SOME EXPANSIONS IN SERIES OF BESSEL FUNCTIONS

By

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A general theorem on generating functions is applied to derive a number of interesting expansions for the generalized hypergeometric ${}_rF_s$ function in series of Bessel functions. Several further expansion formulas, relevant to the present discussion, are also considered. Many of these expansions in series of Bessel functions stem from (or are motivated by) their applicability in various seemingly diverse fields of applied sciences and engineering.

1. INTRODUCTION

In the usual notation, let $I_\nu(z)$ defined by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2n}}{n!\ \Gamma(\nu+n+1)} \qquad (|z| < \infty) \tag{1.1}$$

denote the modified Bessel function of the first kind and of order ν (*cf.*, *e.g.*, Watson [12, p. 77]). Expansions of various hypergeometric functions of one and more variables in series of $I_\nu(z)$ are scattered in the literature. Recently, Shreshtha [5] gave two such expansions for the confluent hypergeometric ${}_1F_1$ function and extended them further to hold true for the generalized hypergeometric ${}_rF_s$ function with r numerator and s denominator parameters. We recall here the most general result in Shreshtha's paper in the form (*cf*. [5, p. 296, Equation (3.2)]:

$$(1.2)\qquad (\tfrac{1}{4}z)^{\lambda}\ {}_rF_s\left[\begin{matrix}(a_r);\\ \\ (b_s);\end{matrix}\ \left(\frac{wz}{m}\right)^m\right] = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)}\, e^{z/2} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda+n)\Gamma(2\lambda+n)}{n!} I_{\lambda+n}(\tfrac{1}{2}z)$$

$$\cdot\ {}_{2m+r}F_{m+s}\left[\begin{matrix}\Delta(m;-n),\Delta(m;2\lambda+n),(a_r);\\ \\ \Delta(m;\lambda+\tfrac{1}{2}),(b_s);\end{matrix}\ w^m\right],$$

where $\lambda \neq 0, -1, -2, \ldots$; (a_r) abbreviates the array of r parameters, with similar interpretations for (b_s), <u>et cetera</u>, and the symbol $\Delta(m;\lambda)$ represents the array of m parameters

$$(\lambda+j-1)/m, \quad j = 1, \ldots, m,$$

m being a positive integer. Throughout this paper it should indeed be understood that exceptional parameter values that would render either side of an expansion formula meaningless or undefined are tacitly excluded. Thus, for example, the parameters $\lambda + \frac{1}{2}$ and $b_1, \ldots, b_s$ in the expansion formula (1.2) are assumed to be neither zero nor a negative integer.

The special case of (1.2) when $m = 1$ is a well-known result recorded, for example, by Luke [2, p. 20, Equation (4)]. A direct proof of this special case was presented by Srivastava [6] who made use of the elementary expansion

$$(1.3)\qquad (\tfrac{1}{2}z)^{\lambda} = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)}\, e^{z} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda+n)\Gamma(2\lambda+n)}{n!} I_{\lambda+n}(z),$$

where, as also in (1.2), $\lambda \neq 0, -1, -2, \ldots$. The object of the present paper is to show that much more general expansions than (1.2) can be deduced by appropriately applying a theorem on generating functions. We also derive several further results analogous to (1.2). The expansions (3.2), (3.5), (3.6), and (3.8) in series of Bessel functions are believed to be new.

2. APPLICATIONS OF A THEOREM ON GENERATING FUNCTIONS

We begin by recalling the following

THEOREM (Srivastava and Panda [11, p. 472, Theorem 2]). Corresponding to the given sequences $\{\Lambda_n\}_{n=0}^{\infty}$ and $\{\Omega_n\}_{n=0}^{\infty}$, let

(2.1) $$P_n^{(\lambda)}(x;m) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} \; (\lambda+n)_{mk}}{k!} \Lambda_k \, x^k$$

and

(2.2) $$\theta_n(t) = \sum_{r=0}^{\infty} \frac{\Omega_{n+r} \; t^r}{(\lambda+2n+1)_r}, \qquad |t| < T_0,$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, m is an arbitrary positive integer, and the complex parameter λ is neither zero nor a negative integer. Suppose also that $G(z)$ is defined by

(2.3) $$G(z) = \sum_{n=0}^{\infty} \Lambda_n \, \Omega_{mn} \frac{z^n}{n!}, \qquad |z| < S_0.$$

Then

(2.4) $$\sum_{n=0}^{\infty} \frac{(-t)^n}{n!(\lambda+n)_n} P_n^{(\lambda)}(x;m)\theta_n(t) = G(xt^m),$$

provided that $\Lambda_0 \, \Omega_0 \neq 0$, $|xt^m| < S_0$, $|t| < T_0$, and the series on the left has a meaning.

