THE IRREDUNDANT RAMSEY NUMBER s(3,7)

E.J. Cockayne, J.H. Hattingh
C.M. Mynhardt

DMS–538–IR May 1990

THE IRREDUNDANT RAMSEY NUMBER s(3,7)

E.J. Cockayne
University of Victoria
Victoria, CANADA

J.H. Hattingh
Rand Afrikaans University
Johannesburg, SOUTH AFRICA

C.M. Mynhardt
University of South Africa
Pretoria, SOUTH AFRICA

ABSTRACT

The irredundant Ramsey number $s(m,n)$ is the smallest integer p such that for every graph G of order p, the complement $\overline{G}$ of G contains an m–element irredundant set or G contains an n–element irredundant set. We prove that $s(3,7) = 18$.

1. INTRODUCTION

Unless stated otherwise, terminology and notation used in this paper follow that of [1]. In particular, all graphs are simple and undirected.

The *closed neighbourhood* $N[X]$ of a set X of vertices in a graph $G = (V,E)$ is defined to be the union of the closed neighbourhoods of its elements, that is, $N[X] = \bigcup_{x \in X} N[x]$. For $x \in X$, the *private neighbourhood* $PN(x,X)$ *of* x *relative to* X consists of all those vertices in the closed neighbourhood of x but not in the closed neighbourhood of the remainder of X; that is, $PN(x,X) = N[x] - N[X-\{x\}]$. The elements of $PN(x,X)$ are the *private neighbours* *of* x (*relative to* X). The set $X \subseteq V$ is *irredundant* if $PN(x,X) \neq \emptyset$ for each $x \in X$. Observe that X is irredundant iff each vertex which is not isolated in $\langle X \rangle$ has a private neighbour in $V - X$. The (*upper*) *irredundance number* IR(G) of the graph G is the largest cardinality among the irredundant sets of vertices of G. It is easy to see that the concept of irredundance extends that of independence, for if X is an independent set of G, then $x \in PN(x,X)$ for each $x \in X$ and hence X is irredundant. Since the classical Ramsey numbers can also be defined using independent sets instead of cliques, the above observation leads to the following definition of irredundant Ramsey numbers:

The *irredundant Ramsey number* $s(q_1,q_2,\cdots,q_t)$ is the smallest integer p such that for any t–edge colouring of K_p and at least one $i \in \{1,2,\cdots,t\}$, the complement $\overline{G}_i$ of the graph G_i induced by the *i*th edge colour class has an irredundant set of size q_i. Hence in the case where $t = 2$, the irredundant Ramsey number s(m,n) is the smallest integer p such that for every graph G with p vertices, $IR(\overline{G}) \geq m$ or $IR(G) \geq n$. This concept was introduced by Brewster, Cockayne and Mynhardt in [2], where it was shown that $s(3,3) = 6$, $s(3,4) = 8$ and $s(3,5) = 12$. In [3], the same authors prove that $s(3,6) = 15$. Short proofs of these results may be found in [10]. The irredundant Ramsey number s(3,3,3) was investigated in [6] and proved to be equal to 13 in [7]. In this paper we prove that $s(3,7) = 18$.

2. PRELIMINARY OBSERVATIONS

In what follows, if G is a graph of order p with $IR(\overline{G}) < m$ and $IR(G) < n$, then G will be called an *(m,n,p)–graph* or an *(m,n)–graph.* Given an (m,n,p)–graph $G = (V,E)$ and a vertex $v \in V$, we can partition V into the three subsets $V = \{v\} \cup N(v) \cup \overline{N[v]}$, where $\overline{N[v]} = V - N[v]$. Note that $\langle N(v) \rangle$ is an $(m-1,n,\deg(v))$-graph and that $\langle \overline{N[v]} \rangle$ is an $(m,n-1,p-1-\deg(v))$-graph. We shall need the following three results of [10], which are stated here without proof.

