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# Bernoulli wavelet method for numerical solution of anomalous infiltration and diffusion modeling by nonlinear fractional differential equations of variable order

Devendra Chouhan [a,*], Vinod Mishra [b], H.M. Srivastava [c,d,e]

[a] Department of Mathematics, Institute of Engineering and Science, IPS Academy, Indore 452012, Madhya Pradesh, India
[b] Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148106, Punjab, India
[c] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, V8W 3R4, Canada
[d] Department of Medical Research, China Medical University Hospital, Taichung 40402, Taiwan, Republic of China
[e] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ 1007, Baku, Azerbaijan

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ABSTRACT

In this paper, generalized fractional-order Bernoulli wavelet functions based on the Bernoulli wavelets are constructed to obtain the numerical solution of problems of anomalous infiltration and diffusion modeling by a class of nonlinear fractional differential equations with variable order. The idea is to use Bernoulli wavelet functions and operational matrices of integration. Firstly, the generalized fractional-order Bernoulli wavelets are constructed. Secondly, operational matrices of integration are derived and utilize to convert the fractional differential equations (FDE) into a system of algebraic equations. Finally, some numerical examples are presented to demonstrate the validity, applicability and accuracy of the proposed Bernoulli wavelet method.


## 1. Introduction

Fractional calculus is an old mathematical topic from 17th century [1]. Fractional calculus are increasingly used in modeling of practical problems in various areas of engineering and physics such as continuum and statistical mechanics [2], fluid mechanics [3] dynamic of viscoelastic materials [4], econometrics [5], electromagnetism [6], propagation of spherical flames [7], $\Psi$-Hilfer problem [8], fractional PDE-constrained optimization problems [9] and in many other fields [10–18]. Due to the fractional order kernel of these FDEs, analytic solutions are usually difficult to obtain [19]. Therefore extensive research has been performed on the development of numerical methods for the solution of FDEs such as variational iteration method [20], Adomian decomposition method [21], fractional differential transform method [22], operational approach [23–25] and wavelet methods like Chebyshev wavelet method [26], Legendre wavelet method [27], Haar wavelet method [28–30], Shannon wavelet method [31], Taylor wavelet collocation method [32] and cubic B-spline wavelet collocation method [33].

Wavelet theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering and other science disciplines. It has many applications in signal analysis, image processing, data compression, detection of submarine and aircrafts, prediction of earthquake and early detection of breast cancer etc.

* Corresponding author.
*E-mail address:* dchouhan100@gmail.com (D. Chouhan).

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Anomalous infiltration and diffusion modeling given by a class of nonlinear FDEs of variable order has many applications [34–36]. This paper aims at numerically solving problems of anomalous infiltration and diffusion modeling which is given by a class of nonlinear FDEs of variable order using Bernoulli wavelet method. Bernoulli wavelet functions have advantages like orthogonality, simple to understand, easy to use and provide better approximated solutions.

In the present paper we try to find the numerical solution of the following nonlinear variable order fractional differential equations for anomalous infiltration and diffusion modeling

$$\mathbb{D}_t^{\beta(t)} f(t) + \lambda_1 f'(t) + \lambda_2 f(t) + \lambda_3 f''(t) f(t) = g(t), \quad f(0) = f_0 \tag{1}$$

where $0 < \beta(t) \leq 1, \quad g(t) \in L^2[0, 1]$ is known and $f(t) \in L^2[0, 1]$ is the unknown function. $\lambda_1, \lambda_2, \lambda_3$ and $f_0$ are all constant.

The present paper is organized as follows. In Section 2, we introduce some basic definitions of fractional calculus, mathematical preliminaries and notations of a class of nonlinear FDEs of variable order. In Section 3, mathematical meaning, existence, uniqueness and well-posedness of the proposed model (1) of nonlinear variable order fractional differential equations for anomalous infiltration and diffusion is explained. In Section 4, we construct the generalized fractional order Bernoulli wavelet functions and function approximation using Bernoulli wavelet. Main results are given in Section 5, where wavelet method is presented. In Section 6, theorems on convergence and error estimation of the proposed method are presented. In Section 7, several numerical examples are given. In the last, conclusions are drawn in Section 8.

## 2. Preliminaries and notations

In this section some necessary definitions and mathematical preliminaries of the fractional calculus theory, Wavelet theory and Bernoulli wavelets are given which will be used further in this paper.

**Definition 2.1.** A real function $\upsilon(z), z > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $\upsilon(z) = z^p \upsilon_1(z)$, where $\upsilon_1(z) \in C[0, \infty]$ and it is said to be in the space $C_\mu^n$ if $\upsilon^{(n)} \in C_\mu, n \in \mathbb{N}$.

**Definition 2.2** (*Fractional Calculus*)**.** The Riemann–Liouville fractional integral operator $\mathbb{I}^\gamma$ of order $\gamma > 0$ on the usual Lebesgue space $L^1[a, b]$ is given as [37],

$$(\mathbb{I}^\gamma \upsilon)(z) = \frac{1}{\Gamma(\gamma)} \int_0^z (z - x)^{\alpha - 1} \upsilon(x) dx, \quad z > 0$$

$$(\mathbb{I}^0 \upsilon)(z) = \upsilon(z)$$

Where $\upsilon \in C_\mu, \quad \mu \geq -1$

Its fractional derivative of order $\gamma > 0$ is given by

$$(\mathbb{D}^\gamma \upsilon)(z) = \left(\frac{d}{dz}\right)^n (\mathbb{I}^{n-\gamma} \upsilon)(z) \quad \text{for} \quad n - 1 < \gamma \leq n$$

where $n$ is an integer and $\upsilon \in C_1^n$

Now by the Riemann–Liouville fractional integral definition, we have

(i) $\mathbb{I}^\gamma z^a = \frac{\Gamma(a+1)}{\Gamma(\gamma+a+1)} z^{\gamma+a}$
(ii) $\mathbb{I}^\gamma \mathbb{I}^\beta \upsilon(z) = \mathbb{I}^{\gamma+\beta} \upsilon(z)$
(iii) $\mathbb{I}^\gamma \mathbb{I}^\beta \upsilon(z) = \mathbb{I}^\beta \mathbb{I}^\gamma \upsilon(z)$
(iv) $\mathbb{I}^\gamma \big(\alpha_1 \upsilon(z) + \alpha_2 u(z)\big) = \alpha_1 \mathbb{I}^\gamma \upsilon(z) + \alpha_2 \mathbb{I}^\gamma u(z)$

where $\gamma, \beta \geq 0, \ z > 0, \ a > -1$ and $\alpha_1, \alpha_2$ are constants.