Setting

$$(2.5)\qquad \Lambda_n = \frac{\prod_{j=1}^{r}(a_j)_n \prod_{j=1}^{v}(d_j)_{mn}}{\prod_{j=1}^{s}(b_j)_n \prod_{j=1}^{u}(c_j)_{mn}} \quad \text{and} \quad \Omega_n = \frac{\prod_{j=1}^{p}(\alpha_j)_n \prod_{j=1}^{u}(c_j)_n}{\prod_{j=1}^{q}(\beta_j)_n \prod_{j=1}^{v}(d_j)_n},$$

and making use of the identity

$$(2.6)\qquad (\lambda)_{mn} = m^{mn} \prod_{j=1}^{m}\left[\frac{\lambda+j-1}{m}\right]_n \qquad (m = 1,2,3,\ldots),$$

we find from the assertion (2.4) that (cf. [7, p. 304, Equation (3.12)])

$$(2.7)\qquad {}_{mp+r}F_{mq+s}\left[\begin{matrix}\Delta[m;(\alpha_p)],\ (a_r); \\ \Delta[m;(\beta_q)],\ (b_s);\end{matrix}\ xt^m m^{m(p-q)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!\ (\lambda+n)_n}\, \Gamma_n[(\alpha_p),(c_u);(\beta_q),(d_v)]$$

$$\cdot\ {}_{p+u}F_{1+q+v}\left[\begin{matrix}(\alpha_p)+n,\ (c_u)+n; \\ \lambda+2n+1,\ (\beta_q)+n,\ (d_v)+n;\end{matrix}\ t\right]$$

$$\cdot\ {}_{m(2+v)+r}F_{mu+s}\left[\begin{matrix}\Delta(m;-n),\ \Delta(m;\lambda+n),\ \Delta[m;(d_v)],\ (a_r); \\ \Delta[m;(c_u)],\ (b_s);\end{matrix}\ xm^{m(2-u+v)}\right],$$

where $\Delta[m;(\alpha_p)]$ represents the array of mp parameters

$$(\alpha_i+j-1)/m, \quad i = 1, \ldots, p \quad \text{and} \quad j = 1, \ldots, m,$$

with similar interpretations for $\Delta[m;(\beta_q)]$, et cetera, and (for convenience)

(2.8)
$$\Gamma_n[(\alpha_p),(c_u);(\beta_q),(d_v)] = \frac{\prod_{j=1}^{p} (\alpha_j)_n \prod_{j=1}^{u} (c_j)_n}{\prod_{j=1}^{q} (\beta_j)_n \prod_{j=1}^{v} (d_j)_n}, \qquad n \geqq 0.$$

Now recall the relationship (cf. [2, p. 22, Equation (2)])

(2.9)
$$I_\lambda(z) = \frac{(\frac{1}{2}z)^\lambda}{\Gamma(\lambda+1)} e^{\pm z} {}_1F_1\left[\begin{matrix}\lambda+\frac{1}{2}; \\ 2\lambda+1;\end{matrix} \mp 2z\right],$$

where, as usual, both upper signs or both lower signs are taken simultaneously. In view of (2.9), the expansion formula (1.2) can easily be rewritten in its equivalent form:

(2.10)
$${}_rF_s\left[\begin{matrix}(a_r); \\ (b_s);\end{matrix} \left(\frac{wz}{m}\right)^m\right] = \sum_{n=0}^{\infty} \frac{(\lambda+\frac{1}{2})_n}{(2\lambda+n)_n} \frac{(-z)^n}{n!} {}_1F_1\left[\begin{matrix}\lambda+n+\frac{1}{2}; \\ 2\lambda+2n+1;\end{matrix} z\right]$$

$$\cdot {}_{2m+r}F_{m+s}\left[\begin{matrix}\Delta(m;-n),\ \Delta(m;2\lambda+n),\ (a_r); \\ \Delta(m;\lambda+\frac{1}{2}),\ (b_s);\end{matrix} w^m\right],$$

where we have also made use of the identity (2.6) with $m = 2$.