Lemma 1 [10]. *If* G *is an* (m,n,p)*-graph, then* $p - s(m,n-1) \leq \delta(G) \leq \Delta(G) \leq s(m-1,n) - 1$. ■

Lemma 2 [10]. *Suppose* G *is a graph with* $IR(\overline{G}) \leq 2$. *For an arbitrary vertex* v *of* G, *let* $X = \{x_1,x_2,\cdots,x_\ell\} \subseteq \overline{N[v]}$ *have the property that at most one of the sets* $Y_i = \{y \in N(v) \mid x_i y \in E(G)\}$, $i \in \{1,2,\cdots,\ell\}$, *is empty. Then* $\langle X \rangle$ *is bipartite.* ■

Observe that if the hypothesis of Lemma 2 is satisfied for a graph G, so that there exists an ℓ-element set $X \subseteq V$ with $\langle X \rangle$ bipartite, then X contains an $\lceil \ell/2 \rceil$-element subset which is independent in G. Thus if $\lceil \ell/2 \rceil \geq n - 1$, this set, together with v, yields an n-element independent set in G.

Lemma 3 [10]. *If* G *is a* $(3,6,14)$*-graph, then* $\delta(G) \geq 2$ *and at least* 12 *vertices of* G *have degree at least three.* ■

We shall also need two results proved in [2]. We state the first of these in a slightly stronger form, the proof of which is implicit in the proof given in [2] and hence omitted here.

Lemma 4 [2]. *Suppose* G *is a graph with* $IR(\overline{G}) \leq 2$. *For an arbitrary vertex* v *of* G, *let* $X \subseteq \overline{N[v]}$ *be such that* $\langle X \rangle \cong P_3$ *with vertex sequence* (x_1,x_2,x_3)

and such that $Y_i = \{y \in N(v) \mid x_i y \in E(G)\}$ *is nonempty for* $i \in \{1,3\}$. *Then at least one of the sets* $Y_1 - Y_3$ *and* $Y_3 - Y_1$ *is empty.* ■

Note that if the hypothesis of Lemma 4 is satisfied for vertices x_1, x_2 and x_3, then by Lemma 4, x_1 and x_3 have at least one common neighbour in $N(v)$, and not both have private neighbours (relative to $\{x_1,x_3\}$) in $N(v)$.

Lemma 5 [2]. *The graph* $\overline{G}$ *has an irredundant set of size three if and only if* G *contains a triangle or* G *contains a 6–cycle with vertex sequence* $(v_1,v_2,\cdots,v_6)$, *where* v_1v_4, v_2v_5, v_3v_6 *are all edges of* $\overline{G}$. ■

The edges v_1v_4, v_2v_5, v_3v_6 are called the *diagonals* of the 6–cycle $v_1,v_2,\cdots,v_6$. Whenever Lemma 5 is used to show that $IR(\overline{G}) \geq 3$ (or to prove the existence or non–existence of certain edges in G if $IR(\overline{G}) \leq 2$), we simply write, for brevity, "by $C(v_1,v_2,\cdots,v_6)$, . . .". A generalisation of Lemma 5 to irredundant sets of any given size can be found in [5].

3. CALCULATION OF s(3,7)

We prove that $s(3,7) = 18$ by proving that no (3,7,18)–graph exists. In order to do so, we henceforth assume G to be a (3,7,18)–graph and 2–colour the edges of K_{18} in the colours red and blue, where the red edges belong to G and the blue edges to $\overline{G}$. Instead of saying that uv is an edge of G ($\overline{G}$, respectively), we shall also say that uv is a red (blue) edge, or that "u sends a red (blue) edge to v", etc. We begin by proving that G is 4–regular. For $v \in V$, let S_v consist of those vertices in $\overline{N[v]}$ which send blue edges to each vertex in $N(v)$. As before, $\deg(v)$ is used to denote the degree of the vertex v in G, *i.e.*, the *red degree* of v.

Theorem 1. *If G is a (3,7,18)–graph, then G is 4–regular.*

Proof. An application of Lemma 1 shows that $3 \leq \deg(v) \leq 6$ for each vertex v of G. To show that $\Delta(G) \leq 4$, we use Lemma 2. If $\deg(v) = 6$, then $S_v = \emptyset$, for otherwise $N(v) \cup S_v$ contains a set of cardinality seven which is independent in G, contradicting $IR(G) \leq 6$. Hence Lemma 2 can be applied with $X = \overline{N[v]}$ to yield an independent set of cardinality $\lceil 11/2 \rceil + 1 = 7$ in G, again a contradiction. Suppose $\deg(v) = 5$. If $|S_v| \geq 3$, then, since $IR(\overline{G}) \leq 2$, Lemma 5 implies that G has no triangles and therefore there exist two nonadjacent vertices, $r,s \in S_v$. But then $N(v) \cup \{r,s\}$ is an independent set of G of cardinality seven, which is impossible. Consequently, $|S_v| \leq 2$ and we can apply Lemma 2 with $|X| = 11$, which yields a contradiction as before.