There are many disadvantages of Riemann–Liouville derivatives when trying to model real world phenomena with fractional differential equations. Therefore we need to introduce a modified fractional differential operator $\mathbb{D}_C^\gamma$ proposed by Caputo.

$$\mathbb{D}_C^\gamma \upsilon(z) = \frac{1}{\Gamma(n - \gamma)} \int_0^z (z - x)^{n-\gamma-1} \upsilon^{(n)}(x) dx, \quad (n - 1 < \gamma \leq n)$$

where $n$ is an integer $z > 0$ and $\upsilon \in C_1^n$

Liouville–Caputo derivative operator $\mathbb{D}_C^\gamma$ has some useful properties, for an integer $n$, $\upsilon \in C_1^n$ and $z > 0$.

(i) $\mathbb{I}^\gamma \mathbb{D}_C^\gamma \upsilon(z) = \upsilon(z)$
(ii) $\mathbb{I}^\gamma \mathbb{D}_C^\gamma \upsilon(z) = \upsilon(z) - \sum_{k=0}^{n-1} \upsilon^{(k)}(0^+) \frac{z^k}{k!}, \qquad n - 1 < \gamma \leq n$
(iii)

$$\mathbb{D}_C^\gamma z^a = \begin{cases} 0, & \gamma \in \mathbb{N}_0, \quad a < \gamma \\ \frac{\Gamma(a+1)}{\Gamma(a-\gamma+1)} z^{a-\gamma}, & \text{otherwise} \end{cases}$$

(iv) $\mathbb{D}_C^\gamma \alpha = 0$

**Definition 2.3** (*Wavelets and Bernoulli Wavelets*)**.** Wavelets is a family of functions constructed from dilation parameter '$a$' and translation parameter '$b$' of a single function called the 'mother wavelet' $\psi(x)$. They are defined by [38]

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \quad a, b \in R,\ a \neq 0$$

Now for the discrete values of $a$ and $b$, $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, where $n$ and $k$ are positive integers. We have the following family of discrete wavelets:

$$\psi_{k,n}(x) = |a|^{-1/2}\psi(a_0^k x - nb_0)$$

where $\psi_{k,n}(x)$ forms a basis of $L^2(R)$ [39].

**Bernoulli Wavelets:** The Bernoulli wavelets $\psi_{nm}(x) = \psi(k, \hat{n}, m, x)$ have four arguments $\hat{n} = n-1, n = 1, 2, 3, ---, 2^{k-1}$, $k$ can be any positive integer, $m$ is the degree of the Bernoulli polynomials and $x$ is the normalized time. They are defined on the interval [0, 1) by [40].

$$\psi_{nm}(x) = \begin{cases} 2^{(k-1)/2}\mathcal{B}_m(2^{k-1}x - \hat{n}) & \text{for} \quad \frac{\hat{n}}{2^{k-1}} \leq x < \frac{\hat{n}+1}{2^{k-1}} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\mathcal{B}_m(x) = \begin{cases} \dfrac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2\mathbb{B}_{2m}}{(2m!)}}}\mathbb{B}_m(x), & m > 0 \\ 1, & m = 0 \end{cases}$$

and $m = 0, 1, 2, ---, M-1$. Here $\mathbb{B}_m(x)$ are the Bernoulli polynomials of order $m$, which are defined on the interval [0,1] as [41]

$$\mathbb{B}_m(x) = \sum_{j=0}^{m}\binom{m}{j}\mathbb{B}_{m-j}x^j$$

where $\mathbb{B}_j = \mathbb{B}_j(0), j = 0, 1, ---, m$ are Bernoulli numbers. The first four Bernoulli polynomials are

$$\mathbb{B}_0(x) = 1, \quad \mathbb{B}_1(x) = x - \frac{1}{2}, \quad \mathbb{B}_2(x) = x^2 - x + \frac{1}{6}, \quad \mathbb{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

Bernoulli polynomials form a complete basis over the interval [0, 1] with the following condition [42],

$$\int_0^1 \mathbb{B}_m(x)\mathbb{B}_n(x)dx = (-1)^{n-1}\frac{(m!)(n!)}{(m+n)!}\mathbb{B}_{m+n}, \quad m, n \geq 1$$

## 3. Mathematical meaning, existence, uniqueness and well-posedness of the proposed model (1)

Below we briefly describe the model presented by Eq. (1). The main purpose of this section is to establish its existence, uniqueness and mathematical well-posedness.

Let us give a less cursory description of the system which we consider, from both a mathematical and physical point of view. We do not enter the physical details and merely mention those which are related to the structure of our equations.

In (1), $\lambda_1$ represents infiltration coefficient, $\lambda_2$ represents diffusion coefficients and $g(t)$ represents the concentration or probability density function of the particles. In many anomalous infiltration–diffusion phenomena, the diffusion behavior changes with the time evolution. Particularly, it becomes more Fickian with the time in some diffusion processes (from anomalous diffusion to normal diffusion). This kind of phenomena extensively exists in experimental measurements of various fields, such as biophysics, plasma physics, and econophysics. In addition, still in some diffusion processes the diffusion rate decreases with the time climbing (from normal diffusion to sub-diffusion). From the traditional viewpoints, most scholars are apt to integer order derivative equations associated with time dependent diffusion coefficient to simulate the time dependent diffusion process. However, we believe this approach does not capture the origin of the problems. Though in some special experimental cases, this method can provide a good data fitting, it cannot be extended to the general formulation for time dependent diffusion processes. As an alternative approach, the time dependent nonlinear variable order fractional differential equations for anomalous infiltration and diffusion model, given in (1), is the right choice to depict this type of anomalous infiltration and diffusion phenomena.

