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New Results Involving the Generalized Krätzel Function with Application to the Fractional Kinetic Equations

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Abstract: Sun is a basic component of the natural environment and kinetic equations are important mathematical models to assess the rate of change of chemical composition of a star such as the sun. In this article, a new fractional kinetic equation is formulated and solved using generalized Krätzel integrals because the nuclear reaction rate in astrophysics is represented in terms of these integrals. Furthermore, new identities involving Fox–Wright function are discussed and used to simplify the results. We compute new fractional calculus formulae involving the Krätzel function by using Kiryakova's fractional integral and derivative operators which led to several new identities for a variety of other classic fractional transforms. A number of new identities for the generalized Krätzel function are then analyzed in relation to the *H*-function. The closed form of such results is also expressible in terms of Mittag-Leffler function. Distributional representation of Krätzel function and its Laplace transform has been essential in achieving the goals of this work.

Keywords: generalized Krätzel function; fractional images; *H*-function; kinetic equation

MSC: 44A10; 44A20; 33C60; 33E12; 26A33; 44A20

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1. Introduction and Motivation

Modern gas theories and astrophysics have significant impacts in the advancement of environmental sciences. Sun plays a major role in global warming research and the evolution of stars such as the sun uses a set of differential equations [1]. The three characteristics, namely temperature, pressure, and mass, describe the internal structure of stars, which is entirely made up of gases. Actually, for the cloud to create a star, there needs be more gravitational force present than internal pressure. The cloud emits light when nuclear fusion occurs, and a protostar forms as a result. Mathematical models and structures are based on equation of state, translucence, and rate of nuclear energy production. Energy in such stars is produced by the process of nuclear reactions. Hence, the rate of change in the chemical composition of stars is described by kinetic equations as the rate of reaction for formation and destruction of each class. Haubold and Mathai [1] used the ordinary kinetic equation to examine such composition $F(t)$ in terms of the rate of destruction $d(F)$ and the rate of production $p(F)$ given by

$$\frac{dF}{dt} = -d(F_t) + p(F_t), \tag{1}$$

where F_t is the function labelled as $F_t(t^*) = F(t - t^*), t^* > 0$. Hence, if the inhomogeneity as well as spatial fluctuation of $F(t)$ with species concentration, $F_j(t = 0) = F_0$ is ignored then we have,

$$\frac{dF_j}{dt} = -c_j F_j(t). \tag{2}$$

Next, by ignoring subscript j and integrating, Equation (2) yields

$$\mathrm{F(t)} - \mathrm{F}_0 = -\mathrm{c}\; I_{0+}^{-1}\mathrm{F(t)}. \tag{3}$$

The non-integer kinetic equation was also studied by Haubold and Mathai [1]

$$\mathrm{F(t)} - \mathrm{F}_0 = -c^{\delta} I_{0+}^{\delta}\mathrm{F(t)}, \tag{4}$$

where $I_{0+}^{\delta}, \delta > 0$ is the R–L fractional integral and c is an arbitrary constant. Then for any integrable function u(t), they modelled a more general equation given as

$$\mathrm{F(t)} - u(\mathrm{t})\mathrm{F}_0 = -\mathrm{d}^{\delta}\; I_{0+}^{\delta}\mathrm{F(t)}. \tag{5}$$

According to a review of the literature, no such equation containing the generalized Krätzel function [2] has been developed. This article's main goal is to formulate and answer this problem by using the new results involving the generalized Krätzel integral which is a multipurpose integral of astronomy and physics [2]. For example, thermonuclear functions of Boltzmann–Gibbs statistical mechanics comprise the Krätzel function and have basic applications in astrophysics [2]. Particular cases involving the original Krätzel function and H-function leading to new formulae are also obtained. Whereas mostly in the literature the original Krätzel function, $Z_{\rho}^{v}(x)$ is studied with respect to the variable x but the results presented in this research are with respect to v. Furthermore, different fractional images including the generalized Krätzel function have been attained under Kiryakova's fractional operators (E–K operators with multiplicity m) defined in ([3], p. 9, Equation (27)). The beauty of these fractional operators lies in the fact that the several frequently used fractional operators are connected with them or can be obtained as special cases [3–6]. For the interest of large audience [7–9], more popular and widely used fractional operators namely Riemann–Liouville (R–L), Saigo as well as Marichev–Saigo–Maeda (M–S–M) fractional operators are computed in this research. For example, novel boundary-value problems involving the Euler–Darboux equation are discussed in [7] by using Saigo fractional operators. Marichev–Saigo–Maeda fractional-calculus operators were used in [8] to establish several new formulas containing the $(p,\, q)$- extended Bessel function. The results are expressed as the Hadamard product of the (p, q)- extended Gauss hypergeometric function and the Fox–Wright function. More interesting results can be found in [9]. Similarly, Krätzel integral transform [10] contains the Laplace and Meijer transforms as special cases. Therefore, diverse aspects of this transformation have been remained an important subject of the literature [11]. In particular, Rao and Debnath [12] have analysed the Krätzel integral with reference to a specific space of distributions. Fractional operators are used by Kilbas and Shlapakov [13] to study the Krätzel function and these outcomes were extended in [14,15]. However, the study of this important function remained limited to real positive variables until the work of Kilbas et al. [16]. They explored the relation of H-function with the Krätzel function to express it as a function of complex argument. The Krätzel function is useful in solar neutrino and nuclear astrophysics as the reaction-rate probability integral [17,18]. The Krätzel function also expresses the inverse Gaussian density and other interesting generalizations of the Krätzel function are discussed in [19,20]. More recently, Tassaddiq [21] has investigated a new representation of the generalized Krätzel function as an infinite series of the complex delta function. As a direct consequence, its response on a suitably chosen function over a specific domain is easily obtained using standard properties of delta function. For new representations of other special functions, a keen reader may like to see [22–29].

This work is organised as follows: Section 2 includes all the necessary preliminary data relating to Kiryakova's fractional operators and the generalized Krätzel function. The generalized Krätzel function is used in computing the fractional images in Section 3.1. Section 3.2 discusses the corresponding fractional derivatives. The methodology and solution of a fractional kinetic equation comprising the generalized Krätzel function is covered in Section 3.3. New integrals of products of special functions are obtained in Section 3.4. A detailed discussion and comparison of the results with other researches is a part of Section 4. Conclusion is given in Section 5.

2. Preliminaries

In this section, we will briefly discuss the basic notions and definitions required for this research.

Definition 1. *Let $\mathbb{C}$ denote the set of complex numbers and $\Re$ denote the real part of any complex number. Then for $v \in \mathbb{C}$, $\Re(v)\langle 0; \rho \in \mathbb{R}$, $\rho \leq 0 \wedge x > 0$ the basic Krätzel function [10] $Z_\rho^v(x)$ is defined as*

$$Z_\rho^v(x) = \int_0^\infty t^{v-1} exp\left(-t^\rho - \frac{x}{t}\right) dt. \quad (6)$$

It is related to the McDonald function [10] $K_v(t)$ for $\rho = 1$; $x = \frac{t^2}{4}$,

$$Z_1^v\left(\frac{t^2}{4}\right) = \left(\frac{t}{2}\right)^v K_v(t). \quad (7)$$

Definition 2. *The Krätzel integral transform [10] defined by*

$$K_\rho^v(f(x)) = \int_0^\infty Z_\rho^v(xt) f(t) dt; (x > 0; \rho \geq 1). \quad (8)$$

Following relations of Krätzel function $Z_\rho^v(x)$ and H-function [16] enables $Z_\rho^v(x)$ to be the functions of complex argument

$$Z_\rho^v(s) = \frac{1}{\rho} H_{0,2}^{2,0}\left[s \middle| (0,1)\left(\frac{v}{\rho}, \frac{1}{\rho}\right)\right]; (\rho > 0, v \in \mathbb{C}; s \in \mathbb{C}, s \neq 0), \quad (9)$$

$$Z_\rho^v(s) = \frac{1}{|\rho|} H_{1,1}^{1,1}\left[s \middle| \begin{matrix} \left(1 - \frac{v}{\rho}, -\frac{1}{\rho}\right) \\ (0,1) \end{matrix}\right]; (\rho < 0, \Re(v) < 0; s \in \mathbb{C}, s \neq 0). \quad (10)$$

Definition 3. *Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of real and positive real numbers respectively then for $\rho \in \mathbb{R}^+; \sigma \in \mathbb{R}$; $s \in \mathbb{C}$ and $a, b > 0$, the generalized Krätzel function, [20], (see also [18,19]) defined by*

$$Z_{\sigma,\rho}^{a;b}(s) = \int_0^\infty t^{s-1} exp\left(-at^\sigma - \frac{b}{t^\rho}\right) dt. \quad (11)$$

For $a = \rho = 1$; $\sigma = \rho; b = x; s = v$, it reduces to Equation (6).