Obviously, the expansion formula (1.2) or (2.10) is a very special case of (2.7), with λ replaced by 2λ, when

$$p = q = 0, \quad u - 1 = v = 0, \quad c_1 = \lambda + \tfrac{1}{2}, \quad t = z, \quad \text{and} \quad x = (w/m)^m.$$

3. FURTHER EXPANSIONS

Making use of the relationship (cf. [12, p. 147, Equation (1)])

$$(3.1) \qquad I_\lambda(z)\ I_\mu(z) = \frac{(\frac{1}{2}z)^{\lambda+\mu}}{\Gamma(\lambda+1)\Gamma(\mu+1)}\ {}_2F_3\left[\begin{matrix}\Delta(2;\lambda+\mu+1); \\ \lambda+1,\ \mu+1,\ \lambda+\mu+1;\end{matrix}\ z^2\right],$$

we find from our expansion formula (2.7) with λ replaced by $\lambda+\mu$, and with

$$\begin{cases} p = q = 0,\ u = v = 2,\ u_1 = \frac{1}{2}(\lambda+\mu+1),\ u_2 = \frac{1}{2}(\lambda+\mu+2), \\ v_1 = \lambda+1,\ v_2 = \mu+1,\ t = z^2, \quad \text{and} \quad x = (w/m)^{2m}, \end{cases}$$

that

$$(3.2) \qquad (\tfrac{1}{2}z)^{\lambda+\mu}\ {}_rF_s\left[\begin{matrix}(a_r); \\ (b_s);\end{matrix}\ \left(\frac{wz}{m}\right)^{2m}\right]$$

$$= \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\lambda+\mu} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda+\mu+2n)(\lambda+\mu)_n}{n!} I_{\lambda+n}(z)\ I_{\mu+n}(z)$$

$$\cdot\ {}_{4m+r}F_{2m+s}\left[\begin{matrix}\Delta(m;-n),\ \Delta(m;\lambda+\mu+n),\ \Delta(m;\lambda+1),\ \Delta(m;\mu+1),\ (a_r); \\ \Delta(m;\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}),\ \Delta(m;\frac{1}{2}\lambda+\frac{1}{2}\mu+1),\ (b_s);\end{matrix}\ w^{2m}\right],$$

whose special case when $m = 1$ is known (cf., e.g., [4, p. 224, Equation (4)]).

The expansion formula (3.2) simplifies considerably in its special case when $\lambda = \mu$. An expansion analogous to (3.2), but in series of the products

$$I_{\lambda+2n}(z)\ J_{\lambda+2n}(z),$$

where [cf. Equation (1.1)]

$$J_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \, (\frac{1}{2}z)^{\nu+2n}}{n! \; \Gamma(\nu+n+1)} \qquad (|z| < \infty) \tag{3.3}$$

denotes the (ordinary) Bessel function of order ν, would also follow from the general result (2.7), since [2, p. 25, Equation (20)]

$$I_{\lambda}(z) \; J_{\lambda}(z) = \frac{(\frac{1}{2}z)^{2\lambda}}{\{\Gamma(\lambda+1)\}^2} \, {}_0F_3\left[\begin{matrix} \text{———}; \\ \Delta(2;\lambda+1), \; \lambda+1; \end{matrix} \; -\frac{z^4}{64}\right]. \tag{3.4}$$

Thus, if in (2.7) we set

$p = q = 0$, $u = v - 2 = 0$, $d_1 = \frac{1}{2}(\lambda+1)$, $d_2 = \frac{1}{2}\lambda + 1$, $t = -\frac{z^4}{64}$, and $x = (w/m)^{4m}$, we find that

$$(\tfrac{1}{2}z)^{2\lambda} \, {}_rF_s\left[\begin{matrix} (a_r); \\ (b_s); \end{matrix} \; \left(-\frac{1}{64}\right)^m \left(\frac{wz}{m}\right)^{4m}\right] \tag{3.5}$$

$$= \Gamma(\lambda+1) \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} \, I_{\lambda+2n}(z) \; J_{\lambda+2n}(z)$$

$$\cdot \; {}_{4m+r}F_s\left[\begin{matrix} \Delta(m;-n), \; \Delta(m;\lambda+n), \; \Delta(m;\frac{1}{2}\lambda+\frac{1}{2}), \; \Delta(m;\frac{1}{2}\lambda+1), \; (a_r); \\ (b_s); \end{matrix} \; w^{4m}\right],$$

whose special case when $m = 1$ does not seem to have been recorded earlier.

Yet another expansion in series of Bessel functions would result from (2.7) if we set

$$p = q = u = v = 0, \quad t = (\tfrac{1}{2}z)^2, \quad \text{and} \quad x = (w/m)^{2m},$$

and apply the definition (1.1). We thus obtain

$$(3.6) \qquad (\tfrac{1}{2}z)^{\lambda}\ {}_rF_s\left[\begin{matrix}(a_r); \\ \\ (b_s);\end{matrix}\ \left[\frac{wz}{2m}\right]^{2m}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} I_{\lambda+2n}(z)$$

$$\cdot\ {}_{2m+r}F_s\left[\begin{matrix}\Delta(m;-n),\ \Delta(m;\lambda+n),\ (a_r); \\ \\ (b_s);\end{matrix}\ w^{2m}\right],$$

whose special case when $m = 1$ yields a known result (<u>cf</u>., <u>e.g</u>., [4, p. 223, Equation (1)]).