Theorem on existence and uniqueness of the solution of the model(1).

**Theorem.** Let $\Omega \in \mathbb{R}^n$, be an open and bounded set with a smooth boundary $\partial\Omega$, which is the disjunct union of $\partial\Omega_0$ and $\partial\Omega_1$. Let $T > 0$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{N}$ and $\mathbb{C}$ is a generic constant. Let $0 < \beta(t) \leq 1$, $g(t) \in L^2[0, 1]$ then model (1) has a unique solution in $f \in \mathcal{L}\big(\Omega; \mathbb{C}(0, T); \beta_{(0,1)}\big)$ in $(0, T)$.

**Proof.** The proof is based on contraction argument. To this purpose we define a suitable metric space. Let $f \in \mathcal{L}\big(\Omega; \mathtt{C}(0,T); \beta_{(0,1)}\big)$ and $\tau \in (0,T)$. We denote by $(\beta_\tau, d)$ a complete metric space.

$$\beta_\tau = \mathcal{L}^\infty\Big(\Omega, \mathtt{C}([0,1]\times[0,\tau];[0,1])\Big) \times \mathtt{C}\big(\overline{\Omega}\times[0,\tau];\mathbb{R}^n\big)$$

where $\mathcal{L}^\infty\Big(\Omega, \mathtt{C}([0,1]\times[0,\tau];[0,1])\Big)$ and $\mathtt{C}\big(\overline{\Omega}\times[0,\tau];\mathbb{R}^n\big)$ are endowed with their natural metrics as normed spaces and $\mathcal{L}\big(\Omega; \mathtt{C}(0,T); \beta_{(0,1)}\big)$ is endowed with the metric

$$\sup_{\Omega} \max_{t\in[0,1]} \mathbb{D}_t^{\beta(t)}\big(f_t, \overline{f_t}\big)$$
$$\big\|f_t - f_{J(t)}\big\|^2_{\Omega_0^2} \to 0 \text{ as } J \to \infty$$

Hence for any function $f(t) \in \mathcal{L}_0^2\big(\Omega; \mathtt{C}(0,T); \beta_{(0,1)}\big)$, we have by dominated convergence theorem

$$f_J(t) = \Omega_{\beta,J} f(t) + \sum_{j=-1}^{J-1}\sum_{k=1}^{n_j} d_{j,k} g_{j,k-2}(t)$$

and the approximation order is $O(2^{-4J})$ if $f(t)$ is sufficiently smooth, where

$$\Omega_{\beta,J} f(t) = \beta_{(0,1)}\mathtt{C}_1\lambda_1(2^J t) + \mathtt{C}_2\lambda_2(2^J t) + \mathtt{C}_3\lambda_3\big(2^J(T-t)\big) + \beta_{(0,1)}\big(2^J(\tau - t)\big)$$

Suppose $N = 2^J\mathtt{C} + 3$ and $\Omega_j(t)$ s a $1\times N$ vector as

$$\begin{aligned}\Omega_J(t) = \Big[&\lambda_1(2^J t), \lambda_2(2^J t), \lambda_3\big(2^J(L-t)\big), \lambda_1\big(2^J(L-t)\big), f_{-1,-1}(t), \psi_{-1,0}(t), ---\\ &f_{-1,n_{-1}-2}(t), f_{0,-1}(t), f_{0,0}(t), f_{0,1}(t), ---, f_{0,L-3}(t), f_{0,n_0-2}(t)\\ &f_{1,-1}(t), f_{1,0}(t), f_{1,1}(t), ---, f_{1,2L-3}(t), f_{1,n_1-2}(t)\\ &---\\ &f_{J-1,-1}(t), f_{J-1,0}(t), f_{J-1,1}(t), --- f_{J-1,n_J-3}(t), f_{J-1,n_{J-1}-2}(t)\Big]\\ \triangleq &\Big[\omega_1(t), \omega_2(t), ---, \omega_N(t)\Big]\end{aligned}$$

$f_J(t)$ can be rewritten as

$$f_J(t) = \sum_{k=1}^{N} \widehat{f}_k \omega_k(t) = \Omega_J(t)\hat{f}$$

where

$$\hat{f} = \Big[\hat{f}_1, \hat{f}_2, ---, \hat{f}_N\Big]^T$$

So by Fubini–Tonelli theorem there exists a solution $f \in \mathcal{L}\big(\Omega; \mathtt{C}(0,T); \beta_{(0,1)}\big)$ of Model (1).

Now it is obvious for Borel regular probability measure $\Omega$ in [0, 1] for $t \in [0,T]$ and for injective function $f \in \big(\Omega; \mathtt{C}(0,T); \beta_{(0,1)}\big)$, that the solution of Model (1) is unique.

## 4. Generalized variable order fractional Bernoulli wavelet functions and function approximation using Bernoulli wavelet

### 4.1. Generalized variable order fractional Bernoulli wavelet functions

Using Bernoulli wavelets, variable order fractional Bernoulli wavelet functions (VOFBWFs) on the interval [0,1) can be defined as follows

$$\psi_{n,m}^{\alpha(t)}(x) = \begin{cases} 2^{(k-1)/2}\mathcal{B}_m(2^{k-1}x^{\alpha(t)} - \hat{n}) & \text{for } \frac{\hat{n}}{2^{k-1}} \le x^{\alpha(t)} < \frac{\hat{n}+1}{2^{k-1}} \\ 0 & \text{otherwise} \end{cases}$$

with

$$\mathcal{B}_m(2^{k-1}x^{\alpha(t)} - \hat{n}) = \begin{cases} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2\mathbb{B}_{2m}}{(2m!)}}}\mathbb{B}_m(2^{k-1}x^{\alpha(t)} - \hat{n}), & m > 0 \\ 1, & m = 0 \end{cases}$$

where $\alpha(t) > 0, \hat{n} = n-1, m = 0, 1, 2, ---, M-1, n = 1, 2, 3, ---, 2^{k-1}$ and $\mathcal{B}_m(x)$ are well known Bernoulli polynomials of order $m$ [43].