The generalized Krätzel function is mainly focused in this research by making use of its distributional representation given by [21]

$$Z_{\sigma,\rho}^{a;b}(s) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \delta(s + \sigma n - \rho r), \quad (12)$$

and

$$Z_{\sigma,\rho}^{a;b}(s) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n(-b)^r(\sigma n - \rho r)^p}{n!r!p!} \delta^{(p)}(s). \quad (13)$$

whereas, for an appropriate function $\wp$ and the constant τ, delta function is defined by the following properties [30]:

$$\langle \delta(s-\tau), \wp(s) \rangle = \wp(\omega);\ \delta(-s) = \delta(s);\ \delta(\tau s) = \frac{\delta(s)}{|\tau|}, \text{ where } \tau \neq 0. \quad (14)$$

The Krätzel function $Z_{\rho}^{v}(x)$ is a well-studied function but its Laplace transform w.r.t variable v was unkown untill the investigation of [21] as a special case of the following result

$$L\left(Z_{\sigma,\rho}^{a;b}(s);\omega\right) = 2\pi\sum_{n,r,p=0}^{\infty} \frac{(-a)^{n}(-b)^{r}(\sigma n-\rho r)^{p}}{n!r!p!}\,\omega^{p} = 2\pi\exp\left(-ae^{\sigma\omega} - be^{-\rho\omega}\right). \tag{15}$$

The purpose of the current research is achieved by making use of the above Equation (15).

Definition 4. [31] *For $\alpha\epsilon\mathbb{C};\Re(\alpha) > 0)$, the Mittag-Leffler function is defined as follows:*

$$E_{\alpha}(z) = \sum_{r=0}^{\infty} \frac{z^{r}}{\Gamma(\alpha r+1)}. \tag{16}$$

However, for $\alpha = 1$, it reduces to the exponential function and $\Gamma(z)$ symbolizes the basic gamma function. Similarly, the Mittag-Leffler function of parameters 2 and 3 i.e., $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha) > 0)$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^{r}}{\Gamma(\alpha r+\beta)};\; E_{\alpha,\beta}^{\gamma}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_{r}z^{r}}{\Gamma(\alpha r+\beta)}. \tag{17}$$

Definition 5. [32] *The Fox–Wright function symbolized by ${}_{p}\Psi_{q}$ is well-defined in the following form*

$$\begin{gathered} {}_{p}\Psi_{q}\left[\begin{matrix}(a_{i},\ A_{i})\\ (b_{j},B_{j})\end{matrix};z\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_{i}+A_{i}m)}{\prod_{j=1}^{q}\Gamma((b_{j}+B_{j}m)}\frac{z^{m}}{m!} \\ \left(a_{i}\epsilon\mathbb{R}^{+}(i=1,\ldots.p);B_{j}\epsilon\mathbb{R}^{+}(j=1,\ldots.q);1+\sum_{i=1}^{q}B_{i}-\sum_{j=1}^{p}A_{j} > 0\right). \end{gathered} \tag{18}$$

Definition 6. [32] *The Fox H-function is defined as*

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_{i},A_{i})\\ (b_{j},\ B_{j})\end{matrix}\right] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_{1},A_{1}),\ldots, & (a_{i},A_{i})\\ (b_{1},\ B_{1}),\ldots, & (b_{j},\ B_{j})\end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_{j}+B_{j}\mathfrak{s}\right)\prod_{i=1}^{n}\Gamma\left(1-a_{j}-A_{j}\mathfrak{s}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_{j}-B_{j}\mathfrak{s}\right)\prod_{i=n+1}^{p}\Gamma\left(a_{j}+A_{j}\mathfrak{s}\right)} z^{-\mathfrak{s}}\,d\mathfrak{s}, \end{aligned} \tag{19}$$

$(1 \le m \le q\wedge;0 \le n \le p\wedge A_{i} > 0\,(i=1,\cdot\cdot,p)\wedge B_{j} > 0\,(j=1,\cdot\cdot,q)\wedge a_{i}\in\mathbb{C}\,(i=1,\cdots,p)\wedge b_{j}\in\mathbb{C}\,(j=1,\cdot\cdot,q))$. $\mathcal{L}$ is a suitable contour (namely Mellin–Barnes form) having the property to split up the poles of $\{\Gamma(b_{j}+B_{j}\mathfrak{s})\}_{j=1}^{m}$ and $\{\Gamma(1-a_{j}-A_{j}\mathfrak{s})\}_{j=1}^{n}$. H-function turn into Meijer G-function [32] *for $A_{p} = B_{q} = 1$ in (14)*

$$\begin{gathered} H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_{1},A_{1}),\ldots, & (a_{i},A_{i})\\ (b_{1},\ B_{1}),\ldots, & (b_{j},\ B_{j})\end{matrix}\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}\Gamma\left(b_{j}+B_{j}m\right)\prod_{i=1}^{n}\Gamma\left(1-a_{i}-A_{i}m\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_{j}-B_{j}m\right)\prod_{i=n+1}^{p}\Gamma\left(a_{i}+im\right)}\frac{z^{m}}{m!} \\ H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_{1},1),\ldots, & (a_{i},1)\\ (b_{1},1),\ldots, & (b_{j},1)\end{matrix}\right] = G_{p,q}^{m,n}\left[z\middle|\begin{matrix}a_{1}\ldots, & a_{i}\\ b_{1},\ldots, & b_{j}\end{matrix}\right]. \end{gathered} \tag{20}$$

and

$${}_{p}\Psi_{q}\left[\begin{matrix}(a_{i},\ A_{i})\\ (b_{j},B_{j})\end{matrix};z\right] = H_{p,q+1}^{1,p}\left[-z\middle|\begin{matrix}(1-a_{1},A_{1}),\ldots, & (1-a_{i},\ A_{i})\\ (0,1),(1-b_{1},\ B_{1}),\ldots, & ((1-b_{j},B_{j})\end{matrix}\right].$$

and also to the hypergeometric functions as

$$\begin{gathered} {}_{p}\Psi_{q}\left[\begin{matrix}(a_{i},\ 1)\\ (b_{j},1)\end{matrix};z\right] = G_{p,q+1}^{1,p}\left[-z\middle|\begin{matrix}(1-a_{1},1),\ldots, & (1-a_{i},1)\\ 0,(1-b_{1},1),\ldots, & (1-b_{j},1)\end{matrix}\right] = {}_{p}F_{q}\left[\begin{matrix}a_{i}\\ b_{j}\end{matrix};z\right].\frac{\Gamma(a_{1})\ldots\Gamma(a_{i})}{\Gamma(b_{1})\ldots\Gamma(b_{j})}; \\ (a_{i} > 0;b_{j}\notin\mathbb{Z}_{0}^{-}), \end{gathered} \tag{21}$$

where $\mathbb{Z}_{0}^{-}$, is the set of negative integers including 0.

