Free-Market Prices Tend to Equilibria

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Abstract. Ignoring the way the exchange of goods takes place and taking into account only the result of the trade, we analyze the relation between demand, supply, and prices. We prove that in a free-market economy ruled only by demand and supply there are infinitely many equilibria, all stable, and that prices tend either to one of the equilibria or to zero. Moreover, we show that if other forces act on the market then, generically, no equilibria exist.

1. Introduction

More than two centuries ago, Adam Smith affirmed that in a free market economy ruled only by demand and supply, prices tend to equilibria. Economists have debated this issue ever since. Equilibria exist for most mathematical models tackled in the literature, the stability of these equilibria, however, is far from obvious. The starting point of these models is the exchange of goods between agents, whereas the law of demand and supply comes only second into play. This setting has two disadvantages: it makes production hard to accommodate into the model and it leads to a trade-dependent analysis that shifts away from the essence of the law of demand and supply.

Therefore we propose here a different point of view. Ignoring any information about the exchange of goods and taking into account only the essence of the trade, we start directly from the law of demand and supply and seek the relation between prices and the difference demand − supply. This leads us to a system of differential equations that allows us to prove the following result:

Theorem. *In a free market economy ruled only by demand and supply there are infinitely many equilibria, all stable, and each price tends either to an equilibrium or to zero. If other forces act on the market then, generically, no equilibria exist.*

This result gives full credit to Adam Smith and shows how external forces influence the market, but as we will see in the last section—as paradoxical as this may seem—it also endorses somewhat the Keynesian point of view as a possible alternative. A close analysis of the equations also offers some insight into inflation and hyperinflation and proves that the latter cannot occur in a free market economy.

In Section 2 we give a mathematical form of the law of demand and supply, define the *free market*, write in a first approximation and then solve the equations that govern the price of a product, and finally prove the above theorem in this particular case. In Section 3 we analyze the whole market by considering n products and possible relations between them. In Section 4 we consider the general equations that fulfill the laws of the free market and see that the results of the previous sections are true even in this setting. In Section 5 we summarize the conclusions and discuss the perspectives of further research along the lines traced here.

2. Price Dynamics for One Product

Let $d, s\colon \mathbb{R} \to [0, \infty)$ be two differentiable[1] functions that express the *demand* and the *supply* of a certain product, where $\mathbb{R}$ denotes the set of real numbers and represents the time. Let $r\colon \mathbb{R} \to [0, \infty)$ be the difference between demand and supply,

$$r = d - s.$$

[1] The hypothesis of differentiability is unessential in the end since the differential equations we develop can be replaced by difference equations.

which we will call *defect* since it measures the imperfection of the market, and let $p: \mathbb{R} \to (0, \infty)$ be another differentiable function, which expresses the price of the product.

According to Adam Smith we have the following *laws of demand and supply*:

— if at a given time demand is lower than supply, prices decrease, so the demand increases, therefore the defect increases;
— if at a given time demand is larger than supply, prices increase, so the supply increases, therefore the defect decreases;
— if at a given time demand equals supply, prices are constant, so the defect remains constant, namely 0.

Denoting by prime the derivative and retaining only the desired information from the above laws, we can rewrite them as follows:

L1. $r(t) < 0 \Rightarrow p'(t) < 0 \Rightarrow r'(t) > 0;$
L2. $r(t) > 0 \Rightarrow p'(t) > 0 \Rightarrow r'(t) < 0;$
L3. $r(t) = 0 \Rightarrow p'(t) = 0 \Rightarrow r'(t) = 0.$

We can thus ignore the change in demand and supply and deal only with the change in defect.

Definition. An economy is a *free market* if the laws **L1**, **L2**, and **L3** are the only ones that rule the variations in price and defect.