Now for using VOFBWFs on the interval $[0, h]$, we define generalized variable order fractional Bernoulli wavelet functions (GVOFBWFs) by changing $x$ into $t/l$, in the following way [44]

$$\psi_{n,m}^{l\alpha(u)}(t) = \begin{cases} 2^{(k-1)/2}\mathcal{B}_m\left(2^{k-1}\frac{t^{\alpha(u)}}{l^{\alpha(u)}} - \hat{n}\right) & \text{for } \frac{\hat{n}}{2^{k-1}} \le \frac{t^u}{l^u} < \frac{\hat{n}+1}{2^{k-1}} \\ 0 & \text{otherwise} \end{cases}$$

with

$$\mathcal{B}_m\left(2^{k-1}\frac{t^{\alpha(u)}}{l^{\alpha(u)}} - \hat{n}\right) = \begin{cases} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2\mathbb{B}_{2m}}{(2m!)}}}\mathbb{B}_m\left(2^{k-1}\frac{t^{\alpha(u)}}{l^{\alpha(u)}} - \hat{n}\right), & m > 0 \\ 1, & m = 0 \end{cases}$$

### *4.2. Generalized variable order fractional Bernoulli functions*

Variable order Fractional Bernoulli functions (VOFBFs) $\mathbb{B}_m^{\alpha(u)}(x)$ can be defined using Bernoulli polynomials as [45]

$$\mathbb{B}_m^{\alpha(u)}(x) = \sum_{j=0}^{m}\binom{m}{j}\mathbb{B}_{m-j}^{\alpha(u)}x^{\alpha(u)j}$$

where $\mathbb{B}_j^{\alpha(u)} = \mathbb{B}_j^{\alpha(u)}(0), j = 0, 1, ---, m$ are Bernoulli numbers. The first few variable order fractional Bernoulli functions are [46]

$$\mathbb{B}_0^{\alpha(u)}(x) = 1, \quad \mathbb{B}_1^{\alpha(u)}(x) = x^{\alpha(u)} - \frac{1}{2}, \quad \mathbb{B}_2^{\alpha(u)}(x) = x^{2\alpha(u)} - x^{\alpha(u)} + \frac{1}{6},$$
$$\mathbb{B}_3^{\alpha(u)}(x) = x^{3\alpha(u)} - \frac{3}{2}x^{2\alpha(u)} + \frac{1}{2}x^{\alpha(u)}$$

Now by changing variable $x$ into $t/l$, we can define generalized variable order fractional Bernoulli functions by using variable order fractional Bernoulli functions on the interval $[0, h]$ as

$$\mathbb{B}_m^{l\alpha(u)}(t) = \sum_{j=0}^{m}\binom{m}{j}\mathbb{B}_{m-j}^{l\alpha(u)}\frac{t^{\alpha(u)j}}{l^{\alpha(u)j}}$$

On the interval $[0, h]$, the first few generalized Variable order fractional Bernoulli functions are [47]

$$\mathbb{B}_0^{l\alpha(u)}(t) = 1, \quad \mathbb{B}_1^{l\alpha(u)}(t) = \frac{t^{\alpha(u)}}{l^{\alpha(u)}} - \frac{1}{2}, \quad \mathbb{B}_2^{l\alpha(u)}(t) = \frac{t^{2\alpha(u)}}{l^{2\alpha(u)}} - \frac{t^{\alpha(u)}}{l^{\alpha(u)}} + \frac{1}{6},$$
$$\mathbb{B}_3^{l\alpha(u)}(t) = \frac{t^{3\alpha(u)}}{l^{3\alpha(u)}} - \frac{3}{2}\frac{t^{2\alpha(u)}}{l^{2\alpha(u)}} + \frac{1}{2}\frac{t^{\alpha(u)}}{l^{\alpha(u)}}$$

On the interval $[0, h]$, the generalized variable order fractional Bernoulli functions satisfy the following relation [48]

$$\int_0^l \mathbb{B}_m^{l\alpha(u)}(t)\mathbb{B}_n^{l\alpha(u)}(t)t^{\alpha(u)-1}dt = \frac{l^{\alpha(u)}}{\alpha(u)}(-1)^{n-1}\frac{(m!)(n!)}{(m+n)!}\mathbb{B}_{m+n}^{\alpha(u)}, \quad m, n \ge 1$$

### *4.3. Function approximation using generalized variable order fractional Bernoulli wavelets*

A function $f(t)$ defined over $L^2[0, l]$ may be expanded by Generalized variable order fractional Bernoulli wavelets as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\psi_{nm}^{l\beta(u)}(t) \tag{2}$$

where $a_{nm} = \left(f(t), \psi_{nm}^{l\beta(u)}(t)\right)$ in which ( , ) denotes the inner product.

If the infinite series in Eq. (2) is truncated, then it can be rewritten as

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\psi_{nm}^{l\beta(u)}(t) = A^T\Psi^{l\beta(u)}(t) \tag{3}$$

where $T$ indicates transposition and $A$ and $\Psi^{l\beta(u)}(t)$ are $\hat{m} = 2^{k-1}M$ column vectors given by

$$A = \left[a_{1,0}, a_{1,1}, \ldots, a_{1,M-1}, a_{2,0}, \ldots, a_{2,M-1}, \ldots, a_{2^{k-1},0}, \ldots, a_{2^{k-1},M-1}\right]^T$$

$$\Psi^{l\beta(u)}(t) = \left[\psi_{1,0}^{l\beta(u)}(t), \psi_{1,1}^{l\beta(u)}(t), \ldots, \psi_{1,M-1}^{l\beta(u)}(t), \ldots, \psi_{2^{k-1},0}^{l\beta(u)}(t), \ldots, \psi_{2^{k-1},M-1}^{l\beta(u)}(t)\right]^T$$