Definition 7. [1] *Kiryakova's fractional operators (E–K operators with multiplicity m) are defined as*

$$I^{(\gamma_k),(v_k)}_{(\beta_k),m} f(z) = \begin{cases} \int_0^1 f(z\sigma)\, H^{m,0}_{m,m}\left[\sigma \middle| \begin{matrix} \left(\gamma_k + v_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{matrix}\right] d\sigma; \sum_k v_k > 0, \\ z^{-1}\int_0^z f(\xi)\, H^{m,0}_{m,m}\left[\frac{\xi}{z} \middle| \begin{matrix} \left(\gamma_k + v_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{matrix}\right] d\xi; \sum_k v_k > 0. \end{cases} \tag{22}$$

Here, $v_k's$ represent the fractional order of integration, $\gamma_k's$ are used only as weights whereas $\beta_k's$ are supplementary parameters and for $v_k = 1$ we have $I^{(\gamma_k),(v_k)}_{(\beta_k),m} f(z) = f(z)$. Since, $H^{m,0}_{m,m}$ vanishes for $|\sigma| > 1$ therefore it is evident to use the limits from 0 to ∞.

Definition 8. [1] *In compliance with (22), the fractional derivative (R–L type) with order $v = (v_1 \geq 0, \ldots, v_m \geq 0)$is defined as*

$$D^{(\gamma_k)_1^m,(v_k)}_{(\beta_k),m}(f(z)) = D_\eta\, I^{(\gamma_k+v_k),(\eta_k-v_k)}_{(\beta_k),m} f(z) = D_\eta \int_0^1 f(z\sigma) H^{m,0}_{m,m}\left[\sigma \middle| \begin{matrix} \left(\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{matrix}\right] d\sigma. \tag{23}$$

where D_η, is defined and stated as [1]

$$D_\eta = \prod_{r=1}^m \prod_{j=1}^{\eta_r} \frac{1}{\beta_r} z\frac{d}{dz} + \gamma_r + j;\ \eta_k = \begin{cases} [v_k] + 1; v_k \notin \mathbb{Z} \\ v_k; v_k \in \mathbb{Z} \end{cases}. \tag{24}$$

Similarly, the fractional derivative of Caputo type is defined and stated as [1]

$$*D^{(\gamma k)_1^m,(vk)}_{(\beta k),m} f(z) = I^{(\gamma k+vk),(\eta_k - v_k)}_{(\beta k),m} D_\eta\, f(z). \tag{25}$$

The fractional operators defined in (22) transform the power function as follows:

$$I^{(\gamma k),(vk)}_{(\beta k),m}\{z^p\} = \prod_{i=1}^m \frac{\Gamma\left(\gamma_i + 1 + \frac{p}{\beta_i}\right)}{\Gamma\left(\gamma_i + v_i + 1 + \frac{p}{\beta_i}\right)} z^p; ([-\beta_k(1+\gamma_k)] < p; v_k \geq 0; k = 1, \ldots, m). \tag{26}$$

It is important to note that the following fractional operators (see Table 1) can be obtained by specifying (m = 1,2,3) in (22), such as R–L and Erdélyi–Kober (E–K) for ($m = 1$); Saigo fractional operators for ($m = 2$) and Marichev–Saigo–Maeda (M–S–M) fractional operators for ($m = 3$).

Table 1. Important cases of Kiryakova's fractional operators [1].

Cases of (22)	Relation between the Kernels of Popular Fractional Transforms and (22)
$(m = 1; \beta = 1;\ \sigma = \frac{t}{x} \wedge \sigma = \frac{x}{t})$ Riemann–Liouville (R–L)	$H^{1,1}_{1,0}\left[\sigma \middle\| \begin{matrix} (\gamma + v, 1) \\ (\gamma, 1) \end{matrix}\right] = G^{1,1}_{1,0}\left[\frac{t}{x} \middle\| \begin{matrix} \gamma + v \\ \gamma \end{matrix}\right] = \frac{(x-t)^{v-1} t^\gamma}{\Gamma(v)}$
(m = 1) Erdélyi–Kober (E–K)	$H^{1,1}_{1,0}\left[\sigma \middle\| \begin{matrix} \left(\gamma + v, \frac{1}{\beta}\right) \\ \left(\gamma, \frac{1}{\beta}\right) \end{matrix}\right] = \beta\sigma^{\beta-1} G^{1,1}_{1,0}\left[\sigma^\beta \middle\| \begin{matrix} \gamma + v \\ \gamma \end{matrix}\right] = \beta\ \frac{\sigma^{\beta\gamma+\beta-1}\left(1-\sigma^\beta\right)^{v-1}}{\Gamma(v)}$
$(m = 2; \beta_1 = \beta_2 = \beta > 0; \sigma = \frac{t}{x} \wedge \sigma = \frac{x}{t})$ Saigo [33–35]	$H^{2,0}_{2,2}\left[\sigma \middle\| \begin{matrix} \left(\gamma_1 + v_1 + 1 - \frac{1}{\beta}, \frac{1}{\beta}\right), \left(\gamma_2 + v_2 + 1 - \frac{1}{\beta}, \frac{1}{\beta}\right) \\ \left(\gamma_1 + 1 - \frac{1}{\beta}, \frac{1}{\beta}\right), \left(\gamma_2 + 1 - \frac{1}{\beta}, \frac{1}{\beta}\right) \end{matrix}\right] = G^{2,0}_{2,2}\left[\sigma^\beta \middle\| \begin{matrix} \gamma_1 + v_1, \gamma_2 + v_2 \\ \gamma_1, \gamma_2 \end{matrix}\right] =$ $\beta\ \frac{\sigma^{\beta\gamma_2}\left(1-\sigma^\beta\right)^{v_1+v_2-1}}{\Gamma(v_1+v_2)}\ {}_2F_1\left(\gamma_2 + v_2 - \gamma_1, v_1; v_1 + v_2; 1 - \sigma^\beta\right)$
$(m = 3; \beta_1 = \beta_2 = \beta_3 = \beta = 1)$ Marichev–Saigo–Maeda (M–S–M) [33–35]	$H^{3,0}_{3,3}\left(\frac{t}{x}\right) = G^{3,0}_{3,3}\left[\frac{t}{x} \middle\| \begin{matrix} \gamma_1' + \gamma_2', & v - \gamma_1, v - \gamma_2 \\ \gamma_1', \gamma_2', & v - \gamma_1 - \gamma_2 \end{matrix}\right] =$ $\frac{x^{-\gamma_1}}{\Gamma(v)}\ (x-t)^{\delta-1} t^{-\gamma_1'} F_3\left(\gamma_1, \gamma_1', \gamma_2, \gamma_2', v; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right)$

Except as otherwise specified, the variables' restrictions will be equivalent to those in Section 2 and the reference materials thereof.

3. Main Results

3.1. New Identities Containing the Fractional Calculus Images of Generalized Krätzel Function

Lemma 1. *For $\rho \in \mathbb{R}^+; \sigma \in \mathbb{R}$; $s \in \mathbb{C}$ and $a, b > 0$, the following identity for the Fox–Wright function can be proved*

$$\sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} {}_0\Psi_0\left[\begin{matrix}-\\-\end{matrix}\middle|(\sigma n-\rho r)\omega\right] = {}_0\Psi_0\left[\begin{matrix}-\\-\end{matrix}\middle|-ae^{\sigma\omega}-be^{-\rho\omega}\right]. \tag{27}$$

Proof. Let us consider Equation (15) then the Laplace transform of the generalized Krätzel function can be modified as

$$L\left(Z_{\sigma,\rho}^{a;b}(s);\omega\right) = 2\pi\sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} {}_0\Psi_0\left[\begin{matrix}-\\-\end{matrix}\middle|(\sigma n-\rho r)\omega\right], \tag{28}$$

and

$$L\left(Z_{\sigma,\rho}^{a;b}(s);\omega\right) = 2\pi\exp\left(-ae^{\sigma\omega}-be^{-\rho\omega}\right) = 2\pi_0\Psi_0\left[\begin{matrix}-\\-\end{matrix}\middle|-ae^{\sigma\omega}-be^{-\rho\omega}\right]. \tag{29}$$

Hence from both of the above Equations (28) and (29), the required result is proved. □

Remark 1. *It is to be remarked that a general result is obvious from Equation (27) as follows*

$$\sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} {}_p\Psi_q\left[\begin{matrix}(a_i,\ A_i)\\(b_j,B_j)\end{matrix}\middle|(\sigma n-\rho r)\omega\right] = {}_p\Psi_q\left[\begin{matrix}(a_i,\ A_i)\\(b_j,B_j)\end{matrix}\middle|-ae^{\sigma\omega}-be^{-\rho\omega}\right]. \tag{30}$$

Similar results will hold for the Mittag-Leffler and other special functions due to the relation between them.