To show that $\delta(G) \geq 4$, suppose v is a vertex of G with $\deg(v) = 3$. If $|S_v| \geq 8$, then since $s(3,4) = 8$ and $IR(\overline{G}) \leq 2$, the graph $\langle S_v \rangle$ contains an irredundant set T with $|T| \geq 4$. But then $N(v) \cup T$ is a 7–element irredundant set in G, a contradiction. Hence $|S_v| \leq 7$ so that $|\overline{N[v]} - S_v| \geq 7$. Since $\Delta(G) \leq 4$, each vertex in $\overline{N[v]} - S_v$ has degree at most three in $H = \langle \overline{N[v]} \rangle$, a (3,6,14)–graph. Lemma 3 implies that H has at least five (and hence at least six) vertices of degree exactly three. However, the computer algorithm described in Section 4 and the analysis given in Sections 5 and 6 show that no (3,6,14)–graph contains a vertex of degree three, a contradiction. ∎

In view of Theorem 1, we may now assume that for any vertex v of G, the graph $H = \langle \overline{N[v]} \rangle$ is a (3,6,13)–graph with 20 edges. We now prove that

$\langle S_v \rangle$ is one of only four graphs.

Lemma 6. *$\langle S_v \rangle$ is a $(3,3,|S_v|)$-graph, where $4 \leq |S_v| \leq 5$.*

Proof. Since G is a (3,7,18)-graph, $IR(\overline{\langle S_v \rangle}) \leq IR(\overline{G}) \leq 2$. Further, if $\langle S_v \rangle$ contains a 3-element irredundant set, then the union of this set and $N(v)$ is a 7-element irredundant set of G, which is impossible. Since $s(3,3) = 6$, this implies that $|S_v| \leq 5$. If $S_v = \emptyset$, we may apply Lemma 2 with $X = \overline{N[v]}$ to obtain a contradiction. If $1 \leq |S_v| \leq 3$, let $s \in S_v$ be arbitrary and apply Lemma 2 with $X = (\overline{N[v]} - S_v) \cup \{s\}$ to obtain a similar contradiction. ∎

Note that Lemma 6 implies that $\langle S_v \rangle \in \{2K_2, P_4, C_4, C_5\}$. The possible degree sequences of H are determined in Lemma 8. In order to do this, we first prove the existence of matchings of certain cardinalities in H. As in the proof of Lemma 6, let $X = (\overline{N[v]} - S_v) \cup \{s\}$ for some $s \in S_v$; recall that $\langle X \rangle$ is bipartite.

Lemma 7. *If $|S_v| = 4$ ($|S_v| = 5$ respectively), then there is a matching consisting of five (four) edges in $\langle X \rangle$.*

Proof. If $|S_v| = 4$, then $|X| = 10$ and since H is a (3,6,13)-graph, $\langle X \rangle$ has no 6-element independent set. Hence $\langle X \rangle$ has bipartition (X_1,X_2) with $|X_1| = |X_2| = 5$. Let M be a maximum matching from X_1 to X_2. If M does not saturate X_2 (say), then by Hall's Theorem (see [1, p. 161]), there is a subset A of X_2 such that $|N(A)| < |A|$, where here $N(A) = (\bigcup_{a \in A} N(a)) \cap X_1$. But then $|(X_1 - N(A)) \cup A| = |X_1| - |N(A)| + |A| > |X_1| = 5$ and $(X_1 - N(A)) \cup A$ is independent in H, which is impossible.

Consequently, M consists of five edges.

Similarly, if $|S_v| = 5$, *i.e* $|X| = 9$, then $\langle X\rangle$ has a matching consisting of four edges. ■

Lemma 8. *If* $|S_v| = 4$ ($|S_v| = 5$ *respectively*), *then* H *has degree sequence* (2,2,2,3,3,3,3,3,3,4,4,4,4) ((2,2,2,2,3,3,3,3,4,4,4,4,4) *respectively*).