We can now write the equations that govern a free market. From the first implication in **L1**, **L2**, and **L3**, we see that a possible way to express these laws is that the rate of change in price is proportional to the defect, so in a first approximation we can write that

$$p' \sim r. \tag{2.1}$$

Condition (2.1) leads to the equation $p' = \alpha r$, where $\alpha > 0$ is a constant. Looking at the relation between r and r' in **L1**, **L2**, and **L3**, we similarly write that

$$r' \sim -r, \tag{2.2}$$

which leads to the equation $r' = -\beta r$, where $\beta > 0$ is a constant. Combining the two equations we obtain the system

$$\begin{cases} p' = \alpha r \\ r' = -\beta r \end{cases} \quad \alpha, \beta > 0. \tag{2.3}$$

This is of course only a first approximation since we do not know if (2.1) and (2.2) truly describe the free market. Though (2.1)-(2.2) verify **L1**, **L2**, and **L3**, the actual relations describing the market might be more complicated. Therefore in Section 4 we will replace (2.3) with the more general system

$$\begin{cases} p' = \alpha f(r) \\ r' = -\beta h(r), \end{cases} \tag{2.4}$$

where f and h are functions whose properties we will describe later.

Returning to system (2.3), we see that the second equation is separable and independent on the first. Its solution is $r(t) = -(c_1/\beta)e^{-\beta t}$, where c_1 is a constant. Substituting this into the first equation we obtain

$$p' = -\frac{\alpha c_1}{\beta} e^{-\beta t}, \tag{2.5}$$

which if integrated yields $p(t) = -c_1(\alpha/\beta)e^{-\beta t} + c_2$, where c_2 is a constant. So the general solution of system (2.3) is

$$\begin{cases} p(t) = -c_1(\alpha/\beta)e^{-\beta t} + c_2 \\ r(t) = -(c_1/\beta)e^{-\beta t}. \end{cases} \tag{2.6}$$

Since $\beta > 0$, $\lim_{t\to\infty} e^{-\beta t} = 0$ and therefore $\lim_{t\to\infty} r(t) = 0$ and $\lim_{t\to\infty} p(t) = c_2$. To have a geometrical image of what this means, let us draw the phase-plane portrait. The equilibria (infinitely many) fill the positive part of the Op axis, (i.e. they are points of the form $(0, p), p > 0$, see Figure 1). Eliminating t between p and r in (2.6), we obtain

$$p = -\frac{\alpha}{\beta} r + c_2, \tag{2.7}$$

so the phase-plane portrait is the one in Figure 1.

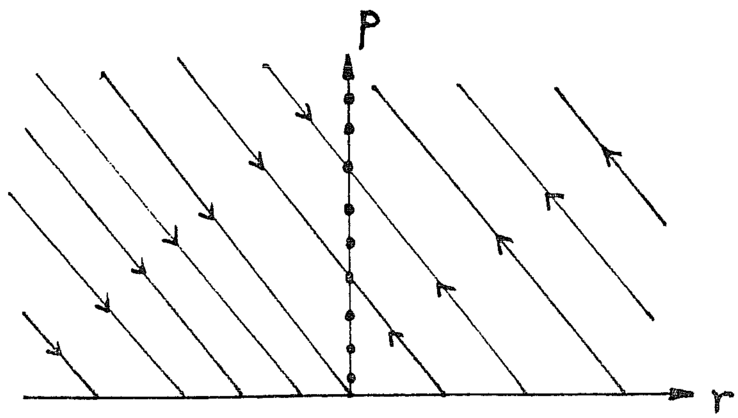

FIGURE 1.
The phase-plane portrait of system (2.3)

This shows that any possible price tends to a constant, which depends on the choice of the initial condition and that the defect tends to zero, thus proving the first part of the Theorem in this particular case. Also notice that all those solutions that do not reach an equilibrium of the Op axis represent prices that tend to zero before demand reaches supply. This happens when a certain product, still existing on the market, doesn't sell anymore.