To get the value of $A$, we suppose

$$f_{ij} = \langle f, \psi_{ij}^{l\beta(u)} \rangle = \int_0^l f(t)\psi_{ij}^{l\beta(u)}(t)t^{\beta(u)-1}dt$$

Using Eq. (3), we have

$$f_{ij} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm} \int_0^l \psi_{nm}^{l\beta(u)}\psi_{ij}^{l\beta(u)}(t)t^{\beta(u)-1}dt = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}p_{nm}^{ij}$$

$i = 1, 2, \ldots, 2^{k-1}, \quad j = 0, 1, 2, \ldots, M-1.$

Here

$$p_{nm}^{ij} = \int_0^l \psi_{nm}^{l\beta(u)}\psi_{ij}^{l\beta(u)}(t)t^{\beta(u)-1}dt$$

Hence

$$f_{ij} = A^T\left[p_{1,0}^{ij}, p_{1,1}^{ij}, \ldots, p_{1,M-1}^{ij}, \ldots, p_{2^{k-1},0}^{ij}, \ldots, p_{2^{k-1},M-1}^{ij}\right]^T$$

Now

$$F^T = A^T P$$

where

$$F = \left[f_{1,0}, f_{1,1}, \ldots, f_{1,M-1}, f_{2,0}, \ldots, f_{2,M-1}, \ldots, f_{2^{k-1},0}, \ldots, f_{2^{k-1},M-1}\right]^T$$

and

$$P = \left[p_{nm}^{ij}\right]$$

Here $P$ is the matrix of order of $2^{k-1}M \times 2^{k-1}M$ and

$$P = \langle \Psi^{l\beta(u)}, \Psi^{l\beta(u)} \rangle = \int_0^l \Psi^{l\beta(u)}(t)\Psi^{l\beta(u)T}(t)t^{\beta(u)-1}dt \tag{4}$$

Finally, $A^T$ in Eq. (3) is given by

$$A^T = F^T P^{-1}$$

## 5. Main results

### 5.1. Transformation of differential operators into matrix forms

In this section we discuss technique to transform differential operators into matrix forms for getting numerical solution of variable order fractional differential equations.

Now

$$\mathbb{D}_t f(t) = Ef(t) = \begin{bmatrix} H & 0 & 0 & \cdots & 0 \\ 0 & H & 0 & \cdots & 0 \\ 0 & 0 & H & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H \end{bmatrix} f(t) \tag{5}$$

where

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m-1 & 0 \end{bmatrix} \tag{6}$$

Here the number of $H$ in $E$ is $n$. Now using (4), (5) and (6) we have following equation

$$\mathbb{D}_t \Psi^{l\beta(u)}(t) = P^{-1}EP\Psi^{l\beta(u)}(t) = W\Psi^{l\beta(u)}(t) \tag{7}$$

Now by (3) and (7), we have following result

$$\mathbb{D}_t f(t) = \mathbb{D}_t\left(A^T\Psi^{l\beta(u)}(t)\right) = A^T(P^{-1}EP)\Psi^{l\beta(u)}(t)$$

Similarly for variable order fractional differential equation we can obtain following

$$\mathbb{D}_t^{\beta(t)} f(t) = Lf(t) = \begin{bmatrix} G & 0 & 0 & \cdots & 0 \\ 0 & G & 0 & \cdots & 0 \\ 0 & 0 & G & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G \end{bmatrix} f(t)$$

where

$$G = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma\left(2-\beta(t)\right)}t^{-\beta(t)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma\left(3-\beta(t)\right)}t^{-\beta(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma\left(m+1-\beta(t)\right)}t^{-\beta(t)} \end{bmatrix}$$

The number of $G$ in matrix $L$ is $n$. Now following equation can be obtained

$$\mathbb{D}_t^{\beta(t)} \Psi^{l\beta(u)}(t) = (P^{-1}LP)\Psi^{l\beta(u)}(t) = R\Psi^{l\beta(u)}(t) \tag{8}$$

Using (3) and (7), we have

$$\mathbb{D}_t^{\beta(t)} f(t) = A^T(P^{-1}LP)\Psi^{l\beta(u)}(t)$$

Finally Eq. (1) can be rewritten into the following matrix form

$$A^TR\Psi^{l\beta(u)}(t) + \lambda_1A^TW\Psi^{l\beta(u)}(t) + \lambda_2A^T\Psi^{l\beta(u)}(t) + \lambda_3A^TW^2\Psi^{l\beta(u)}(t)A^TW\Psi^{l\beta(u)}(t) = g(t) \tag{9}$$

where $W$ and $R$ are given by (7) and (8) respectively. This discretized system can be analyzed for stability and solved by Cubic B-Spline collocation iteration method [30]. By calculating the values of $\Psi^{l\beta(u)}(t)$ and $g$ on [0, 1] and by solving the above system of algebraic equations using computer aided technique like MATLAB, we can obtain the unknown $A$.

## 6. Theorems on convergence and error estimation

In this section, some theorems on convergence analysis and error estimation of proposed method are given.

**Theorem 6.1.** *The solution of problem (1), given by series solution equation (3), using Bernoulli wavelet method converges towards $f(x)$.*

**Proof.** Suppose $\psi_{nm}(x) = 2^{(k-1)/2}\mathcal{B}_m(2^{k-1}x - \hat{n})$, where $\psi_{nm}(x)$ form a basis of $L^2(R)$ and let $L^2(R)$ be a Hilbert space.

Let

$$f(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\psi_{nm}^{l\beta(u)}(x)$$

where

$$a_{nm} = \langle f(x), \psi_{nm}^{l\beta(u)}(x)\rangle, \qquad \text{represents an inner product.}$$

Now

$$f(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} \langle f(x), \psi_{nm}^{l\beta(u)}(x)\rangle\psi_{nm}^{l\beta(u)}(x)$$

Let us denote

$$\psi_{nm}^{l\beta(u)}(x) \text{ as } \psi(x) \text{ and } \alpha_j = \langle f(x), \psi_{nm}^{l\beta(u)}(x)\rangle$$

Suppose $\{S_n\}$ is the sequence of partial sums of $\left(\alpha_j\psi(x_j)\right)$ and $S_n, S_m$ are arbitrary partial sums with $n \geq m$. We prove $\{S_n\}$ is a Cauchy sequence in Hilbert space.

