi(x^2) = \phi(x)^2$ for every $x \in R_1$) is either a homomorphism or an antihomomorphism [5, Theorem 3.1]. The question arises as to what can we say about surjective additive maps preserving n^{th} powers. Following Herstein we call such maps *n-Jordan maps*. Herstein showed that an n-Jordan map ϕ from a unital ring R_1 onto a prime ring R_2 whose characteristic is 0 or larger than n is of the form $\phi(x) = \beta\theta(x)$ where θ is a homomorphism or an antihomomorphism and β is an $(n-1)$-st root of unity lying in the center of R_2 (see

[4, Theorem K] or [5, Theorem 3.2]). The proof is rather short and rests heavily on the presence of a unit element (see its outline below). In fact, the unit element appears implicitly even in the conclusion of the theorem. The goal of this paper is to extend this theorem to non-unital rings and thereby confirm Herstein's conjecture [4, p. 341].

Let us fix the notation. Throughout, R_1 and R_2 will be rings with centers Z_1 and Z_2, respectively. We assume that R_2 is prime. By C_2 we denote the extended centroid of R_2, and by F_2 we denote the field of fractions of Z_2. Of course, $Z_2 \subseteq F_2 \subseteq C_2$. By ϕ we denote an n-Jordan map of R_1 *onto* R_2 where $n \geq 3$. Thus, ϕ is an additive map satisfying

$$\phi(x^n) = \phi(x)^n$$

for every $x \in R_1$. We also assume that the characteristic of R_2 is 0 or larger than $2m(m+1)$ with $m = 4n - 8$. This, of course, is a technical assumption caused by our techniques, which could almost certainly be improved if not removed.

In particular, since the characteristic of R_2 is 0 or larger than n, a linearization of $\phi(x^n) = \phi(x)^n$ gives

$$\sum_{\pi \in S_n} \phi\left(x_{\pi(1)} \cdots x_{\pi(n)}\right) = \sum_{\pi \in S_n} \phi\left(x_{\pi(1)}\right) \cdots \phi\left(x_{\pi(n)}\right)$$

for all $x_1, \cdots, x_n \in R_1$. Setting

$$\{u_1, \cdots, u_n\} = \sum_{\pi \in S_n} u_{\pi(1)} \cdots u_{\pi(n)}$$

we thus have

$$\phi\left(\{x_1, \cdots, x_n\}\right) = \{\phi(x_1), \cdots, \phi(x_n)\}.$$

Let us sketch Herstein's proof in the unital case. Assume that R_1 contains 1, set $\alpha = \phi(1)$ and note that $\alpha^n = \alpha$. Letting $x_1 = x$, $x_2 = \cdots = x_n = 1$ in the relation above one obtains easily that $\alpha \in Z_2$, and hence α is an $(n-1)$-st root of unity. But then, letting

$$x_1 = x,\ x_2 = y,\ x_3 = \cdots = x_n = 1$$

one arrives at $\phi(xy+yx) = \alpha^{n-2}\left(\phi(x)\phi(y)+\phi(y)\phi(x)\right)$, which means that the map $\theta(x) = \alpha^{n-2}\phi(x)$ is a Jordan homomorphism of R_1 onto R_2. But then θ is a homomorphism or an antihomomorphism.

There does not seem to be a natural way of using this method in the non-unital case. However, if R_2 has a nontrivial center something similar can be done. In fact, Killam [6, Theorem] has previously verified Herstein's conjecture in this situation. In Section 2 we present an alternate solution in a special case of this situation. Namely, we show that if ϕ is bijective, R_1 is prime and $Z_1 \neq \{0\}$ and $Z_2 \neq \{0\}$, then ϕ has the form $\phi(x) = \beta\theta(x)$ where θ is a homomorphism or an antihomomorphism of R_1 into R_2F_2 and $\beta \in F_2$ satisfies $\beta^{n-1} = 1$ (Theorem 2.2).

In Section 3 we show that ϕ must be of the same form (however, with C_2 playing the role of F_2) also in the case when ϕ is bijective and R_2 is not a PI ring (Theorem 3.10). The proof of this theorem depends on a characterization of commuting traces of biadditive maps [3].

Using the results of Sections 2 and 3 we can conclude that ϕ has the form as in Theorem 3.10 whenever R_1 is prime and ϕ is bijective. Namely, by this theorem we may assume that R_2 is a PI ring. But then $Z_2 \neq \{0\}$ [8, Theorem 2]. We claim that then Z_1 is nontrivial, too. If R_1 is a PI ring, this is true for the same reason. If R_1 is not a PI ring, then Theorem 3.10 can be applied to the map $\phi^{-1}: R_2 \to R_1$. Thus, $\phi^{-1}(x) = \gamma\psi(x)$ where γ lies in C_1, the extended centroid of R_1, and $\psi: R_2 \to R_1C_1$ is a homomorphism or an antihomomorphism. But then it follows at once that $\phi^{-1}(Z_2) \subseteq Z_1$, showing that $Z_1 \neq \{0\}$.

It remains, therefore, to treat the case when the kernel of ϕ is nontrivial. This is done in Section 4. We introduce the concept an n-ideal of a ring (this extends the concept of a Jordan ideal) and show that under rather mild assumptions any nonzero n-ideal contains a nonzero ideal (Theorem 4.7). Applying this we prove that the kernel of ϕ is a prime ideal of R_1 (Theorem 4.9). This makes it possible for us to obtain a definitive result about the form of ϕ. Namely, ϕ induces a bijective n-Jordan map $\hat{\phi}$ of the prime ring R_1/K, where K is the kernel of ϕ, onto the prime ring R_2. Thus, there is

$\beta \in C_2$ and a homomorphism or an antihomomorphism $\hat{\theta} : R_1/K \to R_2C_2$ such that

$$\beta^{n-1} = 1 \text{ and } \hat{\phi}(\bar{x}) = \beta\hat{\theta}(\bar{x}) \text{ for all } \bar{x} \in R_1/K.$$

Letting $\nu : R_1 \to R_1/K$ denote the natural homomorphism of R_1 onto R_1/K, we then have

$$\phi(x) = \hat{\phi}\nu(x) = \beta(\hat{\theta}\nu)(x) \quad \text{for all} \quad x \in R_1.$$

Noting that $\theta = \hat{\theta}\nu$ is a homomorphism or an antihomomorphism of R_1 into R_2C_2 we have $\phi(x) = \beta\theta(x)$, $x \in R_1$.