Proof. By assumption, H has exactly $|S_v|$ vertices which are nonadjacent to every vertex in N(v). Since G is 4–regular, these are precisely the vertices of H of degree four. Since H is a (3,6,13)–graph and $s(3,5) = 12$, every vertex of H has degree at least one in H. Since G is 4–regular, each vertex in N(v) is adjacent to three vertices in $\overline{N[v]} - S_v$. If $\deg_H(u) = 1$, then u is adjacent to three vertices in N(v) and there are nine edges joining vertices in N(v) to vertices in $B = \overline{N[v]} - \{u\} - S_v$. Since $7 \leq |B| \leq 8$, there is a vertex $w \in B$ which is adjacent to at least two vertices in N(v) and hence $\deg_H(w) \leq 2$. Note that u and w have a common neighbour in N(v) and hence, since G is triangle–free, u and w are nonadjacent. Let $C = N_H[u] \cup N_H[w]$. Then $|C| \leq 5$ and since $s(3,4) = 8$, H – C contains a 4–element irredundant set X. Clearly, $X \cup \{u,w\}$ is irredundant in H, which is impossible. Hence H only has vertices of degrees two, three and four. Since H has 13 vertices and 20 edges, the result easily follows by solving the simultaneous equations

$$\left.\begin{aligned} 2a + 3b + 4c &= 40 \\ a + b + c &= 13 \end{aligned}\right\},$$

where a, b and c are the number of vertices of degree two, three and four respectively and $c = |S_v|$. ■

Lemma 9. G *does not contain a subgraph isomorphic to* K(2,3).

Proof. Suppose F is a subgraph of G isomorphic to K(2,3) and let u, w be the vertices of F of degree three. Then u has degree at most one in $\langle\overline{N[w]}\rangle$, contradicting Lemma 8. ■

In order to prove that no (3,7,18)–graph G exists, we prove that for any $v \in V(G)$, the graph $\langle S_v\rangle \notin \{2K_2,P_4,C_4,C_5\}$. We henceforth assume that $V(K_{18}) = \{0,1,\cdots,17\}$ and choose v to be the vertex 0, while the red neighbours of v form the set $N(v) = \{14,15,16,17\}$ so that $V(H) = \{1,2,\cdots,13\}$.

Assume first that $|S_v| = 4$, say 10, 11, 12, 13 send no red edges to N(v). By Lemma 2, $\langle\{1,2,\cdots,9,10\}\rangle$ is bipartite and we may assume without loss of generality that $\langle\{1,2,\cdots,9,10\}\rangle$ has bipartition (X,Y) with $X = \{1,\cdots,5\}$ and $Y = \{6,\cdots,10\}$ and that $M = \{\{i,i+5\} \mid i \in \{1,2,\cdots,5\}\}$ is the red matching from X to Y guaranteed by Lemma 7.

Lemma 10. *If* $|S_v| = 4$, *then* $\langle S_v\rangle \cong C_4$.

Proof. If $\langle S_v\rangle \in \{2K_2,P_4\}$, then $\langle S_v\rangle$ has at least two nonadjacent vertices u and w of degree one. Since G is 4–regular, there exist vertices $u_1,u_2,u_3,w_1,w_2,w_3 \in \{1,2,\cdots,9\}$ such that uu_i, ww_i are red for each $i \in \{1,2,3\}$. Now any two vertices u_i, u_j, with $i \neq j$ are distinct and form the end vertices of a common P_3 with vertex u in H and hence by Lemma 4, have a common red neighbour in N(v). By Lemma 9, no vertex in N(v) is adjacent to all three vertices u_1, u_2 and u_3. Hence we may assume without losing generality that

$u_1 14$, $u_2 14$, $u_2 15$, $u_3 15$, $u_1 16$, $u_3 16$ are red edges, so that u_i, $i \in \{1,2,3\}$, are the vertices of degree two in H. However, the vertices w_1, w_2 and w_3 also do not all have a common neighbour in N(v). Since 17 is the only vertex in N(v) that can possibly be adjacent to two distinct vertices in $\{w_1,w_2,w_3\} - \{u_1,u_2,u_3\}$, it follows that $\{u_1,u_2,u_3\} \cap \{w_1,w_2,w_3\} \neq \emptyset$.