Let

$$S_n = \sum_{j=1}^{n} \alpha_j\psi(x_j)$$

So

$$\begin{aligned}\langle f(x), S_n\rangle &= \langle f(x), \sum_{j=1}^{n} \alpha_j\psi(x_j)\rangle \\ &= \sum_{j=1}^{n} \bar{\alpha}_j\langle f(x), \psi(x_j)\rangle \\ &= \sum_{j=1}^{n} \bar{\alpha}_j\alpha_j = \sum_{j=1}^{n} \left|\alpha_j\right|^2\end{aligned}$$

Now we claim that

$$\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} \left|\alpha_j\right|^2 \quad \text{for} \;\; n > m.$$

For this we have

$$\begin{aligned}\left\|\sum_{j=m+1}^{n} \alpha_j\psi(x_j)\right\|^2 &= \langle \sum_{i=m+1}^{n} \alpha_i\psi(x_i), \sum_{j=m+1}^{n} \alpha_j\psi(x_j)\rangle \\ &= \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} \alpha_i\bar{\alpha}_j\langle\psi(x_i), \psi(x_j)\rangle \\ &= \sum_{j=m+1}^{n} \alpha_j\bar{\alpha}_j \\ &= \sum_{j=m+1}^{n} \left|\alpha_j\right|^2\end{aligned}$$

Hence $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} \left|\alpha_j\right|^2 \quad \text{for} \;\; n > m.$

By Bessel's inequality, we have $\sum_{j=1}^{\infty} \left|\alpha_j\right|^2$ is convergent and hence $\|S_n - S_m\|^2 \to 0$ as $m, n \to \infty$.

Now $\|S_n - S_m\|$ converges to 0 and $\{S_n\}$ is a Cauchy sequence so suppose it converges to '$s$'. We need to prove that $f(x) = s$

$$\begin{aligned}\langle s - f(x), \psi(x_j)\rangle &= \langle s, \psi(x_j)\rangle - \langle f(x), \psi(x_j)\rangle \\ &= \langle \lim_{n\to\infty} S_n, \psi(x_j)\rangle - \alpha_j \\ &= \lim_{n\to\infty} \langle S_n, \psi(x_j)\rangle - \alpha_j \\ &= \alpha_j - \alpha_j\end{aligned}$$

$$\Rightarrow \langle s - f(x), \psi(x_j)\rangle = 0$$

Hence $f(x) = s$ and $\sum_{j=1}^{n} \alpha_j\psi(x_j)$ converges to $f(x)$.

### 6.1. Error estimation

Error estimation for the approximate solution of Eq. (1) is discussed in this part.

Suppose $\bar{f}(x)$ is the approximate solution for $f(x)$ and $E_n(x) = f(x) - \bar{f}(x)$ is the error function.

$$\bar{f}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\psi_{nm}^{l\beta(u)}(x) + H_n(x) = A^T\Psi^{l\beta(u)}(t) + H_n(x)$$

where $H_n(x)$ is the perturbation term.

$$H_n(x) = \bar{f}(x) - A^T \Psi^{l\beta(u)}(t) \tag{10}$$

Now find an approximation $\bar{E_n}(x)$ to the error function $E_n(x)$ in the same way as we did before the solution of the problem.

Subtract Eq. (10) from Eq. (3), the error function $E_n(x)$ satisfies the problem,

$$E_n(x) + A^T \Psi^{l\beta(u)}(t) = -H_n(x) \tag{11}$$

Now Eq. (11) is recalculated in the same way as we did before the solution of equation (9) for the construction of $\bar{E_n}(x)$ to $E_n(x)$.

Hence the stability of Bernoulli wavelet method is established through this convergence theorem and error estimation.

### 6.2. Accuracy and stability of the proposed method

The convergence rate of LWM [49] is two in case of variable order fractional differential equations for anomalous infiltration and diffusion modeling. Herein more detailed analysis of the error terms is performed for the solution of (1). In this section we try to estimate the accuracy and stability of the results obtained by applying the proposed Bernoulli wavelet method.

**Theorem 1.** *In case of variable order fractional differential equations for anomalous infiltration and diffusion modeling, the order of convergence of the proposed Bernoulli wavelet method is four comparing to two in case of LWM.*

**Proof.** Let us assume that $\mathbb{D}_t^{\beta}(t)f(t) \in L^2(R)$ is a continuous function on [0, 1] and $\forall\, x \in [0, 1]$, $\exists \eta : |f'(x)| \le \eta$. The error at the M$^{th}$ level resolution can be written as

$$|E_M| = |f(x) - f_M(x)| = \left| \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm} \psi_{nm}^{l\beta(u)}(x) \right|$$

therefore the quadrate of the $L^2$-norm of the error function can be expanded as

$$\begin{aligned} \|E_M\|_2^2 &= \int_0^1 \left( \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm} \psi_{nm}^{l\beta(u)}(x) \right)^2 dx \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{r=1}^{2^{r-1}} \sum_{s=0}^{2^{r-1}} a_{nm} a_{rs} \int_0^1 \psi_{nm}^{l\beta(u)}(x) \psi_{rs}^{l\beta(u)}(x) dx \end{aligned} \tag{12}$$

Here the coefficients $a_{nm}$ are bounded by

$$a_{nm} \le \eta \frac{1}{2^{n+1}} \tag{13}$$

The integrals of Bernoulli wavelets are monotonically increasing in interval [0, 1]. Thus the maximal value of the $\psi_{nm}^{l\beta(u)}(x)$ is reached in the interval $x \in \left[\xi_j(i), 1\right]$, where $\psi_{nm}^{l\beta(u)}(x)$ can be expanded as

$$\psi_{nm}^{l\beta(u)}(x) = \frac{1}{n!} \sum_{j=2}^{n} \binom{n}{j} (x - \xi_j)^{n-j} \left( \frac{1}{2^m + 1} \right)^j$$