Theorem 1. *The Kiryakova's fractional transform of the generalised Krätzel function $Z_{\sigma,\rho}^{a;b}(s)$ is given by*

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = 2\pi\omega^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]; \tag{31}$$

$$([-\beta_k(1+\gamma_k)]< p;\delta_k\geq 0;k=1,\ldots,m;\ \rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0).$$

Proof. Let us first consider the action of Equation (22) on Equation (15)

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}2\pi\sum_{n,r,p=0}^{\infty}\frac{(-a)^n(-b)^r(\sigma n-\rho r)^p}{n!r!p!}\omega^p\right), \tag{32}$$

then exchanging the summation and integration

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = 2\pi\sum_{n,r,p=0}^{\infty}\frac{(-a)^n(-b)^r(\sigma n-\rho r)^p}{n!r!p!}I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}\omega^p\right), \tag{33}$$

and then by using Equation (26) yields

$$I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = 2\pi\sum_{n,r,p=0}^{\infty}\frac{(-a)^n(-b)^r(\sigma n-\rho r)^p}{n!r!p!}\prod_{i=1}^{m}\frac{\Gamma\left(\gamma_i+1+\frac{\chi+p-1}{\beta_i}\right)}{\Gamma\left(\gamma i+\delta i+1+\frac{\chi+p-1}{\beta i}\right)}\omega^{p+\chi-1}, \tag{34}$$

which after modifications by using Equation (18) leads to the following

$$I^{(\gamma k),(\delta k)}_{(\beta k),m}\left(\omega^{\chi-1}\mathrm{L}\left\{\mathrm{Z}^{\mathrm{a;b}}_{\sigma,\rho}(s);\omega\right\}\right)=2\pi\omega^{\chi-1}\sum_{\mathrm{n,r}=0}^{\infty}\frac{(-a)^n(-b)^r}{\mathrm{n!r!}}\,{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix};(\sigma\mathrm{n}-\rho\mathrm{r})\omega\right]; \tag{35}$$

$$([-\beta_k(1+\gamma_k)]< p;\delta_k\geq 0;k=1,\ldots,m;\ \rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0).$$

Lastly, an application of Lemma 1, leads to the required simplified form. □

Corollary 1. *For*$(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$*, the Kiryakova's fractional transform of the H-Function is given by*

$$I^{(\gamma k),(\delta k)}_{(\beta k),m}\left(\omega^{\chi-1}\mathrm{L}\left\{\frac{1}{\rho}H^{2,0}_{0,2}\left[s\middle|(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}\right)=2\pi\omega^{\chi-1}\,{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix};-\mathrm{e}^{\rho\omega}-x\mathrm{e}^{-\omega}\right]; \tag{36}$$

$$([-\beta_k(1+\gamma_k)]<p;\delta_k\geq 0;k=1,\ldots,m).$$

Proof. It can be proved by taking $a=\rho=1;\ \sigma=\rho;b=\mathrm{x}$ in Equation (31) and then using Equation (9). □

Corollary 2. *For* $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$,*the Kiryakova's fractional transform of the H-Function is given by*

$$I^{(\gamma k),(\delta k)}_{(\beta k),m}\left(\omega^{\chi-1}L\left\{\frac{1}{|\rho|}H^{1,1}_{1,1}\left[s\middle|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right)=2\pi\omega^{\chi-1}\,{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right] \tag{37}$$

$$([-\beta_k(1+\gamma_k)]<p;\delta_k\geq 0;k=1,\ldots,m).$$

Proof. It can be proved by taking $a=\rho=1;\ \sigma=\rho;\ b=\mathrm{x}$ in Equation (31) and then using Equation (9). □

Continuing in this way, we obtain the following Table 2 of fractional integrals formulae involving the generalised Krätzel function $\mathrm{Z}^{\mathrm{a;b}}_{\sigma,\rho}(\mathrm{s})$ and H-Function for the Marichev–Saigo–Maeda ($m=3$), Saigo ($m=2$), Erdélyi–Kober and Riemann–Liouville (R–L) ($m=1$) fractional integrals

Table 2. Fractional integrals formulae involving the generalised Krätzel function $\mathrm{Z}^{\mathrm{a;b}}_{\sigma,\rho}(\mathrm{s})$ and H-function.

***m* = 3**	**Marichev–Saigo–Maeda Fractional Integrals**
$\mathrm{I}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}_{0+}\left(\omega^{\chi-1}\mathrm{L}\left\{\mathrm{Z}^{\mathrm{a;b}}_{\sigma,\rho}(s);\omega\right\}\right)=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\,{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\(\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & (\chi+\delta-\gamma_1'-\gamma_2,1)\end{matrix};-\mathrm{a}\mathrm{e}^{\sigma\omega}-b\mathrm{e}^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.	
$\mathrm{I}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}_{0+}\left(\omega^{\chi-1}\mathrm{L}\left\{\frac{1}{\rho}H^{2,0}_{0,2}\left[s\middle|(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}\right)=$ $2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\,{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\(\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & (\chi+\delta-\gamma_1'-\gamma_2,1)\end{matrix};-\mathrm{e}^{\rho\omega}-x\mathrm{e}^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$	
$I^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}_{0+}\left(\omega^{\chi-1}L\left\{\frac{1}{\lvert\rho\rvert}H^{1,1}_{1,1}\left[s\middle|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right)=$ $2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\,{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\(\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & (\chi+\delta-\gamma_1'-\gamma_2,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$.	

Table 2. *Cont.*

***m* = 3**	**Marichev–Saigo–Maeda Fractional Integrals**
	$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s)\right\};\omega\right) = 2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\,{}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',-1) & (1-\chi+\gamma_1+\gamma_2'-\delta,-1) & (1-\chi-\gamma_1,-1)\\ (1-\chi,-1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,-1) & (1-\chi+\gamma_1-\gamma_2,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.
	$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\vert\begin{matrix}-\\(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\,{}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',-1) & (1-\chi+\gamma_1+\gamma_2'-\delta,-1) & (1-\chi-\gamma_1,-1)\\ (1-\chi,-1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,-1) & (1-\chi+\gamma_1-\gamma_2,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$
	$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\lvert\rho\rvert}H_{1,1}^{1,1}\left[s\middle\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\,{}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',-1) & (1-\chi+\gamma_1+\gamma_2'-\delta,-1) & (1-\chi-\gamma_1,-1)\\ (1-\chi,-1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,-1) & (1-\chi+\gamma_1-\gamma_2,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$
m = 2	Saigo fractional integrals
	$I_{0+}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(z);\omega\right\}\right) = 2\pi\omega^{\chi-\gamma_1-1}\,{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.
	$I_{0+}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\vert\begin{matrix}-\\(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-\gamma_1-1}\,{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\delta+\gamma_2,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$
	$I_{0+}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\lvert\rho\rvert}H_{1,1}^{1,1}\left[s\middle\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-\gamma_1-1}\,{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\delta+\gamma_2,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$
	$I_{-}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = 2\pi\omega^{\chi-\gamma_1-1}\,{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,-1) & (\gamma_2-\chi+1,-1)\\(1-\chi,-1) & ((\gamma_1+\gamma_2+\delta-\chi+1,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$
	$I_{-}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\vert\begin{matrix}-\\(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-\gamma_1-1}\,{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,-1) & (\gamma_2-\chi+1,-1)\\(1-\chi,-1) & ((\gamma_1+\gamma_2+\delta-\chi+1,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$
	$I_{-}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\lvert\rho\rvert}H_{1,1}^{1,1}\left[s\middle\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-\gamma_1-1}\,{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,-1) & (\gamma_2-\chi+1,-1)\\(1-\chi,-1) & ((\gamma_1+\gamma_2+\delta-\chi+1,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$
***m* = 1**	Erdélyi–Kober fractional integrals
	$I_{0+}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = 2\pi\omega^{\chi-1}\,{}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\delta,1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.
	$I_{0+}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\vert\begin{matrix}-\\(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-1}\,{}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\delta,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$
	$I_{0+}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\lvert\rho\rvert}H_{1,1}^{1,1}\left[s\middle\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-1}\,{}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\delta,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$
	$I_{0-}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right) = 2\pi\omega^{\chi-1}\,{}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\delta-\chi+1,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$
	$I_{0-}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\vert\begin{matrix}-\\(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-1}\,{}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\delta-\chi+1,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq 0)$
	$I_{0-}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\lvert\rho\rvert}H_{1,1}^{1,1}\left[s\middle\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right) = 2\pi\omega^{\chi-1}\,{}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\delta-\chi+1,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq 0)$