Thus, Theorem 2.2, Theorem 3.10, and Theorem 4.9 together result in the following theorem, which is the main object of this paper.

Main Theorem. *Let $n \geq 3$ and let ϕ be an n-Jordan map of R_1 onto R_2, where R_1 is an arbitrary ring and R_2 is a prime ring of characteristic 0 or larger than $2m(m+1)$ with $m = 4n - 8$. Then there exists $\beta \in C_2$, the extended centroid of R_2, such that $\beta^{n-1} = 1$ and a homomorphism or an antihomomorphism $\theta : R_1 \to R_2C_2$ such that $\phi(x) = \beta\theta(x)$, $x \in R_1$.*

2. The Case with Nontrivial Centers

Throughout this section we assume, in addition to the general assumptions made in Section 1, that R_1 is also a prime ring and that ϕ is a bijection.

Lemma 2.1. *If $Z_2 \neq 0$, then $\phi(Z_1) \subseteq Z_2$.*

Proof. Pick $c \in Z_1$. We have

$$\{cx_1, x_2, \cdots x_n\} = \{x_1, cx_2, x_3, \cdots, x_n\}$$

and so

$$\{\phi(cx_1), \phi(x_2), \cdots, \phi(x_n)\} = \{\phi(x_1), \phi(cx_2), \phi(x_3), \cdots \phi(x_n)\}$$

for all $x_i \in R_1$. We may choose $\phi(x_3) = \cdots = \phi(x_n)$ to be a nonzero element c_0 in Z_2. Therefore we have

$$\begin{aligned} &\frac{n!}{2}\, c_0^{n-2} \Big(\phi(cx_1)\phi(x_2) + \phi(x_2)\phi(cx_1) \\ &\qquad - \phi(x_1)\phi(cx_2) - \phi(cx_2)\phi(x_1) \Big) = 0. \end{aligned} \tag{2.1}$$

Since char. $R_2 > n$ and $c_0 \neq 0$ we conclude from (2.1) that

$$\phi(cx_1)\phi(x_2) + \phi(x_2)\phi(cx_1) - \phi(x_1)\phi(cx_2) - \phi(cx_2)\phi(x_1) = 0. \tag{2.2}$$

Setting $u = \phi(x_1)$, $y = \phi(x_2)$ (whence $\phi(cx_1) = \phi\left(c\phi^{-1}(u)\right)$ and $\phi(cx_2) = \phi\left(c\phi^{-1}(y)\right)$ we may rewrite (2.2) as

$$\phi\left(c\phi^{-1}(u)\right) y + y\phi\left(c\phi^{-1}(u)\right) - u\,\phi\left(c\phi^{-1}(y)\right) - \phi\left(c\phi^{-1}(y)\right) u = 0. \tag{2.3}$$

Without loss of generality we may assume that R_2 is noncommutative. In view of (2.3) the map $f : R_2 \to R_2$ given by $f(u) = \phi\left(c\phi^{-1}(u)\right)$ satisfies the conditions of [1, Corollary 4.10], and so there exists $\lambda(c) \in C_2$ such that $\phi\left(c\phi^{-1}(u)\right) = \lambda(c)u$, $u \in R_2$. In other words (setting $x = \phi^{-1}(u)$)

$$\phi(cx) = \lambda(c)\,\phi(x) \qquad x \in R_1. \tag{2.4}$$

We shall show that $a = \phi(c) \in Z_2$. We have

$$\begin{aligned} n!\,\phi(c^{n-1}x) &= \phi\{c, \cdots, c,\, x\} = \{a, \cdots, a,\, \phi(x)\} \\ &= (n-1)!\, \left(a^{n-1}\,\phi(x) + a^{n-2}\phi(x)a + \cdots + \phi(x)a^{n-1}\right). \end{aligned} \tag{2.5}$$

On the other hand from (2.4) we have

$$n\,\phi(c^{n-1}x) = n\,\lambda(c^{n-1})\phi(x). \tag{2.6}$$

Comparing (2.5) and (2.6) we obtain

$$a^{n-1}y + a^{n-2}ya + \cdots + ya^{n-1} = \mu y \qquad \mu \in C_2, \tag{2.7}$$

where $\mu = n\,\lambda(c^{n-1})$. In particular

$$na^n = \mu a, \quad i.e., \quad a^n = \frac{\mu}{n}\, a. \tag{2.8}$$

Multiplying (2.7) first on the left and then from the right by a we get (using (2.8))

$$\begin{aligned} \mu a y &= a^n y + a^{n-1} y a + \cdots + a y a^{n-1} \\ &= \frac{\mu}{n} a y + a^{n-1} y a + \cdots + a y a^{n-1} \\ \mu y a &= a^{n-1} y a + \cdots + a y a^{n-1} + y a^n \\ &= a^{n-1} y a + \cdots + a y a^{n-1} + \frac{\mu}{n} y a. \end{aligned} \tag{2.9}$$

Comparing both relations in (2.9) we conclude that

$$\left(\frac{n-1}{n}\right) \mu [a, y] = 0$$

for all $y \in R_2$. Thus $a \in Z_2$ unless $\mu = 0$. But if $\mu = 0$ then from (2.8) we see that $a^n = 0$. In this case $\phi(c^n) = 0$ which implies that $c^n = 0$ and in turn that $c = 0$ and hence $a = 0 \in Z_2$. We have thereby proved that $\phi(Z_1) \subseteq Z_2$.