Let $x \in \{u_1,u_2,u_3\} \cap \{w_1,w_2,w_3\}$. Note that x is adjacent to two distinct vertices in N(v), to the distinct vertices $u,w \notin N(v)$ and to a vertex y in the matching M from X to Y. Since G is 4–regular, the only possibility is when $x = 5$, $y = 10$ and one of u and w, say u, is the vertex 10. Say $x = u_1 = w_1$. Then the common neighbour of x and w_2 (x and w_3) in N(v) is 14 (16) (say), while w_2 and w_3 are both adjacent to 17. Now w_2 and w_3 are both adjacent to two vertices in N(v) and hence also of degree two in H, which contradicts Lemma 8. ∎

Let $\langle S_v \rangle$ have vertex sequence (10,11,12,13). Note that 10 12 and 11 13 are blue. Hence to avoid a blue K_6, 12 sends a red edge to X, while 11 and 13 send two independent red edges to X – N[10]. Note that 10 is adjacent to exactly two vertices in X. Without loss of generality, assume that 10 4, 11 3, 13 2 are red. By Lemma 4, vertices 4 and 5 have a common red neighbour, say 14, in N(v). Note that v sends no red edge to N(10). Hence if $H' = \langle \overline{N[10]} \rangle$ (and S_{10} is the set of all vertices of H' which send no red edges to N(10)), then exactly two vertices in {15,16,17} send no red edges to N(10) since $\langle S_{10} \rangle \in \{C_4,C_5\}$. In particular, two vertices in {15,16,17}, say 16 and 17, send no red edges to 4 and 5. We formulate a more general result.

Lemma 11. *Let* u *be any vertex of* H *which sends no red edges to* N(v). *Then* $v \in S_u$ *and exactly two vertices in* N(v) *are in* S_u.

Proof. If u sends no red edges to $N(v)$, then v sends no red edges to $N(u)$, hence $v \in S_u$. Since $\langle S_u \rangle \in \{C_4, C_5\}$, $|N(v) \cap S_u| = 2$. ■

We next prove that $1 \in S_{10}$.

LEMMA 12. *Vertex* 1 *sends no red edges to* $N(10)$.

Proof. If $1 \notin S_{10}$, then 1 is adjacent to 11 or 13; say 1 11 is red. But then 12 sends no red edges to $\{1,2,3\}$, so that 12 4 or 12 5 is red. However, in this case 12 sends three red edges to $N(10)$, contradicting Lemma 9. ■

Note that, by the proof of Lemma 12, the edge 12 1 is red, while 5 12 and 4 12 are blue. Since $\{v,16,17,1\} \subseteq S_{10}$, we may assume that 1 16 (say) is red.

LEMMA 13. *Vertex* 1 *has no common neighbours with* 4 *and* 5 *in* $N(v)$.

Proof. Let $x \in \{4,5\}$ and suppose 1 and x have a common neighbour y in $N(v)$. Note that x 12 is blue, as are y 11 and 1 10. Hence by $C(x,y,1,12,11,10)$, $IR(\overline{G}) \geq 3$, a contradiction. ■

COROLLARY. *The edges* 1 15, 4 6, 5 6, 1 9 *are blue.*

Proof. Since 16 and 17 are the only vertices in $N(v)$ which send no red edges to $N(10)$, 15 sends a red edge to $\{4,5\}$, hence, by Lemma 13, 1 15 is blue. If 4 6 (or 5 6) is red, then by Lemma 4, vertices 4 (or 5) and 1 have a common neighbour in N(v); therefore 4 6 and 5 6 are blue. Similarly, 1 9 is blue. ∎

Lemma 14. *Vertices* 2 *and* 3 *have no common neighbour in* N(v).

Proof. Note that 2 11 and 3 13 are blue, for otherwise 11 would have degree 1 in $\langle\overline{N[13]}\rangle$. If $x \in N(v)$ and 2 x, 3 x are red, then by C(2,13,10,11,3,x), $IR(\overline{G}) \geq 3$. ∎

In order to prove that $S_{10} = \{v,16,17,1\}$, we prove in the next three lemmas that $\{6,7,8\} \cap S_{10} = \emptyset$.

Lemma 15. *Vertex* 6 *sends a red edge to* N(10).

Proof. Note that 6 12 is blue. By Lemma 13, the edge 1 14 is blue. If 6 sends only blue edges to N(10), then, since $deg_H(6) \geq 2$, vertex 6 sends a red edge to 2 or 3. Without loss of generality, let 2 6 be red. Also, in this case $\langle S_{10}\rangle \cong C_5$, so 6 is adjacent to 17. By Lemma 4, vertices 1 and 2 have a common neighbour in N(v) and this is 16. To avoid a K(2,3), 1 7 is blue. Therefore 1 and 3 have a common neighbour in N(v) and this must be 16. However, 2 16 is red, contradicting Lemma 14. ∎

Lemma 16. *Vertex* 7 *sends a red edge to* N(10).