Table 2. *Cont.*

m = 1	Riemann–Liouville (R–L) Fractional Integrals
	$\mathrm{I}_{0+}^{\delta}\left(\omega^{\chi-1}\mathrm{L}\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]=\Gamma(\chi)\mathrm{E}_{1,\delta+\chi}^{\chi}(-ae^{\sigma\omega}-be^{-\rho\omega})$ $(\rho\in\mathbb{R}^{+};\sigma\in\mathbb{R};\,s\in\mathbb{C};\,a,b>0)$
	$\mathrm{I}_{0+}^{\delta}\left(\omega^{\chi-1}\mathrm{L}\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\|(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]=\Gamma(\chi)\mathrm{E}_{1,\delta+\chi}^{\chi}(-e^{\rho\omega}-xe^{-\omega})$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$
	$\mathrm{I}_{0+}^{\delta}\left(\omega^{\chi-1}\mathrm{L}\left\{\frac{1}{\|\rho\|}H_{1,1}^{1,1}\left[s\middle\|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]=\Gamma(\chi)\mathrm{E}_{1,\delta+\chi}^{\chi}(-e^{\rho\omega}-xe^{-\omega})$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$
	$\mathrm{I}_{-}^{\delta}\left(\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^{+};\sigma\in\mathbb{R};\,s\in\mathbb{C};\,a,b>0)$
	$\mathrm{I}_{-}^{\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\|(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$
	$\mathrm{I}_{-}^{\delta}\left(\omega^{\chi-1}L\left\{\frac{1}{\|\rho\|}H_{1,1}^{1,1}\left[s\middle\|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$

3.2. New Identities Containing the Fractional Calculus Derivatives of Generalized Krätzel Function

One can find the generalized fractional derivatives involving the generalized Krätzel function by following the steps of the proof of Theorem 1 and using the new representation of the generalized Krätzel function. This fact is significant that we compute these fractional derivatives using the wide-ranging result ([3], Theorem 4) stated as

$$D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\left\{z^c\,{}_p\Psi_q\left[\begin{matrix}(a_i,\ \alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix};\lambda z^{\mu}\right]\right\}=z^c\left\{{}_{p+m}\Psi_{q+m}\left[\begin{matrix}(a_i,\ \alpha_i)_1^p,\left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\(b_j,\beta_j)_1^q,\left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix};\lambda z^{\mu}\right]\right\}. \quad (38)$$

Theorem 2. *The Kiryakova's fractional derivative of the generalised Krätzel function $Z_{\sigma,\rho}^{a;b}(s)$ is given by*

$$\left(\mathrm{D}_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\omega^{\chi-1}\mathrm{L}\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\}\right)=2\pi\omega^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\\left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]; \quad (39)$$
$$(\rho\in\mathbb{R}^{+};\sigma\in\mathbb{R};\,s\in\mathbb{C};\,a,b>0).$$

Proof. These generalized fractional derivatives involving the *generalized Krätzel function* can be obtained by using Equations (27–29) and (38). □

Corollary 3. *For $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$, the following result containing H-Function holds true:*

$$D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\left(\omega^{\chi-1}\mathrm{L}\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle|(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}\right)=2\pi\omega^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]. \quad (40)$$

Proof. It can be proved by taking $a=\rho=1;\ \sigma=\rho;b=x$ in Equation (39) and then using Equation (9). □

Corollary 4. *For $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$, the following result containing the H-Function holds true:*

$$D^{(\gamma k)^m_1,(\delta k)}_{(\beta k),m}\left(\omega^{\chi-1}L\left\{\frac{1}{|\rho|}H^{1,1}_{1,1}\left[s\left|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right.\right];\omega\right\}\right)=2\pi\omega^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)^m_1\\\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)^m_1\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]. \quad (41)$$

Proof. It can be proved by taking $a=\rho=1;\ \sigma=\rho;\ b=x$ in Equation (39) and then using Equation (9). □

Continuing in this way, we obtain the following Table 3 of fractional derivatives formulae involving the generalised Krätzel function $Z^{a;b}_{\sigma,\rho}(s)$ and H-Function for the Marichev–Saigo–Maeda ($m=3$), Saigo ($m=2$), Erdélyi–Kober and Riemann–Liouville (R–L) ($m=1$) fractional derivatives.

Table 3. Fractional derivatives formulae involving the generalised Krätzel function $Z^{a;b}_{\sigma,\rho}(s)$ and H-function.

***m* = 3**	**Marichev–Saigo–Maeda Fractional Derivatives**
	$D^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}_{0+}\omega^{\chi-1}L\left\{Z^{a;b}_{\sigma,\rho}(s);\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & (\chi-\delta+\gamma_1+\gamma_1'+\gamma_2',1)\\(\chi-\gamma_2,1) & (\chi-\delta+\gamma_1++\gamma_2',1) & (\chi-\delta+\gamma_1+\gamma_1',1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.
	$D^{\gamma_1{}',\gamma_2,\gamma_2',\delta}_{0+}\omega^{\chi-1}L\left\{\frac{1}{\rho}H^{2,0}_{0,2}\left[s\vert(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\(\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & (\chi+\delta-\gamma_1'-\gamma_2,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$
	$D^{\gamma_1{}',\gamma_2,\gamma_2',\delta}_{0+}\omega^{\chi-1}L\left\{\frac{1}{\vert\rho\vert}H^{1,1}_{1,1}\left[s\left\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right.\right];\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\(\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & (\chi+\delta-\gamma_1'-\gamma_2,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$.
	$D^{\gamma_1{}',\gamma_2,\gamma_2',\delta}_{-}\omega^{\chi-1}L\left\{Z^{a;b}_{\sigma,\rho}(s);\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\left[\begin{matrix}(1-\chi+\gamma_2',-1) & (1+\gamma_2'-\chi-\gamma_2+\gamma_1,-1) & (1+\delta-\chi-\gamma_1-\gamma_1',-1)\\(1-\chi,-1) & (1-\chi-\gamma_1'+\gamma_2',-1) & (1-\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.
	$D^{\gamma_1{}',\gamma_2,\gamma_2',\delta}_{-}\omega^{\chi-1}L\left\{\frac{1}{\rho}H^{2,0}_{0,2}\left[s\vert(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\left[\begin{matrix}(1-\chi+\gamma_2',-1) & (1+\gamma_2'-\chi-\gamma_2+\gamma_1,-1) & (1+\delta-\chi-\gamma_1-\gamma_1',-1)\\(1-\chi,-1) & (1-\chi-\gamma_1'+\gamma_2',-1) & (1-\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$
	$D^{\gamma_1{}',\gamma_2,\gamma_2',\delta}_{-}\omega^{\chi-1}L\left\{\frac{1}{\vert\rho\vert}H^{1,1}_{1,1}\left[s\left\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right.\right];\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\left[\begin{matrix}(1-\chi+\gamma_2',-1) & (1+\gamma_2'-\chi-\gamma_2+\gamma_1,-1) & (1+\delta-\chi-\gamma_1-\gamma_1',-1)\\(1-\chi,-1) & (1-\chi-\gamma_1'+\gamma_2',-1) & (1-\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$
***m* = 2**	Saigo fractional derivatives
	$D^{\gamma_1,\gamma_2,\delta}_{0+}\omega^{\chi-1}L\left\{Z^{a;b}_{\sigma,\rho}(s);\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\(\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho\in\mathbb{R}^+;\sigma\in\mathbb{R};\ s\in\mathbb{C};\ a,b>0)$.
	$D^{\gamma_1,\gamma_2,\delta}_{0+}\omega^{\chi-1}L\left\{\frac{1}{\rho}H^{2,0}_{0,2}\left[s\vert(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\(\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$
	$D^{\gamma_1,\gamma_2,\delta}_{0+}\omega^{\chi-1}L\left\{\frac{1}{\vert\rho\vert}H^{1,1}_{1,1}\left[s\left\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right.\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\(\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$
	$D^{\gamma_1,\gamma_2,\delta}_{-}\omega^{\chi-1}L\left\{Z^{a;b}_{\sigma,\rho}(s);\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,-1) & (1-\chi+\delta+\gamma_1,-1)\\(1-\chi+\delta-\gamma_2,-1) & (1-\chi,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$
	$D^{\gamma_1,\gamma_2,\delta}_{-}\omega^{\chi-1}L\left\{\frac{1}{\rho}H^{2,0}_{0,2}\left[s\vert(0,1)\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,-1) & (1-\chi+\delta+\gamma_1,-1)\\(1-\chi+\delta-\gamma_2,-1) & (1-\chi,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho>0,v\in\mathbb{C};s\in\mathbb{C},s\neq0)$
	$D^{\gamma_1,\gamma_2,\delta}_{-}\omega^{\chi-1}L\left\{\frac{1}{\vert\rho\vert}H^{1,1}_{1,1}\left[s\left\vert\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right.\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,-1) & (1-\chi+\delta+\gamma_1,-1)\\(1-\chi+\delta-\gamma_2,-1) & (1-\chi,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho<0,\Re(v)<0;s\in\mathbb{C},s\neq0)$