Theorem 2.2. *Let ϕ be an n-Jordan isomorphism ($n \geq 3$) of R_1 onto R_2, where R_1, R_2 are prime rings with $Z_1 \neq 0$, $Z_2 \neq 0$, and* char. $R_2 > n$. *Then there exists $\beta \in F_2$ such that $\beta^{n-1} = 1$ and a homomorphism or antihomomorphism $\theta : R_1 \to R_2 C_2$ such that $\phi(x) = \beta \theta(x)$, $x \in R_1$.*

Proof. Let $c \in Z_1$ and $x \in R_1$. By Lemma 2.1 we then have

$$\begin{aligned} n!\, \phi(c^{n-1} x) &= \phi\{c, \cdots, c, x\} = \{\phi(c), \cdots, \phi(c), \phi(x)\} \\ &= n!\, \phi(c)^{n-1} \phi(x) \\ n!\, \phi(c^{n-2} x^2) &= \phi\{c, \cdots, c, x, x\} = \{\phi(c), \cdots, \phi(c), \phi(x), \phi(x)\} \\ &= n!\, \phi(c)^{n-2} \phi(x)^2. \end{aligned} \tag{2.10}$$

Thus

$$\begin{aligned} \phi(c^{n-1} x) &= \phi(c)^{n-1} \phi(x) \\ \phi(c^{n-2} x^2) &= \phi(c)^{n-2} \phi(x)^2 \end{aligned} \qquad c \in Z_1, \;\; x \in R_1. \tag{2.11}$$

Now fix $0 \neq c \in Z_1$. Then from (2.11) we have

$$\begin{aligned} \phi\left(c^{(n-1)(n-2)} x^2\right) &= \phi\left((c^{n-2})^{n-1} x^2\right) = \phi(c^{n-2})^{n-1} \phi(x^2) \\ \phi\left(c^{(n-1)(n-2)} x^2\right) &= \phi\left((c^{n-1})^{n-2} x^2\right) = \phi(c^{n-1})^{n-2} \phi(x)^2. \end{aligned} \tag{2.12}$$

From (2.12) we see that

$$\phi(x^2) = \alpha\,\phi(x)^2, \quad 0 \neq \alpha = \phi(c^{n-1})^{n-2}\left[\phi(c^{n-2})^{n-1}\right]^{-1} \in F_2.$$

It follows at once that the map $\theta : R_1 \to R_2F_2$ given by $\theta(x) = \alpha\,\phi(x)$ is a Jordan homomorphism whose image is αR_2. It is then clear from the proof of Herstein's theorem [5, Theorem 3.1] that we may conclude that θ is either a homomorphism or an antihomomorphism (for details the reader may consult Killam's paper [6, Lemma 10]). Thus $\phi(x) = \beta\theta(x)$, $\beta = \alpha^{-1} \in F_2$. Since R_2 is prime we may choose $x \in R_1$ such that $\phi(x)^n \neq 0$. Now from

$$\phi(x^n) = \beta\,\theta(x^n) = \beta\,\theta(x)^n = \phi(x)^n = \beta^n\,\theta(x)^n$$

we conclude that $\beta^{n-1} = 1$ and the proof is complete. ∎

3. The Non-PI Case

We begin this section by establishing some general results of a combinatorial nature for an arbitrary ring R which involve the n-ary composition $\{x_1, x_2, \cdots, x_n\}$. These will appear as Lemmas 3.1–3.5 and will be instrumental in the proof of Lemma 3.6.

Lemma 3.1.

$$\{x, x, \cdots, x, y\} = (n-1)! \sum_{p+q=n-1} x^p y x^q, \qquad x, y \in R.$$

Lemma 3.2.

$$\{x, x, \cdots, x, z, w\} =$$

$$(n-2)! \sum_{i+j+k=n-2} (x^i z x^j w x^k + x^k w x^j z x^i) \qquad x, z, w \in R.$$

Proof. Fix an i, j, k position (schematically $\underset{i}{—}\ z\ \underset{j}{—}\ w\ \underset{k}{—}$). For each such there is a reverse position $\underset{k}{—}\ w\ \underset{j}{—}\ z\ \underset{i}{—}$). In each of the above positions there are exactly $(n-2)!$ arrangements of the x's, and the lemma is thereby proved. ∎

Lemma 3.3.

$$\{u, \cdots, u, v, \{u, \cdots, u, r\}\} =$$

$$(n-1)!\,(n-2)! \sum_{\substack{i+j+k=n-2 \\ p+q=n-1}} \left(u^i v u^{j+p} r u^{q+k} + u^{k+p} r u^{q+j} v u^i\right) \quad u, v, r \in R.$$

Proof. Setting $w = \{u, \cdots, u, r\}$ and using Lemma 3.2 we have

$$\{u, \cdots, u, v, \{u, \cdots, u, r\}\} = \{u, \cdots, u, v, w\}$$

$$= (n-2)! \sum_{i+j+k=n-2} (u^i v u^j w u^k + u^k w u^j v u^i)$$

which in turn equals (applying Lemma 3.1 to w)

$$(n-1)!(n-2)! \sum_{\substack{i+j+k=n-2 \\ p+q=n-1}} \left(u^i v u^{j+p} r u^{q+k} + u^{k+p} r u^{q+j} v u^i\right).$$

Lemma 3.4.

$$\{u, \cdots, u, \{u, \cdots, u, v, r\}\} =$$

$$(n-1)!(n-2)! \sum_{\substack{i+j+k=n-2 \\ p+q=n-1}} \left(u^{p+i} v u^j r u^{k+q} + u^{p+k} r u^j v u^{i+1}\right)$$

$$u, v, r \in R.$$

Proof. Setting $w = \{u, \cdots, u, v, r\}$ and using Lemma 3.1 we have

$$\{u, \cdots, u, \{u, \cdots, u, v, r\}\} = \{u, \cdots, u, w\} = (n-1)! \sum_{p+q=n-1} u^p\, w\, u^q$$

which in turn equals (applying Lemma 3.2 to w)

$$(n-1)!(n-2)! \sum_{\substack{i+j+k=n-1 \\ p+q=n-2}} u^p \left(u^i v u^j r u^k + u^k r u^j v u^i\right) u^q$$

$$= (n-1)!(n-2)! \sum_{\substack{i+j+k=n-1 \\ p+q=n-2}} \left(u^{p+i} v u^j r u^{k+q} + u^{p+k} r u^j v u^{i+q}\right).$$

Lemma 3.5.

$$\{x,\cdots,x,x^2,\{x,\cdots,x,y\}\}$$
$$= \{x,\cdots,x,\{x,\cdots,x,x^2,y\}\} \qquad x,y\in R.$$

Proof. The result follows immediately by replacing u by x, v by x^2, and r by y in Lemmas 3.3 and 3.4. ∎

For the remainder of this section we shall assume that ϕ is an n-Jordan isomorphism ($n \geq 3$) of R_1 onto R_2, where R_1, R_2 are prime rings, char. $R_2 = 0$ or is larger than $2m(m+1)$, $m = 4n-8$, and R_2 is not PI.