But what happens if other forces act on the market? Let u_0 denote the sum of all forces that affect the rate of change of the price and let v_0 denote the sum of all forces that affect the rate of change of the defect. Then the new system that describes the dynamics of the market is

$$\begin{cases} p' = u_0 + \alpha r \\ r' = v_0 - \beta r. \end{cases} \tag{2.8}$$

Remark. It seems more natural to consider instead of u_0 and v_0 some time-dependent functions u and v. But at this point we will be contented with the more restrictive assumption of constancy.

In general equations (2.8) have no equilibria since the algebraic system

$$\begin{cases} u_0 + \alpha r = 0 \\ v_0 - \beta r = 0, \end{cases} \tag{2.9}$$

has solutions only if

$$\alpha v_0 + \beta u_0 = 0. \tag{2.10}$$

If condition (2.10) is fulfilled, equations (2.8) have again infinitely many equilibria, this time along the half-line $r = -u_0/\alpha$, $p > 0$. However, in the set of all possible forces $F = \{(u_0, v_0) | u_0, v_0 \in \mathbb{R}\}$, those that fulfill condition (2.10) form a subset of Lebesgue measure zero (like a line in a plane). So it is very unlikely to choose such forces when picking randomly some pair from this set. Therefore, *generically* (i.e. except for a set of forces of measure zero), system (2.8) has no equilibria. This proves the second part of the Theorem.

On the other hand system (2.8) has the general solution

$$\begin{cases} p(t) = -c_1(\alpha/\beta)e^{-\beta t} + [(\alpha/\beta)v_0 + u_0]t + c_2 \\ r(t) = c_1 e^{-\beta t} + v_0/\beta. \end{cases} \tag{2.11}$$

This shows that in the unlikely case that condition (2.10) is fulfilled, all prices tend either to zero or to an equilibrium, whereas if condition (2.10) is not fulfilled two things can happen:

— if $\alpha v_0 + \beta u_0 < 0$, then prices tend to zero after a sufficiently long interval of time: this is when there is no more demand for the given product and it occurs when one of the two forces, u_0 or v_0, exceeds the other (proportionally with respect to α and β) or when both forces work in the same direction such that the resulting force acts towards decreasing the rate of change in price or defect.

— if $\alpha v_0 + \beta u_0 > 0$, then prices rise indefinitely: this is the case of hyperinflation and it occurs when one of the two forces, u_0 or v_0, exceeds the other (proportionally with respect to α and β) or when both forces work in the same direction such that the resulting force acts towards increasing the rate of change in price or defect.

System (2.3) reflects an ideal situation of a free market in which demand and supply respond smoothly to mutual changes. In reality these changes are not smooth but rather discontinuous. The decision to increase production leads to a jump in supply, so though system (2.3) does not reflect this situation, supply can exceed demand even if demand had previously exceeded supply, and conversely. This implies that in reality a free market economy tends to reach a stage at which prices oscillate slightly around the equilibria.

However, free markets as defined above do not exist in reality. External forces always perturb this ideal situation, so system (2.8) gives a better description of real markets. But system (2.8) must be understood only locally in time because the values of u_0 and v_0 are not fixed forever. External forces are rather of the type $u(t), v(t)$, a case more difficult to study in general if not specifying the expressions of these functions. In spite of this, system (2.8) shows what happens if external forces point invariably and without opposition in only one direction. Moreover, system (2.8) proves that the desirable stage to be reached is when the external forces fulfill condition (2.10), a case in which the market acts as if it were free. The only difference is that prices stabilize at a point at which the defect is nonzero.

Though dealing only with the price dynamics of one product, equations (2.3), (2.8), and (2.10) describe the main features of the market. To see this, let us consider now the case of n market products.