Table 3. *Cont.*

$m = 1$	Erdélyi–Kober fractional derivatives
	$D_{0+}^{\gamma,\delta}\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\} = 2\pi\omega^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\(\gamma+\chi,1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho \in \mathbb{R}^+;\sigma \in \mathbb{R};\ s \in \mathbb{C};\ a,b > 0)$.
	$D_{0+}^{\gamma,\delta}\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\|(0,1)\bar{}\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\} = 2\pi\omega^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\(\gamma+\chi,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho > 0, v \in \mathbb{C}; s \in \mathbb{C}, s \neq 0)$
	$D_{0+}^{\gamma,\delta}\omega^{\chi-1}L\left\{\frac{1}{\|\rho\|}H_{1,1}^{1,1}\left[s\middle\|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\} = 2\pi\omega^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\(\gamma+\chi,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho < 0, \Re(v) < 0; s \in \mathbb{C}, s \neq 0)$
	$D_{-}^{\gamma,\delta}\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,-1)\\(1-\chi+\gamma,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$ $(\rho \in \mathbb{R}^+;\sigma \in \mathbb{R};\ s \in \mathbb{C};\ a,b > 0)$.
	$D_{-}^{\gamma,\delta}\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\|(0,1)\bar{}\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,-1)\\(1-\chi+\gamma,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho > 0, v \in \mathbb{C}; s \in \mathbb{C}, s \neq 0)$
	$D_{-}^{\gamma,\delta}\omega^{\chi-1}L\left\{\frac{1}{\|\rho\|}H_{1,1}^{1,1}\left[s\middle\|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,-1)\\(1-\chi+\gamma,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho < 0, \Re(v) < 0; s \in \mathbb{C}, s \neq 0)$
$m = 1$	Riemann–Liouville (R–L) Fractional Derivatives
	$D_{0+}^{\delta}\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\chi-\delta,1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right] = \Gamma(\chi)E_{1,\delta+\chi}^{\chi}\left(-ae^{\sigma\omega}-be^{-\rho\omega}\right)$ $(\rho \in \mathbb{R}^+;\sigma \in \mathbb{R};\ s \in \mathbb{C};\ a,b > 0)$
	$D_{0+}^{\delta}\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\|(0,1)\bar{}\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\chi-\delta,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right] = \Gamma(\chi)E_{1,\delta+\chi}^{\chi}\left(-e^{\rho\omega}-xe^{-\omega}\right)$ $(\rho > 0, v \in \mathbb{C}; s \in \mathbb{C}, s \neq 0)$
	$D_{0+}^{\delta}\omega^{\chi-1}L\left\{\frac{1}{\|\rho\|}H_{1,1}^{1,1}\left[s\middle\|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\chi-\delta,1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right] = \Gamma(\chi)E_{1,\delta+\chi}^{\chi}\left(-e^{\rho\omega}-xe^{-\omega}\right)$ $(\rho < 0, \Re(v) < 0; s \in \mathbb{C}, s \neq 0)$
	$D_{-}^{\delta}\omega^{\chi-1}L\left\{Z_{\sigma,\rho}^{a;b}(s);\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\delta-\chi+1,-1)\\(1-\chi,-1)\end{matrix};-ae^{\sigma\omega}-be^{-\rho\omega}\right]$
	$D_{-}^{\delta}\omega^{\chi-1}L\left\{\frac{1}{\rho}H_{0,2}^{2,0}\left[s\middle\|(0,1)\bar{}\left(\frac{v}{\rho},\frac{1}{\rho}\right)\right];\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\delta-\chi+1,-1)\\(1-\chi,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho > 0, v \in \mathbb{C}; s \in \mathbb{C}, s \neq 0)$
	$D_{-}^{\delta}\omega^{\chi-1}L\left\{\frac{1}{\|\rho\|}H_{1,1}^{1,1}\left[s\middle\|\begin{matrix}\left(1-\frac{v}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right];\omega\right\} = 2\pi\omega^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\delta-\chi+1,-1)\\(1-\chi,-1)\end{matrix};-e^{\rho\omega}-xe^{-\omega}\right]$ $(\rho < 0, \Re(v) < 0; s \in \mathbb{C}, s \neq 0)$

3.3. Non-Integer order Kinetic Equation Comprising of the Generalized Krätzel Function

Srivastava and collaborators have significant contributions [36–39] for fractional calculus. Various general families of fractional kinetic equations were investigated earlier in the references [40–42]. In [41,42] Srivastava investigated much more general functions than the various multi-parameter extensions of the Mittag-Leffler function and the Hurwitz–Lerch function. According to a review of the literature, no such equation containing the generalized Krätzel function has been developed. This section's main goal is to formulate and answer this problem. Next, by following Equations (1)–(5), we state and prove Theorem 3.