Lemma 3.6. *If $x \in R_1$ is such that $\phi(x)$ is not algebraic over C_2 of degree $\leq m$, then $[\phi(x^2), \phi(x)] \in Z_2$.*

Proof. Let $x, y \in R_1$ and set $u = \phi(x)$, $v = \phi(x^2)$, $r = \phi(y)$. Applying ϕ to Lemma 3.5 we have

$$\{u,\cdots,u,v,\{u,\cdots,u,r\}\} = \{u,\cdots,u,\{u,\cdots,u,v,r\}\}.$$

Now, by subtracting the result of Lemma 3.4 from that of Lemma 3.3, we see that

$$(n-1)!(n-2)!\sum\left(u^i[v,u^p]\,u^j r u^{k+q} + u^{k+p} r u^j [u^q,v] u^i\right) = 0,$$

that is (in view of [7, Theorem 5]),

$$\sum_{\substack{i+j+k=n-2\\ p+q=n-1}} \left(u^i[v,u^p]u^j \underset{C_2}{\otimes} u^{k+q} + u^{k+p} \underset{C_2}{\otimes} u^j[u^q,v]u^i\right) = 0. \tag{3.1}$$

We may assume that $k+q < 2n-3$ since otherwise we would have $k = n-2$, $q = n-1$ whence $p = 0$. Likewise we may assume that $k+p < 2n-3$. By assumption $1, u, u^2, \cdots, u^{2n-4}$ are C_2-independent. We claim that the left-hand "coefficient" of the term $(\quad)\otimes u^{2n-4}$ in (3.1) is just $[v,u]$. Indeed, from $k+q = 2n-4$ we see that either $k = n-3$ and $q = n-1$ or $k = n-2$

and $q = n-2$. But $k = n-3$, $q = n-1$ forces $p = 0$. Hence we are left with $k = n-2$, $q = n-2$ which forces $i = j = 0$ and $p = 1$, and the claim is established. From properties of the tensor product we may then conclude that $[v,u]$ lies in the C_2-span of $1, u, \cdots, u^{2n-4}$ and we write

$$[v,u] = \lambda_0 + \lambda_1 u + \cdots + \lambda_{2n-4} u^{2n-4} \qquad \lambda_i \in C_2. \tag{3.2}$$

In particular $[v,u]$ commutes with u, with the result that

$$[v, u^p] = p u^{p-1} [v,u].$$

Setting $k + q = 2n - 3 - \ell$ we have

$$\ell = 2n - 3 - k - q = (n - 2 - k) + (n - q) = (i + j) + p.$$

We now write (3.1) as

$$\sum_{\ell=1}^{2n-3} z_\ell u^{\ell-1} [v,u] \otimes u^{2n-3-\ell} + \sum u^{k+p} \otimes u^j [u^q, v]\, u^i = 0 \tag{3.3}$$

where z_ℓ is a suitable element in the prime field of C. At this point we are only interested in determining z_ℓ when $\ell = 2n-3$. In this case we have $p = n-1$ and $i + j = n-2$, $i = 0, 1, \cdots, n-2$ and so we conclude that $z_{2n-3} = (n-1)^2$. From tensor product considerations it follows that

$$(n-1)^2\, u^{2n-4} [v,u] \in \mathrm{sp}\left\{1, u, \cdots, u^{2n-4}\right\}$$

whence

$$u^{2n-4}[v,u] \in \mathrm{sp}\left\{1, u, \cdots, u^{2n-4}\right\} \tag{3.4}$$

since char. $R > n-1$. Multiplication of (3.2) by u^{2n-4} yields

$$u^{2n-4}[v,u] = \lambda_0 u^{2n-4} + \lambda_1 u^{2n-3} + \cdots + \lambda_{2n-4}\, u^{4n-8}. \tag{3.5}$$

Comparing (3.4) and (3.5) we have

$$\lambda_0 u^{2n-4} + \lambda_1 u^{2n-3} + \cdots + \lambda_{2n-4} u^{4n-8} \in \mathrm{sp}\left\{1, u, \cdots, u^{2n-4}\right\}.$$

Since u is not algebraic of degree $\leq 4n-8$ it follows that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{2n-4} = 0,$$

whence

$$[\phi(x^2), \phi(x)] = [v, u] = \lambda_0 \in Z_2$$

and the proof is complete. ∎

Lemma 3.7. *Let $s \in \mathbb{N}$ such that $s \leq m$. Then for every $u \in R_2$ there exists $r \in R_2$ such that none of the elements r, $r+u$, $r+2u$, $r+3u$ are algebraic of degree $\leq s$ over C_2.*

Proof. Suppose the lemma is not true, *i.e.*, for every $r \in R_2$ at least one of the elements r, $r+u$, $r+2u$, $r+3u$ is algebraic of degree $\leq s$. Then R_2 satisfies the GPI

$$\prod_{k=0}^{3} S_{s+1}\left(x_k y_k,\ x_k(z+ku)\, y_k, \cdots, x_k(z+ku)^s y_k\right) = 0.$$

Here u is fixed, the variables z, x_k, y_k, $k = 0,1,2,3$ are free, and S_{s+1} denotes the standard identity in $s+1$ variables. Rewriting this polynomial as $\sum_{i=0}^{2s(s+1)} T_i$, where each T_i is homogeneous in z of degree i, we see (using the assumption that char. $R > 2m(m+1)$ that R_2 satisfies each T_i. In particular R_2 satisfies $T_{2s(s+1)}$ which is a PI, a contradiction to our assumption that R_2 is not PI.

Lemma 3.8.

$$[\phi(x^2), \phi(x)] \in Z_2 \textit{ for all } x \in R_1.$$

Proof. Pick any $x \in R_1$. By Lemma 3.7 (with $s = m$) there exists $r = \phi(y) \in R_2$ such that for each $k = 0,1,2,3$, $r + k\phi(x) = \phi(y+kx)$ is not algebraic of degree $\leq m = 4n-8$. Therefore by Lemma 3.6 we have

$$\left[\phi\left((y+kx)^2\right),\ \phi(y+kx)\right] \in Z_2, \qquad k = 0,1,2,3. \tag{3.6}$$

Writing (3.6) in the form $a_0 + ka_1 + k^2a_2 + k^3a_3 \in Z_2$ it follows easily that $a_i = 0$, $i = 0, 1, 2, 3$. In particular $a_3 = [\phi(x^2), \phi(x)] \in Z_2$.