3. Price Dynamics for the Market

The price dynamics of n products in a free market is described by a system of $2n$ equations:

$$\begin{cases} p_i' = \alpha_i r_i \\ r_i' = -\beta r_i \end{cases} \quad i = 1, 2, \ldots, n, \tag{3.1}$$

where p_i denotes the price and r_i is the defect of the ith product. If no other relations exist between the variables, each couple of equations corresponding to the same i is decoupled from the others, so the general solution is the same as in (2.5), namely

$$\begin{cases} p_i(t) = -c_{i1}(\alpha_i/\beta_i)e^{-\beta_i t} + c_{i2} \\ r_i(t) = -(c_{i1}/\beta_i)e^{-\beta_i t} \end{cases} \quad i = 1, 2, \ldots, n, \tag{3.2}$$

where c_{i1}, c_{i2} are constants. The same happens with system (2.8), which generalizes to

$$\begin{cases} p_i' = \alpha_i r_i + u_{i0} \\ r_i' = -\beta r_i + v_{i0} \end{cases} \quad i = 1, 2, \ldots, n, \tag{3.3}$$

whose solution is analogous to (2.11),

$$\begin{cases} p_i(t) = -c_{i1}(\alpha_i/\beta_i)e^{-\beta_i t} + [(\alpha_i/\beta_i)v_{i0} + u_0]t - c_{i2} \\ r_i(t) = c_{i1}e^{-\beta_i t} + v_{i0}/\beta_i \end{cases} \quad i = 1, 2, \ldots, n. \tag{3.4}$$

Condition (2.10) now changes into a system of n conditions, all independent of each other,

$$\beta_i u_{i0} + \alpha_i v_{i0} = 0, \quad i = 1, 2, \ldots, n. \tag{3.5}$$

For the equations (3.1), the conclusions are the same as for the equations (2.3) in the previous section. Regarding equations (3.3), if for only one i we have $(u_{i0}, v_{i0}) \neq (0, 0)$, then all the equilibria disappear, except for the unlikely case that all conditions in (3.5) are fulfilled.

In practice, however, there exist relations between commodities, which are usually expressed by algebraic equations or inequalities. If such relations occur, and if their number is not too large, this translates into a lower dimensional phase space. In general this does not alter the above conclusions about the equilibria.

4. The General Case

We will now analyze the more general case when p' and r' do not necessarily vary linearly with r. We have already seen in Section 2 that the equations that describe the market dynamics for one product are

$$\begin{cases} p' = \alpha f(r) \\ r' = -\beta h(r), \end{cases} \tag{4.1}$$

where $f, h: \mathbb{R} \to \mathbb{R}$ are differentiable functions that fulfill the conditions

C1. $f, h < 0$ for $r < 0$,
C2. $f, h > 0$ for $r > 0$,
C3. $f(0) = h(0) = 0$,
C4. $\lim_{r\to 0} \frac{\partial f}{\partial r}(r) < \infty$.

Conditions **C1**, **C2**, and **C3** were considered such that the equations (4.1) are in accord with the laws **L1**, **L2**, and **L3**, whereas condition **C4** was imposed to exclude cases in which the change of prices has nothing to do with the chosen function. For example $f(r) = 1/r$ is in this category: the rate of change of prices is not inversely proportional with the defect. We could have avoided this by asking that f is increasing; condition **C4**, however, allows more generality, for example functions that are mostly increasing but also decrease on some intervals. In some cases markets might behave naturally like this even if no external forces occur. At least the law of demand and supply as defined above does not exclude this possibility

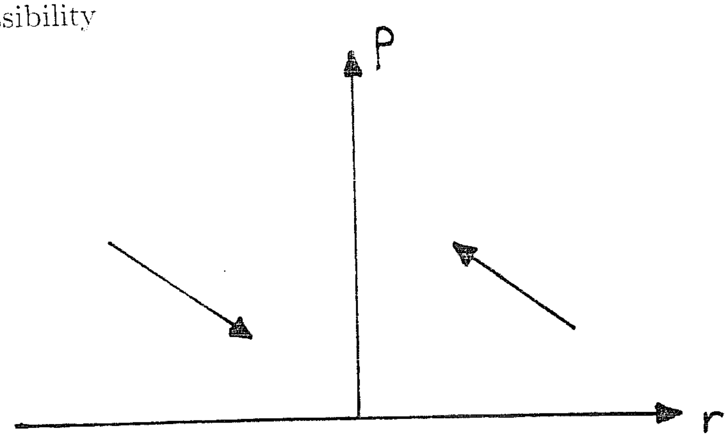

FIGURE 2.
The direction of the flow for the equations (4.1)