The integral included in quadrate of the $L^2$-norm of the error function can be bounded as

$$\int_0^1 \psi_{nm}^{l\beta(u)}(x) \psi_{rs}^{l\beta(u)}(x) dx \le \left( \frac{1}{n!} \sum_{j=2}^{n} \binom{n}{j} (x - \xi_j)^{n-j} \left( \frac{1}{2^m + 1} \right)^j \left( \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} (x - \xi_j)^{r-j} \left( \frac{1}{2^s + 1} \right)^j \right) \right) \tag{14}$$

Inserting (13) and (14) in (12), we have,

$$\begin{aligned} \|E_M\|_2^2 \le \eta^2 & \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{r=1}^{2^{k-1}} \sum_{s=0}^{2^{r-1}} \frac{1}{2^{n+1}} \frac{1}{2^{r+1}} \\ & \times \left( \frac{1}{n!} \sum_{j=2}^{n} \binom{n}{j} (x - \xi_j)^{n-j} \left( \frac{1}{2^m + 1} \right)^j \right. \\ & \left. \left( \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} (x - \xi_j)^{r-j} \left( \frac{1}{2^s + 1} \right)^j \right) \right) \end{aligned}$$

Finally, the error bound can be written as, by applying geometric progression and factorization, at the number of collocation points $N = 2^k M$,

$$\|E_M\|_2 \leq \frac{\eta}{12}\left[\left(\frac{1}{N}\right)^2 + \frac{1}{28}\left(\frac{1}{N}\right)^4\right]$$

This proves Theorem 1 and establishes the accuracy and stability of the proposed BWM. It also shows that the proposed method is more convenient than other methods like LWM as it has less computational cost and more accuracy.

## 7. Numerical examples

In order to illustrate the effectiveness of the proposed method, some numerical examples are given in this section. The examples presented have exact solutions and have also been solved by some other numerical methods like Legendre Wavelet Method (LWM) [49], Method of Approximate Particular Solution (MAPS) [50], Explicit Finite Difference Approximation (EFDA) [51]. This allows us to compare numerical results obtained by the proposed BWM with the analytical solutions or those obtained by the other known methods like LWM. Absolute errors between approximate solutions $y_N$ and the corresponding exact solutions $y$, i.e. $N_e = |y_N - y|$ are considered for both proposed BWM and other known method of LWM.

**Example 1.** Let us first consider the following variable order fractional differential equation representing problems of anomalous infiltration and diffusion modeling,

$$\mathbb{D}_t^{\beta(t)} y(t) - 10y'(t) + y(t) = g(t), \quad t \in [0, 1] \tag{15}$$

where $\beta(t) = \frac{1+2e^t}{7}$, $g(t) = 10\left(\frac{t^{(2-\beta(t))}}{\Gamma(3-\beta(t))} + \frac{t^{(1-\beta(t))}}{\Gamma(2-\beta(t))}\right) + 5t^2 - 90t - 95$ and the initial condition is $y(0) = 5$. By applying the proposed method for $t_i = \frac{2i-1}{2^{k+1}M}$, for $i = 1, 2, ---, 2^k M$ with $k = 2$ and $M = 4$, numerical solutions of (15) can be obtained easily.

Table 1, shows the comparisons between the absolute errors for exact solution and numerical solutions obtained by proposed BWM and other known methods for $k = 2$ and $M = 4$.

Fig. 1, shows the exact solution and numerical solutions for $k = 4$ and $M = 4$.

**Example 2.** Now consider the following nonlinear variable order fractional differential equation which is used in problems of anomalous infiltration and diffusion modeling

$$\mathbb{D}_t^{\beta(t)} y(t) - 7y'(t) + 5y(t) - 6y''(t) = g(t), \quad t \in [0, 1] \tag{16}$$

where $\beta(t) = \frac{3(\cos t + \sin t)}{5}$, $g(t) = 5\left(\frac{2t^{(2-\beta(t))}}{\Gamma(3-\beta(t))} + \frac{3t^{(1-\beta(t))}}{\Gamma(2-\beta(t))}\right) - 275t^2 - 895t - 105$ and the initial condition $y(0) = 0$. By applying the proposed method for $t_i = \frac{2i-1}{2^{k+1}M}$, for $i = 1, 2, ---, 2^k M$ with $k = 2$ and $M = 4$, numerical solutions of (16) can be obtained easily.

Table 2, shows the comparisons between the absolute errors for exact solution and numerical solutions obtained by proposed BWM and other known methods for $k = 2$ and $M = 4$.

Fig. 2, shows the exact solution and numerical solutions for $k = 4$ and $M = 4$.

**Example 3.** Let us consider the following nonlinear variable order fractional differential equation representing problems of anomalous infiltration and diffusion modeling,

$$\mathbb{D}_t^{\beta(t)} y(t) - y'(t) + \big(y(t)\big)^2 = g(t), \quad t \in [0, 1] \tag{17}$$

where $\beta(t) = \frac{1}{(t+1)^2}$, $g(t) = 2\left(\frac{t^{(2-\beta(t))}}{\Gamma(3-\beta(t))} + \frac{3t^{(1-\beta(t))}}{\Gamma(2-\beta(t))}\right) + 2t^2 - 15t + 25$ and the initial condition is $y(0) = 0$. By applying the proposed method for $t_i = \frac{2i-1}{2^{k+1}M}$, for $i = 1, 2, ---, 2^k M$ with $k = 2$ and $M = 4$, numerical solutions of (17) can be obtained easily. Table 3, shows the comparisons between the absolute errors for exact solution and numerical solutions obtained by proposed BWM and other known methods for $k = 2$ and $M = 4$.

Fig. 3, shows the exact solution and numerical solutions for $k = 4$ and $M = 4$.