Lemma 3.9. *There exist $\lambda \in C_2$ and maps $\alpha, \beta : R_1 \to C_2$, α additive, such that*

$$\phi(x^2) = \lambda\,\phi(x)^2 + \alpha(x)\,\phi(x) + \beta(x) \text{ for all } x \in R_1.$$

Proof. By Lemma 3.8 we have $[\phi(x^2), \phi(x)] \in Z_2$, $x \in R_1$, which may be rewritten as

$$\left[\phi\left(\left(\phi^{-1}(y)\right)^2\right),\ y\right] \in Z_2 \text{ for all } y \in R_2.$$

By [2, Theorem 2] we get $\left[\phi\left(\phi^{-1}(y)^2\right), y\right] = 0$ for all $y \in R_2$. Since R_2 does not satisfy the standard identity S_4 we may then invoke [3, Theorem 1] to conclude that there exist $\lambda \in C_2$ and maps $\mu, \gamma : R_2 \to C_2$, μ additive, such that

$$\phi\left(\phi^{-1}(y)^2\right) = \lambda y^2 + \mu(y)y + \gamma(y), \qquad y \in R_2.$$

In other words $\phi(x^2) = \lambda\,\phi(x)^2 + \mu\left(\phi(x)\right)\phi(x) + \gamma\left(\phi(x)\right)$ for all $x \in R_1$. With $\alpha = \mu\phi$ and $\beta = \gamma\phi$ the proof is complete. ∎

Theorem 3.10. *Let ϕ be an n-Jordan isomorphism $(n \geq 3)$ of R_1 onto R_2, where R_1, R_2 are prime rings,* char. $R_2 = 0$ *or is larger than $2m(m+1)$, $m = 4n - 8$, and R_2 is not* PI. *Then there exists $\beta \in C_2$ such that $\beta^{n-1} = 1$ and a homomorphism or antihomormorphism $\theta : R_1 \to R_2C_2$ such that $\phi(x) = \beta\theta(x)$, $x \in R_1$.*

Proof. We have $\phi(x^2) = \lambda\,\phi(x)^2 + \alpha(x)\,\phi(x) + \beta(x)$, with λ, α, β as according to Lemma 3.9. We first claim that $\lambda \neq 0$. Suppose $\lambda = 0$, *i.e.* $\phi(x^2) = \alpha(x)\phi(x) + \beta(x)$. Linearizing, we then have

$$\phi(xy + yx) - \alpha(x)\phi(y) - \alpha(y)\phi(x) \in C_2, \quad x, y \in R_1.$$

We make the subclaim that $\phi(x^k)$ lies in sp $\{\phi(x), 1\}$ for every $k \geq 1$. For $k = 2$ this is certainly true. Now suppose it is true for k. As

$$\begin{aligned} &2\phi(x^{k+1}) - \alpha(x)\phi(x^k) - \alpha(x^k)\phi(x) \\ &\quad = \phi(x^kx + xx^k) - \alpha(x)\phi(x^k) - \alpha(x^k)\phi(x) \in C_2, \end{aligned}$$

it follows that $\phi(x^{k+1}) \in \mathrm{sp}\,\{\phi(x),\, 1\}$. The subclaim is thereby established. In particular $\phi(x)^n = \phi(x^n) \in \mathrm{sp}\{\phi(x),\, 1\}$ for all $x \in R_1$. As $\phi(x)$ is an arbitrary element of R_2 this implies that R_2 is PI, contrary to assumption, and therefore our claim is verified.

We now define $\theta : R_1 \to R_2C_2 + C_2$ by the rule

$$\theta(x) = \lambda\, \phi(x) + \frac{\alpha(x)}{2}, \qquad x \in R_1. \tag{3.7}$$

Clearly θ is additive and it is straightforward to verify that

$$\epsilon(x) = \theta(x^2) - \theta(x)^2$$

lies in C_2. Linearizing, we then have

$$\theta(xy + yx) - \theta(x)\theta(y) - \theta(y)\theta(x) \in C_2, \qquad x, y \in R_1.$$

In particular, $2\theta(x^3) - \theta(x)\theta(x^2) - \theta(x^2)\theta(x) \in C_2$, *i.e.*,

$$2\theta(x^3) - \theta(x)\left(\theta(x)^2 + \epsilon(x)\right) - \left(\theta(x)^2 + \epsilon(x)\right)\theta(x) \in C_2, \ \ i.e.,$$

$$\epsilon(x)\theta(x) \in C_2, \qquad x \in R_1.$$

Thus for each $x \in R_1$ either $\epsilon(x) = 0$ or $\theta(x) \in C_2$. We claim that either $\epsilon(x) = 0$ for all $x \in R_1$ or $\theta(x) \in C_2$ for all $x \in R_1$. If not there exist $x, y \in R_1$ such that $\epsilon(x) \neq 0$, $\theta(x) \in C_2$ and $\epsilon(y) = 0$, $\theta(y) \notin C_2$. From the definition of ϵ and the additivity of θ it is straightforward to show that $\epsilon(x+y)+\epsilon(x-y) = 2\epsilon(x)+2\epsilon(y) = 2\epsilon(x) \neq 0$. Therefore either $\epsilon(x+y) \neq 0$ or $\epsilon(x-y) \neq 0$. On the other hand $\theta(x \pm y) = \theta(x) \pm \theta(y) \notin C_2$, a contradiction is reached, and our claim is established. If $\theta(x) \in C_2$ for all $x \in R_1$ it follows from (3.7) that $\phi(x) \in C_2$ for all $x \in R_1$. Thus R_2 is commutative, in contradiction to R_2 being not PI, and so we must conclude that $\epsilon(x) = 0$ for all $x \in R_1$, *i.e.*, θ is a Jordan homomorphism. In particular $\theta(x^n) = \theta(x)^n$ for all $x \in R_1$. Therefore

$$\begin{aligned}\lambda\, \phi(x)^n + \frac{\alpha(x^n)}{2} &= \lambda\, \phi(x^n) + \frac{\alpha(x^n)}{2} \\ &= \phi(x^n) = \left(\lambda\, \phi(x) + \frac{\alpha(x)}{2}\right)^n \\ &= \lambda^n \phi(x)^n + n\lambda^{n-1}\, \frac{\alpha(x)}{2}\, \phi(x)^{n-1} + \cdots + \frac{\alpha(x)^n}{2^n}.\end{aligned} \tag{3.8}$$