Though we cannot find an explicit solution for these equations since we do not have the expressions of f and h, we can still understand how the phase-plane portrait (the flow) looks like. Conditions **C1**, **C2**, and **C3** reflected in the equations (4.1) imply that the direction of the flow in the phase plane is as in Figure 2. Indeed, points of the form

$(0,p)$, $p > 0$ represent all possible equilibria, then for $r < 0$ we have $p' < 0$ and $r' > 0$, and finally for $r > 0$ we have $p' > 0$ and $r' < 0$. Condition **C4** does not allow any solution with $r > 0$ to run off the half-plane before reaching an equilibrium and keeps solutions with $r < 0$ away from tending only to $p = 0$. So the flow of equations (4.1) is qualitatively similar (topologically equivalent) with the flow of equations (2.3) in Figure 1, thus the first part of the Theorem follows for the equations (4.1) too.

The analogue of equations (2.8) is in this case given by

$$\begin{cases} p' = u_0 + \alpha f(r) \\ r' = v_0 - \beta h(r). \end{cases} \tag{4.2}$$

and the conditions for the existence of equilibria are

$$\begin{cases} u_0 + \alpha f(r) = 0 \\ v_0 - \beta h(r) = 0. \end{cases} \tag{4.3}$$

which, independently on any reasonable[2] form of f and h, lead to at least one equation between u_0, v_0, α, and β. Again, this makes the existence of the equilibria unlikely and proves the second part of the Theorem.

The step to n products is now easy to make by considering n functions like f and h, which fulfill conditions similar to **C1-C4**. The flow is topologically equivalent to the one of the equations (3.1)-(3.3) and the Theorem follows in this case too. The same comments made in Section 3 regarding relations between commodities also hold.

5. Conclusions and Further Research

On one hand the above results give credit to the *laissez-faire* adepts by showing that in a free market economy prices tend to an equilibrium. On the other hand they support the Keynesian view of the government's regulating role by pointing out that if forces work correctly, then the natural equilibrium is recovered. But the problem is: do governments know in what direction to act or are they simply following the public opinion?

A free market economy as defined above has probably never existed. The nature of human beings will make this difficult to happen in some forseeable future. Only a society of totally honest, responsible, unselfish individuals can reach the premises of such an economy. As long as self-interest groups fight to achieve advantages by breaking the natural mechanism, only opposite forces can establish an equilibrium. But we do not wish to discuss social and moral issues here. We would only like to see what the science of economy can do to improve the world's economic situation. We think that its main goal should be to identify and understand the mechanisms of the forces acting on the market. This is a difficult task, in which progress might be difficult to make without the help of other sciences.

[2] By *reasonable* we mean functions that reflect a possible economic reality, rather excluding exotic cases whose graphs are, say, curves dense in the plane, etc.

We have seen that establishing an equilibrium after the natural one is broken is improbable to achieve since the set of suitable forces has measure zero. But it may be true that markets are also endowed with an *action-reaction* law similar to the one encountered in physics. The sum of forces of certain self-interest groups may trigger an opposite reaction of approximately the same strength, such that the resultant reaches (or oscillates about) an equilibrium (i.e. some condition similar to (2.10) tends to be fulfilled). If this is true, then things are not as bad as they seem. However, the history of economics shows that at certain times this hypothetical *action-reaction* law reacts too slowly, giving rise to periods of crisis and depression. Are such events the result of the loss of the market equilibrium or are they due to the fact that certain interest groups become much stronger than others and exceed the opposing forces? Shall "giants" be set free and allowed to prosper? Further mathematical research along the lines described here might soon provide an insight into these and into similar questions.