Theorem 3. *For* $\sigma \in \mathbb{R}$; $\rho \in \mathbb{R}^+ \wedge a,b,d,\delta > 0$, *the subsequent fractional kinetic equation comprising the generalized Krätzel function*

$$F(t) - F_0 Z_{\sigma,\rho}^{a;b}(t) = -d^{\delta}\, I_{0+}^{\delta} F(t), \tag{42}$$

has a solution given by

$$F(t) = \frac{2\pi F_0}{t}\sum_{n,r,p=0}^{\infty}\frac{(-a)^n(-b)^r\left(\frac{\sigma n-\rho r}{t}\right)^p}{n!r!p!}E_{\delta,-p}\left(-d^{\delta}t^{\delta}\right). \tag{43}$$

Proof. In the first step, let us apply the Laplace transform (see [1]) on both sides of Equation (42)

$$L\{\mathrm{F}(t)\} - \mathrm{F}_0 L\left\{\mathrm{Z}^{\mathrm{a;b}}_{\sigma,\rho}(\mathrm{t})\right\} = L\left\{-\mathrm{d}^{\delta}\ I^{\delta}_{0+}\mathrm{F}(t)\right\}, \tag{44}$$

where

$$\mathrm{F}(s) = L[\mathrm{F}(t) : s] = \int_0^{\infty} e^{-st}\mathrm{F}(t)dt, \Re(s) > 0, \tag{45}$$

and

$$L\left\{\ I^{\delta}_{0+}\mathrm{F}(t); \omega\right\} = \omega^{-\delta}\mathrm{F}(\omega). \tag{46}$$

Next, by making use of Equation (15)

$$\mathrm{F}(\omega) = 2\pi \mathrm{F}_0 \sum\nolimits_{\mathrm{n,r,p}=0}^{\infty} \frac{(-a)^n(-b)^r(\sigma\mathrm{n} - \rho\mathrm{r})^{\mathrm{P}}}{\mathrm{n!r!p!}}\ \omega^p - \left(\frac{\omega}{d}\right)^{-\delta}\mathrm{F}(\omega), \tag{47}$$

implies that

$$\mathrm{F}(\omega)\left[1 + \left(\frac{s}{d}\right)^{-\delta}\right] = 2\pi \mathrm{F}_0 \sum\nolimits_{\mathrm{n,r,p}=0}^{\infty} \frac{(-a)^n(-b)^r(\sigma\mathrm{n} - \rho\mathrm{r})^{\mathrm{P}}}{\mathrm{n!r!p!}}\ \omega^p. \tag{48}$$

After some simple calculation, one can obtain

$$\mathrm{F}(\omega) = 2\pi \mathrm{F}_0 \sum\nolimits_{\mathrm{n,r,p}=0}^{\infty} \frac{(-a)^n(-b)^r(\sigma\mathrm{n} - \rho\mathrm{r})^{\mathrm{P}}}{\mathrm{n!r!p!}}\ \omega^p \sum\nolimits_{m=0}^{\infty}\left[-\left(\frac{\omega}{d}\right)^{-\delta}\right]^m. \tag{49}$$

Additionally, suppose that $\delta\mathrm{m} - \mathrm{p} > 0; \delta > 0$ and use $L^{-1}\{\omega^{-\delta}; t\} = \frac{t^{\delta-1}}{\Gamma(\delta)}$ to calculate L^{-1} (the inverse Laplace transform) of Equation (49) as follows

$$\mathrm{F}(t) = 2\pi \mathrm{F}_0 \sum\nolimits_{\mathrm{n,r,p}=0}^{\infty} \frac{(-a)^n(-b)^r(\sigma\mathrm{n} - \rho\mathrm{r})^{\mathrm{P}}}{\mathrm{n!r!p!}} t^{-p-1} \times \sum\nolimits_{\mathrm{m}=0}^{\infty} \frac{\left(-\mathrm{d}^{\delta}\mathrm{t}^{\delta}\right)^{\mathrm{m}}}{\Gamma(\delta\mathrm{m} - \mathrm{p})}. \tag{50}$$

Lastly, Equation (43) can be obtained by using Equation (17) in Equation (50). □

Proof. This can be obtained by using Equation (6) and taking $a = \rho = 1 \wedge\ \sigma = \rho \wedge b = \mathrm{x}$ in Equations (42)–(43). □

Corollary 5. *The fractional kinetic equation comprising of the H function when* $\rho > 0$

$$\mathrm{F}(t) - \mathrm{F}_0 \frac{1}{\rho} H^{2,0}_{0,2}\left[z\middle|\begin{matrix} - \\ (0,1)\left(\frac{t}{\rho}, \frac{1}{\rho}\right)\end{matrix}\right] = -d^{\delta}\ I^{\delta}_{0+}\mathrm{F}(t); (z, \rho > 0, d, \delta > 0), \tag{51}$$

has the solution

$$\mathrm{F}(t) = \frac{2\pi \mathrm{F}_0}{t} \sum\nolimits_{n,r,p=0}^{\infty} \frac{(-)^n(z)^r\left(\frac{\rho n - r}{t}\right)^p E_{\delta,-p}(-d^v t^v)}{n!p!}. \tag{52}$$

Proof. This is obtainable by using (9) and taking $a = \rho = 1 \wedge\ \sigma = \rho \wedge b = \mathrm{x}$ in Equations (42) and (43). □

Corollary 6. *The fractional kinetic equation comprising of the H function when* $\rho < 0$

$$\mathrm{F}(t) - \mathrm{F}_0 \frac{1}{|\rho|} H^{1,1}_{1,1}\left[z\middle|\begin{matrix}\left(1 - \frac{t}{\rho}, -\frac{1}{\rho}\right) \\ (0,1)\end{matrix}\right] = -d^{\delta}\ I^{\delta}_{0+}\mathrm{F}(t); \quad (\rho< 0; x, d, \delta >0), \tag{53}$$

has the solution

$$\mathrm{F}(t) = \frac{2\pi \mathrm{F}_0}{t} \sum\nolimits_{n,r,p=0}^{\infty} \frac{(-)^n (z)^r \left(\frac{\rho n - r}{t}\right)^p E_{\delta,-p}\left(-d^{\delta} t^{\delta}\right)}{n! p!}. \tag{54}$$

Proof. This is obtainable by using Equation (9) and taking $a = \rho = 1 \wedge \sigma = \rho \wedge b = x$ in Equations (42) and (43). □

Remark 2. *One can note that the solution procedure is classical* [1] *as well as the reaction rate* $\mathrm{F}(t)$ *is the function of fractional parameter* δ*. Usually, it is represented in terms of Mittag-Leffler function* [1] *and the same can be observed in the above solution. Hence, it is remarkable that the subsequent infinite triple summation of the coefficients* $C^{a;b}_{\sigma,\rho}(t)$ *in Equation (43) has a closed form*

$$C^{a;b}_{\sigma,\rho}(t) = \sum\nolimits_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r \left(\frac{\sigma n - \rho r}{t}\right)^p}{n! r! p!} = exp\left(-ae^{\frac{\sigma}{t}} - be^{-\frac{\rho}{t}}\right). \tag{55}$$

Similarly,

$$\lim_{t\to\infty} C^{a;b}_{\sigma,\rho}(t) = exp(-a-b); \quad (a,b>0), \tag{56}$$

and

$$\lim_{\substack{t\to\infty \\ b\to\infty}} C^{a;b}_{\sigma,\rho}(t) = 0 = \lim_{\substack{t\to\infty \\ k\to\infty}} C^{a;b}_{\sigma,\rho}(t) = \lim_{t\to 0} C^{a;b}_{\sigma,\rho}(t) = 0.$$

3.4. New Integrals of Products Involving Special Functions

It is worth noting that the subsequent results involving the products of a wide range of special functions are evaluated by taking Equations (10) and (39)

$$\begin{aligned} &\int_0^{\omega} \xi^{\chi-1} \exp\left(-\mathrm{a}e^{\sigma\xi} - b\mathrm{e}^{-\rho\xi}\right) H^{m,0}_{m,m}\left[\frac{\xi}{\omega}\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right] d\xi \\ &= 2\pi\omega^{\chi-2}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}; -\mathrm{a}e^{\sigma\omega} - b\mathrm{e}^{-\rho\omega}\right], \end{aligned} \tag{57}$$

and

$$\begin{aligned} &\int_0^{\omega} \xi^{\chi-1} \exp\left(-\mathrm{e}^{\rho\xi} - x\mathrm{e}^{-\xi}\right) H^{m,0}_{m,m}\left[\frac{\xi}{\omega}\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right] d\xi \\ &= 2\pi\omega^{\chi-2}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}; -\mathrm{e}^{\rho\xi} - x\mathrm{e}^{-\xi}\right]. \end{aligned} \tag{58}$$