Since R_2 is not a PI ring it contains elements that are not algebraic of degree $\leq n$. It follows from (3.8) that $\lambda^n = \lambda$ and accordingly, since $\lambda \neq 0$, that $\lambda^{n-1} = 1$. Also (3.8) shows that for every $x \in R_1$ either $\alpha(x) = 0$ or $\phi(x)$ is algebraic of degree $\leq n-1$. Suppose that, for some $x \in R_1$, $\phi(x)$ is algebraic of degree $\leq n-1$. By Lemma 3.7 (with $s = n-1$) there exists $r = \phi(y) \in R_2$ such that neither r nor $r + \phi(x)$ is algebraic of degree $\leq n-1$. Therefore $\alpha(y) = 0$ and $\alpha(x+y) = 0$. Since α is additive we see that $\alpha(x) = 0$, and so we have shown that $\alpha(x) = 0$ for all $x \in R_1$. Thus we have $\theta(x) = \lambda\,\phi(x)$, $\lambda^{n-1} = 1$, where θ is a Jordan homomorphism of R_1 into R_2C_2 whose image is λR_2. As we have noted earlier (in the proof of Theorem 2.2) this suffices to conclude (in view of [6, Lemma 10]) that θ is either a homomorphism or an antihomomorphism. Setting $\beta = \lambda^{-1}$ we have $\beta^{n-1} = 1$, $\phi(x) = \beta\,\theta(x)$, $x \in R_1$ and the proof is complete. ∎

4. n-Rings and the General Case

The results of the preceding two sections (Theorem 2.2 and Theorem 3.10) essentially yield the Main Theorem in the special case that ϕ is an n-Jordan isomorphism from a prime ring R_1 onto a prime ring R_2. In order to reduce the general case (*i.e.*, ϕ is an n-Jordan homomorphism of an arbitrary ring R_1 onto a prime ring R_2) to the above special case, the key result needed is that $\ker(\phi)$ is a prime ideal of R_1 (Theorem 4.9). This will therefore be our goal in the present section and will in fact result as a corollary to a general analysis of n-rings (which we hope may be of independent interest in its own right).

We begin by defining an n-ring to be a set R with an addition $x + y$ and an n-ary operation $\{x_1, x_2, \cdots, x_n\}$ such that

(i) $R, +$ is an abelian group,

(ii) $\{\,,\cdots,\,\}$ is symmetric, *i.e.*

$$\{x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}\} = \{x_1, x_2, \cdots, x_n\} \text{ for all } x_i \in R,\ \sigma \in S_n.$$

(iii) $\{ , \cdots , \}$ is n-additive, *i.e.*, for $i = 1, 2, \cdots, n$

$$\{x_1, \cdots, x_{i-1},\ y_i + z_i,\ x_{i+1}, \cdots, x_n\}$$
$$= \{x_1, \cdots, x_{i-1},\ y_i,\ x_{i+1}, \cdots, x_n\}$$
$$+ \{x_1, \cdots, x_{i-1},\ z_i,\ x_{i+1}, \cdots, x_n\}$$

for all $x_j, y_i, z_i \in R$.

An additive subgroup U of R will be called an n-ideal if

$$\{x_1, \cdots, x_{i-1},\ u, x_{i+1}, \cdots, x_n\} \in U$$

for all $x_j \in R$, $u \in U$. For us the important example of an n-ring is formed by replacing the ordinary multiplication of an associative ring R by the operation

$$\{x_1, x_2, \cdots, x_n\} = \sum_{\sigma \in S_n} x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n},$$

the n-ary operation already used in Section 1. Clearly any ideal of R is an n-ideal. A 2-ideal is just a Jordan ideal of R, and in this situation Herstein has proved definitive results concerning the relationship between Jordan ideals and ordinary ideals. Most of our efforts in this section will be to develop an analogous theory relating n-ideals to ideals. For the following series of lemmas (Lemmas 4.1–4.6) R will denote an arbitrary ring and U an n-ideal of R.

Lemma 4.1. *For $x \in R$, $u \in U$*

$$(n-1)!\ \left(xu^{n-1} + uxu^{n-2} + \cdots + u^{n-1}x\right) \in U.$$

Proof. The conclusion follows immediately from the observation that for each $k = 0, 1, \cdots, n-1$, $u^k x u^{n-1-k}$ appears exactly $(n-1)!$ times in

$$\{x, u, \cdots, u\} \in U. \quad ▮$$

Replacement of x by xu in Lemma 4.1 yields

Lemma 4.2. *For $x \in R$, $u \in U$*

$$(n-1)!\,\left(xu^n + uxu^{n-1} + \cdots + u^{n-1}xu\right) \in U.$$

Lemma 4.3. *For $x \in R$, $u \in U$*

$$(n-1)!\,[x, u^n] \in U.$$

Proof.

$$(n-1)!\,[x,u^n] = (n-1)!\,\Big([x,u]u^{n-1}$$

$$+ u[x,u]u^{n-2} + \cdots + u^{n-1}[x,u]\Big) \in U$$

by Lemma 4.1 (with $[x,u]$ playing the role of x). ∎

For $n = 2$ we recall the following result due to Herstein [5, Theorem 1.1].

Lemma 4.4. *For $n = 2$, $u \in U$*

$$4Ru^2R \subseteq U.$$

Proof. First we note that

$$2(xu^2 - u^2x) = 2\left([x,u]u + u[x,u]\right) \in U.$$

Then from $2u^2 \in U$ we have $2(xu^2 + u^2x) \in U$. Combining these remarks we see that $4xu^2 \in U$. It follows that $4xu^2y + y(4xu^2) \in U$ where $x, y \in R$. Thus $4xu^2y \in U$ and the proof is complete. ∎

Lemma 4.5. *For $n \geq 3$, $x \in R$, $u \in U$*

$$n!\,xu^n \in U \quad \textit{and} \quad n!\,u^nx \in U.$$

Proof. Since U is an n-ideal we see first of all that $\{x, u^2, u, \cdots, u\} \in U$. Expanding we write

$$\sum_{k=0}^{n} \alpha_k u^k x u^{n-k} \in U, \qquad \alpha_k \in \mathbb{Z},$$

and proceed to determine α_k. For $k = 0$ or $k = n$ it is easy to see that $\alpha_0 = \alpha_n = (n-1)!$. For $k = 1$ (or $k = n-1$) we ask in how many ways does uxu^{n-1} appear, and a counting argument shows this to be $(n-2)(n-2)!$ ways. For $1 < k < n-1$ we consider $u^k x u^{n-k}$ and note that two cases arise:

(a) u^k is the product of u^2, $\underbrace{u, \cdots, u}_{k-2}$ and u^{n-k} is the product of $\underbrace{u, u, \cdots, u}_{n-k}$. A counting argument shows that u^k may be chosen in $\binom{n-2}{k-2}(k-1)!$ ways and for each of these ways u^{n-k} may be chosen in $(n-k)!$ ways. The total number of ways is therefore $\binom{n-2}{k-2}(k-1)!(n-k)! = (n-2)!(k-1)$.