## 8. Conclusions

In this paper, numerical solution of anomalous infiltration and diffusion modeling by a class of nonlinear fractional differential equations with variable order has been obtained using generalized fractional-order Bernoulli wavelet functions based on Bernoulli wavelets. Generalized fractional-order Bernoulli wavelets and operational matrices of integration are used to convert the given fractional differential equations into a system of algebraic equations. Theorems in Section 6

**Table 1**
Absolute errors with $k = 2$ and $M = 4$.

| t | Absolute errors by MAPS | Absolute errors by EFDA | Absolute errors by LWM | Absolute errors by BWM |
|---|---|---|---|---|
| 1/32 | 1.230249e−05 | 3.542340e−07 | 8.091305e−12 | 5.23904e−15 |
| 7/32 | 3.140257e−05 | 9.250157e−07 | 2.024535e−09 | 6.06405e−13 |
| 15/32 | 2.141203e−06 | 7.354206e−08 | 9.564669e−10 | 2.14056e−12 |
| 23/32 | 3.254502e−07 | 6.329045e−08 | 1.696030e−10 | 1.60504e−12 |
| 31/32 | 8.235615e−06 | 8.302604e−07 | 1.734222e−10 | 7.24015e−14 |

**Table 2**
Absolute errors with $k = 2$ and $M = 4$.

| t | Absolute errors by MAPS | Absolute errors by EFDA | Absolute errors by LWM | Absolute errors by Proposed BWM |
|---|---|---|---|---|
| 1/32 | 4.280975e−04 | 3.781460e−08 | 1.199041e−14 | 3.450125e−16 |
| 7/32 | 6.146078e−05 | 7.651381e−09 | 1.421085e−14 | 4.234520e−16 |
| 15/32 | 1.651078e−05 | 8.304157e−09 | 2.842171e−14 | 7.580860e−15 |
| 23/32 | 7.265015e−04 | 20145023e−08 | 1.669775e−13 | 1.706807e−16 |
| 31/32 | 9.231450e−04 | 5.462018e−08 | 2.273737e−13 | 5.098056e−15 |

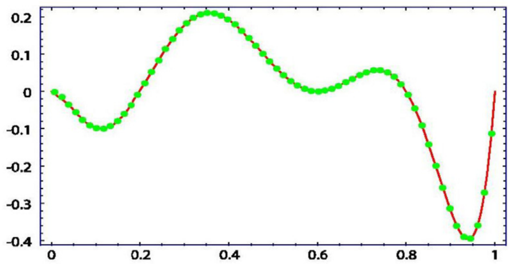


**Fig. 1.** Exact Solution and Numerical Solutions for $k = 4$ and $M = 4$ (Exact ▬▬ Numerical ●●●).

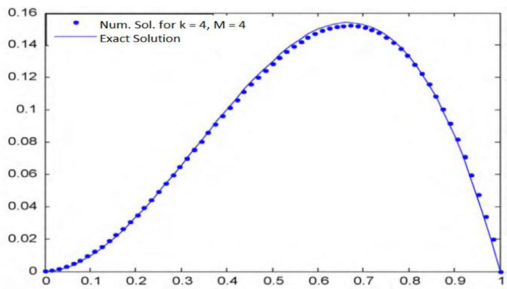


**Fig. 2.** Exact Solution and Numerical Solutions for $k = 4$ and $M = 4$.

**Table 3**
Absolute errors with $k = 2$ and $M = 4$.

| t | Absolute errors by MAPS | Absolute errors by EFDA | Absolute errors by LWM | Absolute errors by Proposed BWM |
|---|---|---|---|---|
| 1/32 | 3.380460e−04 | 8.794059e−06 | 4.370914e−08 | 6.025018e−15 |
| 7/32 | 1.724068e−04 | 2.047306e−06 | 2.705618e−08 | 8.306271e−15 |
| 15/32 | 5.907061e−05 | 6.819204e−07 | 7.208091e−08 | 7.371094e−16 |
| 23/32 | 4.127054e−05 | 9.620725e−07 | 5.740361e−09 | 3.749037e−15 |
| 31/32 | 7.530470e−04 | 1.148076e−06 | 3.506801e−08 | 5.091650e−14 |

show that the proposed BWM is better than LWM in terms of computational cost and accuracy. Numerical examples are illustrated to demonstrate the validity, accuracy and correctness of the proposed BWM. The absolute errors, obtained by proposed BWM and other known methods like MAPS, EFDA and LWM, are compared. Tables 1–3 prove that the proposed

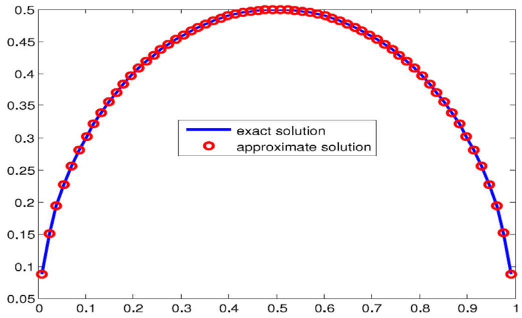


**Fig. 3.** Exact Solution and Numerical Solutions for $k = 4$ and $M = 4$.

BWM is better than other known methods like LWM, EFDA, MAPS etc. in terms of computational cost, accuracy and simplicity.

**Future Scope:** The work presented in this paper can be extended in several directions. In future research study some other wavelets like Second generation wavelets, Periodized Shannon wavelets, Legendre multi-wavelets, Empirical wavelets etc. can also be used to analyze and solve impulsive fractional integro-differential equations, neutral fractional functional differential equation, neutral stochastic functional differential equations, linear neutral multi-delay integro differential equations, delay pantograph differential equation and ill-posed spherical pseudo differential equation etc. which are used in different field of science and engineering.

Based on the proposed work, second generation wavelet can be used in geographical data analysis. One can think of topography of the earth as a function value defined on a sphere. Because of the flexibility of the lifting scheme, it is possible to create wavelets that live on a sphere. In this way, the topographic data of the earth can be compressed and manipulated much like a 1-D signal.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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