By making use of Equations (12) and (22) along with the characterization of the Dirac delta function, subsequent new integrals of products of special functions are calculated:

$$\begin{aligned}
&\int_0^1 \mathrm{Z}_{\sigma,\rho}^{\mathrm{a;b}}(\omega\xi)H_{m,m}^{m,0}\left[\xi\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]d\xi\\
&=2\pi\omega^{-1}\sum_{\mathrm{n,r}=0}^{\infty}\frac{(-a)^n(-b)^r}{\mathrm{n!r!}}\int_0^1\delta(\omega\xi+\sigma\mathrm{n}-\rho\mathrm{r})H_{m,m}^{m,0}\left[\xi\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]d\xi\\
&=2\pi\omega^{-1}\sum_{\mathrm{n,r}=0}^{\infty}\frac{(-a)^n(-b)^r}{\mathrm{n!r!}}H_{m,m}^{m,0}\left[\frac{\rho\mathrm{r}-\sigma\mathrm{n}}{\omega}\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]\\
&=2\pi\omega^{-1}H_{m,m}^{m,0}\left[-\mathrm{a}e^{-\sigma/\omega}-b\mathrm{e}^{\rho/\omega}\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right].
\end{aligned}\tag{59}$$

and

$$\int_0^1 \mathrm{Z}_{\rho}^{\xi}(\omega\xi)H_{m,m}^{m,0}\left[\xi\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]d\xi=\frac{2\pi}{\omega}H_{m,m}^{m,0}\left[-\mathrm{e}^{-\rho/\omega}-\mathrm{x}\mathrm{e}^{1/\omega}\left|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right].\tag{60}$$

and further new integrals of products of special functions are computable by using the relation of Fox-H function $H_{m,m}^{m,0}\left[\frac{\xi}{\omega}\left|\begin{matrix}\left(\gamma k+\delta_k+1-\frac{1}{\beta k},\frac{1}{\beta k}\right)_1^m\\ \left(\gamma k+1-\frac{1}{\beta k},\frac{1}{\beta k}\right)_1^m\end{matrix}\right.\right]$ with other special functions as mentioned in Equations (11)–(16) for G-function, Fox–Wright function and Mittag-Leffler function. For example,

$$\begin{aligned}
&\int_0^1 Z_{\sigma,\rho}^{a;b}(\omega\xi)G_{m,m}^{m,0}\left[\xi\left|\begin{matrix}(\gamma_k+\delta_k)_1^m\\(\gamma_k)_1^m\end{matrix}\right.\right]d\xi=2\pi\omega^{-1}\sum_{n,r=0}^{\infty}\frac{(-a)^n(-b)^r}{n!r!}\int_0^1\delta(\omega\xi+\sigma n-\rho r)G_{m,m}^{m,0}\left[\xi\left|\begin{matrix}(\gamma_k+\delta_k)_1^m\\(\gamma_k)_1^m\end{matrix}\right.\right]d\xi\\
&=2\pi\omega^{-1}\sum_{n,r=0}^{\infty}\frac{(-a)^n(-b)^r}{n!r!}G_{m,m}^{m,0}\left[\frac{\rho r-\sigma n}{\omega}\left|\begin{matrix}(\gamma_k+\delta_k)_1^m\\(\gamma_k)_1^m\end{matrix}\right.\right]=2\pi\omega^{-1}G_{m,m}^{m,0}\left[-ae^{-\sigma/\omega}-be^{\rho/\omega}\left|\begin{matrix}(\gamma_k+\delta_k)_1^m\\(\gamma_k)_1^m\end{matrix}\right.\right].
\end{aligned}\tag{61}$$

Similarly, ([27], Equation (32)) can be rewritten, by using Equation (22), as

$$\begin{aligned}
&\int_0^1\frac{(z\omega)^{\rho-1}}{\exp(e^{z\omega})-1}H_{m,m}^{m,0}\left[\omega\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]d\omega\\
&=\omega^{-1}\int_0^{\omega}\frac{z^{\rho-1}}{\exp(e^{z})-1}H_{m,m}^{m,0}\left[\frac{z}{\omega}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]dz\\
&=\omega^{\rho-1}\sum_{n=0}^{\infty}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m\end{matrix}\left|-(n+1)e^{\omega}\right.\right].
\end{aligned}$$

4. Discussion

In this research, we have followed the recommendations of [3] to compute various fractional formulae containing the generalized Krätzel functions $Z_{\sigma,\rho}^{a;b}(\xi)$ and their simpler cases by using Kiryakova's fractional operators in the form of Theorem 1. Specifications of these results are discussed as a consequence of Kiryakova's fractional operators, considering multiplicity $m=3\wedge\ m=2\wedge\ m=1$ separately. It produced generalized fractional calculus images of the Krätzel function. Following the conclusion of [3] one needs to check if the considered special function can be presented as a Wright g.h.f. ${}_p\Psi_q$ or as simpler ${}_pF_q$-function; in more complicated cases, or if it is a Fox H-function or a Meijer G-function. It is worth noting that the Krätzel function can only be expressed in the form of these functions, as shown in Equations (9) and (10), for different domains, with no relationship between them for the entire domain. However, as suggested in [3] apply a general result such as ([3], Theorems 3 and 4) (or, more broadly, ([3], Theorem 2) and their special cases ([3], Lemmas 1–4). It can be seen that it is not the case. If it is the case then integration will be possible only w.r.t the power series variable z of H-function $H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),\ldots, & (a_p,A_p)\\(b_1,\ B_1),\ldots, & (b_q,\ B_q)\end{matrix}\right.\right]$ and here first time in this research the integration is performed with respect to the co-efficients $\mathrm{A_i}$; $\mathrm{B_i}$. For example, the above Equations (51)–(53) involve the integration with respect to co-efficient variable t of $H^{1,1}_{1,1}\left[z\left|\begin{matrix}\left(1-\frac{t}{\rho},-\frac{1}{\rho}\right)\\(0,1)\end{matrix}\right.\right]$ and $H^{2,0}_{0,2}\left[z\left|(0,1)\overline{\left(\frac{t}{\rho},\frac{1}{\rho}\right)}\right.\right]$ in contrast to the existing literature where mostly the integration is performed with respect to the variable z. Similarly, the involved Laplace transform used throughout in this paper is with respect to this co-efficient variable. Hence it is concluded that the results of this article became possible only due to a new representation [29] of Krätzel functions $\mathrm{Z}^{a;b}_{\sigma,\rho}(\xi)$ as a series of complex delta functions. Hence after, a general result such as ([3], Theorems 3 and 4) (or more generally, ([3], Theorem 2)) and their special cases ([3], Lemmas 1–4) is applicable. It is easy to check that the foremost outcome (31) and its quite a few special cases are entirely confirmable with the above-mentioned theorems.

5. Conclusions

The new fractional transformations of the generalized Krätzel function have been computed by using the multiple E–K operators of the generalized fractional calculus. Hence, the novel fractional images for the widespread non-integer operators are gained by considering the specific cases. It became possible only because the Laplace transform of the generalized Krätzel function is investigated in recent research [21]. Furthermore, while quite a lot of researchers have looked into this family of functions no research has been performed on the fractional kinetic equation. Non-integer phenomena have recently become popular in various engineering and physical science domains due to memory effects. This practice may open an avenue to put up changes in typical solar model. The generalized Krätzel function is used to formulate and solve a new fractional kinetic equation [1]. Specific cases concerning the original Krätzel function are discussed as corollaries. Thereafter, the connection of the generalized Krätzel function with H-function is used to study novel results. A novel representation of Krätzel function as well as the Laplace transform of Krätzel function have played a fundamental role to accomplish the goal of this study. It can be concluded that this work is substantial to develop the application of Krätzel function beyond its original domain and the considered fractional operators and equations can be extensively used for scientific modelling of evolutionary systems by means of memory effects on their dynamics. Furthermore, by using the fractional derivatives of delta function the solution for fractional kinetic equation can be obtained in a larger domain. It gives insights that the used approach will enhance the further applications of delta function, for example authors may use it to improve the fractional-order fuzzy control technique [43] and will also be applicable to the other trending sciences such as big data, machine learning and artificial intelligence [44].

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