(b) u^k is the product of $\underbrace{u, u, \cdots, u}_{k}$ and u^{n-k} is the product of $u^2, u, \cdots, u$. A counting argument shows that u^k may be chosen in $\binom{n-2}{k}k!$ ways and for each of these ways u^{n-k} may be chosen in $(n-k-1)!$ ways. The total number of ways is therefore

$$\binom{n-2}{k} k!\,(n-k-1)! = (n-2)!(n-k-1).$$

Adding the results of (a) and (b) we see that

$$(n-2)!(k-1) + (n-2)!(n-k-1) = (n-2)!(n-2).$$

Putting the preceding calculations together we have

$$\begin{aligned} \{x, u^2, \underbrace{u, \cdots, u}_{n-2}\} &= (n-1)!(xu^n + u_n x) + (n-2)(n-2)! \\ &\cdot (uxu^{n-1} + \cdots + u^{n-1}xu) \in U. \end{aligned} \tag{4.1}$$

Rearrangement of (4.1) yields

$$\begin{aligned} &(n-1)!\,xu^n + [(n-1)! - (n-2)(n-2)!]\,u^n x \\ &+ (n-2)(n-2)!\,\left[u^n x + u^{n-1}xu + \cdots + uxu^{n-1}\right] \in U. \end{aligned} \tag{4.2}$$

Making use of $(n-1)! - (n-2)(n-2)! = (n-2)!$ and multiplying (4.2) by $n-1$ we see that

$$\begin{aligned} &(n-1)(n-1)!\,xu^n + (n-1)!\,u^n x + (n-2)(n-1)! \\ &\cdot \left[u^n x + \cdots + uxu^{n-1}\right] \in U \end{aligned} \tag{4.3}$$

whence by Lemma 4.2

$$(n-1)(n-1)!\,xu^n + (n-1)!\,u^n x \in U. \tag{4.4}$$

By Lemma 4.3 we have

$$(n-1)!\,xu^n - (n-1)!\,u^n x \in U. \tag{4.5}$$

Addition of (4.4) and (4.5) gives us

$$n!\,xu^n \in U. \tag{4.6}$$

By symmetry $n!\,u^n x \in U$ and the proof is complete. ▮

Lemma 4.6. *For $n \geq 3$, $u \in U$*

$$(n!)^2 R u^{n(n-1)} R \subseteq U.$$

Proof. Let $x, y \in R$, $u \in U$. Since $n!\,u^n \in U$ we see that

$$n!\left\{xu^n, y, \underbrace{u^n, \cdots, u^n}_{n-2}\right\} \in U. \tag{4.7}$$

In the expansion of (4.7) we note by Lemma 4.5 that all summands ending or beginning with u^n lie in U. Using a counting argument we conclude that $n!(n-2)!\,xu^{n(n-1)}y \in U$. In particular $(n!)^2 Ru^{n(n-1)}R \subseteq U$. ▮

Our next result generalizes Herstein's result (Lemma 4.4) and may be of independent interest.

Theorem 4.7. *Let $n > 1$ be given and let R be a semiprime ring of characteristic $> n(n-1)$. If $U \neq 0$ is an n-ideal of R then there exists k, $1 \leq k \leq n(n-1)$ such that $0 \neq (n!)^2 Ru^k R \subseteq U$ for some $u \in U$ (hence every nonzero n-ideal of R contains a nonzero ideal of R).*

Proof. We consider first the case when $n = 2$. By Lemma 4.4, $4Ru^2R \subseteq U$ for all $u \in U$. Suppose $4Ru^2R = 0$ for all $u \in U$. Thus $Ru^2R = 0$ whence

$u^2 = 0$ for all $u \in U$. Linearizing, we have $uv + vu = 0$ for all $u, v \in U$. In particular

$$u(ux + xu) + (ux + xu)u = 2uxu = 0$$

for all $u \in U$, $x \in R$, a contradiction to the semiprimeness of R.

By Lemma 4.6 there is a least positive integer $k \leq n(n-1)$ such that $(n!)^2 Ru^k R \subseteq U$ for all $u \in U$. Suppose $(n!)^2 Ru^k R = 0$ for all $u \in U$, and hence $Ru^k R = 0$ for all $u \in U$. Since R is semiprime we conclude

$$u^k = 0 \text{ for all } u \in U. \tag{4.8}$$

We note that $k > 1$ since $U \neq 0$. Linearization of (4.8) produces

$$\{u_1, u_2, \cdots, u_k\} = 0 \text{ for all } u_i \in U. \tag{4.9}$$

In particular we see from (4.9) that

$$\{\underbrace{u, \cdots, u}_{k-1}, v\} = 0,$$

where

$$v = \Big\{u,\, u^{k-2}x,\, \underbrace{u^{k-1}x, \cdots, u^{k-1}x}_{n-2}\Big\} \tag{4.10}$$

for all $u \in U$, $x \in R$.

In any summand s of v the factor u can only appear either (i) at the end of s or (ii) directly in front of the factor $u^{k-2}x$.