Next we deduce a confluent case of the expansion formula (2.7) when t is replaced by λt, and x by x/λ^m, and $\lambda \to \infty$. Thus we have (<u>cf</u>. [7, p. 305, Equation (3.13)])

$$(3.7) \qquad {}_{mp+r}F_{mq+s}\left[\begin{matrix}\Delta[m;(\alpha_p)],\ (a_r); \\ \\ \Delta[m;(\beta_q)],\ (b_s);\end{matrix}\ xt^m\, m^{m(p-q)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \Gamma_n[(\alpha_p),(c_u);(\beta_q),(d_v)]\ {}_{p+u}F_{q+v}\left[\begin{matrix}(\alpha_p)+n,\ (c_u)+n; \\ \\ (\beta_q)+n,\ (d_v)+n;\end{matrix}\ t\right]$$

$$\cdot\ {}_{m(1+v)+r}F_{mu+s}\left[\begin{matrix}\Delta(m;-n),\ \Delta[m;(d_v)],\ (a_r); \\ \\ \Delta[m;(c_u)],\ (b_s);\end{matrix}\ xm^{m(1-u+v)}\right],$$

which indeed would follow also by applying another result given by Srivastava and Panda [11, p. 468, Theorem 1], using (2.5) and (2.6).

If in (3.7), we set

$$p = q = 0,\ u = v - 1 = 0,\ d_1 = \lambda + 1,\ t = (\tfrac{1}{2}z)^2, \quad \text{and} \quad x = (w/m)^{2m},$$

and make use of the definition (1.1), we shall obtain the expansion formula:

$$(\tfrac{1}{2}z)^{\lambda}\ {}_rF_s\left[\begin{matrix}(a_r);\\ \\ (b_s);\end{matrix}\ \left(\frac{wz}{2m}\right)^{2m}\right] = \Gamma(\lambda+1)\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}z)^n}{n!}\, I_{\lambda+n}(z) \tag{3.8}$$

$$\cdot\ {}_{2m+r}F_s\left[\begin{matrix}\Delta(m;-n),\ \Delta(m;\lambda+1),\ (a_r);\\ \\ (b_s);\end{matrix}\ w^{2m}\right],$$

which, when $m = 1$, yields a known result (<u>cf</u>., <u>e.g</u>., [4, p. 224, Equation (2)]).

Finally, we give a mild generalization of the expansion formula (3.7). Incidentally, this generalization of (3.7), given by Equation (3.9) below, is not contained in (2.7). Indeed, it would follow readily if we apply yet another result of Srivastava and Panda [11, p. 472, Theorem 3], again using (2.5) and (2.6). We thus find that (<u>cf</u>. [7, p. 305, Equation (3.14)])

$$(3.9)\qquad {}_{mp+r}F_{mq+s}\left[\begin{matrix}\Delta[m;(\alpha_p)],\ (a_r);\\ \Delta[m;(\beta_q)],\ (b_s);\end{matrix}\ xt^m\, m^{m(p-q)}\right]$$

$$= \beta \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (1-\alpha n+\beta)_{n-1}\, \Gamma_n[(\alpha_p),(c_u);(\beta_q),(d_v)]$$

$$\cdot\ {}_{1+p+u}F_{q+v}\left[\begin{matrix}(1-\alpha)n+\beta,\ (\alpha_p)+n,\ (c_u)+n;\\ (\beta_q)+n,\ (d_v)+n;\end{matrix}\ t\right]$$

$$\cdot\ {}_{m(2+v)+r}F_{m(2+u)+s}\left[\begin{matrix}\Delta(m;-n),\ \Delta(m;1+\beta/(1-\alpha)),\ \Delta[m;(d_v)],\ (a_r);\\ \Delta(m;\beta/(1-\alpha)),\ \Delta(m;1-\alpha n+\beta),\ \Delta[m;(c_u)],\ (b_s);\end{matrix}\ xm^{m(v-u)}\right],$$

which, for $\alpha = 0$, corresponds essentially to the expansion formula (3.7).

For numerous further applications of the special cases of the expansion formulas (2.7) and (3.7) when $m = 1$, see Erdélyi et al. [1, Chapter 7], Luke ([2, Chapters 1 and 7], [3, Chapter 9], [4, Chapter 5]), and Watson [12, Chapters 5 and 11]. As a matter of fact, each of the expansions (2.7), (3.7) and (3.9) as well as their various multivariable extensions have been considered rather systematically in the literature (see, for example, [7], [8], [9, Chapter 9], and [10]).

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