In (i) the factor $u^{k-2}x$ must appear at the beginning of s (since otherwise any summand in (4.10) involving s will be 0). Therefore the totality of summands of v of the form (i) is $(n-2)!(u^{k-2}x)(u^{k-1}x)^{n-2}u$. Consequently the totality of summands of (4.10) involving summands of v of type (i) is

$$\begin{aligned} &(k-1)!(n-2)!\, u(u^{k-2}x)(u^{k-1}x)^{n-2}u\, u^{k-2} \\ &= (k-1)!(n-2)!(u^{k-1}x)^{n-1}u^{k-1}. \end{aligned} \tag{4.11}$$

The totality of summands of v of type (ii) is clearly

$$(n-1)(n-2)!(u^{k-1}x)^{n-1}$$

and consequently the totality of summands in (4.10) involving summands of v of type (ii) is

$$(k-1)!(n-1)(n-2)!(u^{k-1}x)^{n-1}u^{k-1}. \tag{4.12}$$

Adding the results of (4.11) and (4.12) we see that

$$\begin{aligned} 0 &= \{u, \cdots, u, v\} \\ &= n(k-1)!(n-2)!(u^{k-1}x)^{n-1}u^{k-1}. \end{aligned} \tag{4.13}$$

Since char. $R > n(n-1)$ we conclude from (4.13) that

$$(u^{k-1}x)^n = 0 \text{ for all } u \in U,\ x \in R.$$

Thus $u^{k-1}R$ is a nil right ideal of bounded degree $\leq n$. Since R is semiprime it follows that $u^{k-1} = 0$ for all $u \in U$ [5, Lemma 1.1]. But this is in contradiction to the minimality of k and so the proof is complete. ∎

We now define an n-ring R to be n-prime if for any n-ideals

$$U_1, U_2, \cdots, U_n,\ \{U_1, U_2, \cdots, U_n\} = 0 \text{ implies } U_i = 0$$

for some i. Accordingly, we define an n-ideal U to be an n-prime ideal of R in case R/U is an n-prime ring.

Lemma 4.8. *If R is a prime ring of characteristic $>n(n\text{-}1)$ then R is n-prime.*

Proof. Suppose to the contrary that $\{U_1, U_2, \cdots, U_n\} = 0$ where $0 \neq U_i$ is an n-ideal, $i = 1, 2, \cdots, n$. By Theorem 4.7 each U_i contains an ideal $J_i \neq 0$. Setting $J = \bigcap_{i=1}^{n} J_i$ we have $\{J, J, \cdots, J\} = 0$ where $J \neq 0$. In particular $n!x^n = \{x, x, \cdots, x\} = 0$ for all $x \in J$, whence $x^n = 0$ for all $x \in J$. Thus J is nil of bounded degree n, a contradiction to the semiprimeness of R. ∎

We are now ready to apply the results of this section to n-Jordan maps. Clearly any n-Jordan map preserves the n-ary product $\{x_1, x_2, \cdots, x_n\}$.

Theorem 4.9. *Let ϕ be an n-Jordan map of R_1 onto R_2, where R_1 is an arbitrary ring and R_2 is a prime ring of characteristic $> n(n-1)$. Then* $\ker(\phi)$ *is a prime ideal of R_1.*

Proof. We set $K = \ker(\phi)$ and let $\mathcal{S}$ be the set of all ideals of R_1 contained in K. $\mathcal{S}$ is nonempty since $\{0\} \in \mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a maximal member J. We note that if J' is also a maximal member of $\mathcal{S}$ then $J + J' \subseteq K$, which shows that $J = J'$ is unique.

We claim first that J is a semiprime ideal. If not suppose U is an ideal of R_1 properly containing J and such that $U^2 \subseteq J$. We know by the maximality of J that $U \not\subseteq K$. Then

$$\{U, U, R_1, \cdots, R_1\} \subseteq U^2 \subseteq J \subseteq K. \tag{4.14}$$

But the n-ring R/K is n-isomorphic to the n-ring R_2. By Lemma 4.8 R_2 is an n-prime ring, and hence K is an n-prime ideal of R. From (4.14) we obtain the contradiction that $U \subseteq K$, and so we must conclude that J is a semiprime ideal of R_1.

We next claim that J is in fact a prime ideal of R_1. If not, suppose U, V are ideals of R_1properly containing J and such that $UV \subseteq J$. Again we note by the maximality of J that $U \not\subseteq K$ and $V \not\subseteq K$. We also remark that $VU \subseteq J$ since $(VU)^2 \subset J$ and J is a semiprime ideal of R_1. Thus

$$\{U, V, R_1, \cdots, R_1\} \subseteq UV + VU \subseteq J \subseteq K,$$

a contradiction since K is an n-prime ideal. Therefore J is a prime ideal of R_1.

We are now in the situation where R_1/J is a prime ring and K/J is an n-ideal of R_1/J with the property that K/J contains no nonzero ideals of R_1/J. We claim that char. $R_1/J > n(n-1)$. Suppose to the contrary that $mx \in J$ for some $0 < m \leq n(n-1)$ and $x \in J$. Since $J \subseteq K$ we have $mx \in K$ and so $x \in K$ since R_1/K is n-isomorphic to R_2 and hence of characteristic $> n(n-1)$. Thus the set $I_m = \{r \in R_1 \mid mr \in J\}$ is an ideal of R_1 contained in K, and we have a contradiction since I_m properly contains

J. Therefore char. $R_1/J > n(n-1)$. The conditions of Theorem 4.7 (applied to the semiprime ring R_1/J) are now satisfied and so we may conclude that if $K/J \neq 0$ then K/J contains a nonzero ideal of R_1/J. In other words, there is an ideal of R_1 properly containing J and which is contained in K, a contradiction to the maximality of J. We must therefore conclude that $K/J = 0$, *i.e.* $K = J$ is a prime ideal of R_1. ∎

With three key results now in place (Theorem 2.2, Theorem 3.10, and Theorem 4.9) we now refer the reader back to Section 1 for the proof of the Main Theorem.

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REFERENCES

[1] M. Brešar, On generalized biderivations and related maps, *J. Algebra* **172** (1995), 764–786.

[2] M. Brešar, On a generalization of the notion of centralizing mappings, *Proc. Amer. Math. Soc.* **114**(1992), 641–649.

[3] M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), 525–546.

[4] I.N. Herstein, Jordan homomorphisms, *Trans. Amer. Math. Soc.* **81**(1956), 331–341.

[5] I.N. Herstein, *Topics in Ring Theory.* The University of Chicago Press, Chicago, 1969.

[6] E. Killam, A note on n-Jordan mappings, *Com. in Algebra.* To appear.

[7] W.S. Martindale, Lie isomorphisms of prime rings, *Trans. Amer. Math. Soc.* **142**(1969), 437–455.

[8] L.H. Rowen, Some results on the center of a ring with polynomial identity, *Bull. Amer. Math. Soc.* **79**(1973), 219–223.