Proof. Since 6 sends a red edge to N(10) and 4 6, 5 6 are blue, 6 sends a red edge to {11,13}; assume without loss of generality that 6 13 is red.

Then 1 7 is blue, for otherwise (1,7,2,13,6) is a 5–cycle, contradicting the bipartiteness of $\langle\{1,2,\cdots,9,13\}\rangle$ (see Lemma 2). Therefore $7 \notin S_{10}$, for otherwise $|S_{10}| = 5$ but $\langle S_{10}\rangle \ncong C_5$.∎

As in the proof of Lemma 16, we henceforth assume that 6 13 is red. Note that then 2 6 is blue.

Lemma 17. *Vertex* 8 *sends a blue edge to* N(10).

Proof. Suppose $8 \in S_{10}$. Then $S_{10} = \{v,16,17,1,8\}$, so that 1 8, 8 17 and also (by Lemma 4) 3 16 are red. Let S_1 denote the set of vertices nonadjacent to N(1). Clearly, $\{4,5,10\} \subseteq S_1$. Since 2 4, 2 5 are blue and $\langle S_1\rangle \in \{C_4,C_5\}$, 2 sends a red edge to {6,8,12,16}. Note that 2 12 is blue and by Lemma 14, the edge 2 16 is blue. Since 2 6 is blue, 2 8 is red, so that 1 and 2 have a common neighbour in N(v). But 2 16 is blue and by the 4–regularity of G, vertex 1 is nonadjacent to each vertex in N(v) − {16}, a contradiction. ∎

By Lemmas 15, 16 and 17, $S_{10} = \{v,16,17,1\}$, for all other vertices send red edges to N(10). However, we prove that this is not possible, and consequently that for any $v \in V(G)$, at least five vertices in $\overline{N[v]}$ send no red edges to N(v).

Lemma 18. $S_{10} \neq \{v,16,17,1\}$.

Proof. Suppose $S_{10} = \{v,16,17,1\}$. Then 1 17 is red and 8 sends a red edge to {4,5,11}. To avoid triangles, both 6 17 and 6 16 are blue so that the common neighbour of 2 and 6 in N(v) is 15. Since 3 11 and 3 8 are red, 8 11 is blue and so 8 sends a red edge to $a \in \{4,5\}$. Lemma 13 implies that the common neighbour of a and 3 is 14. By considering S_{13}, we see that $\{v,14\} \subseteq S_{13}$, hence exactly one of 16 and 17 belongs to S_{13}. Vertex 2 therefore sends a red edge to, without losing generality, 16. Note that by Lemma 14, the edge 3 16 is blue. Also, 3 17 is blue, for otherwise H would have at least four vertices of degree two, namely 1, 2, 3 and whichever vertex of 4, 5 is adjacent to 15. But now $\{3,4,5,14\}$ sends no red edges to $N(1) = \{6,12,16,17\}$ and $\langle\{3,4,5,14\}\rangle \cong K(1,3)$, which is impossible. ■

We have therefore proved the following theorem:

Theorem 2. *If* G *is a* (3,7,18)*–graph with* $v \in V(G)$ *and* S_v *is the set of vertices which are not adjacent to any vertices in* $N(v)$, *then* $\langle S_v\rangle \cong C_5$. ■

Let G be a (3,7,18)–graph and consider any vertex v of G. By Theorem 2 and Lemma 8, the graph $H = \langle\overline{N[v]}\rangle$ has degree sequence (2,2,2,2,3,3,3,3,4,4,4,4,4). Let X and Y be the sets of vertices of degree two and three respectively. By Theorem 2, $\langle S_v\rangle$ has five edges and there are ten edges joining the vertices in S_v to the vertices in $X \cup Y$. Consequently, there are five edges in $\langle X \cup Y\rangle$. Further, each vertex in X (Y) sends two edges (one edge) to $N(v)$. We next determine the number of edges in $\langle X\rangle$ and $\langle Y\rangle$ respectively.

Lemma 19.

(a) *The bipartite subgraph* F *of* G *induced by the red edges from* X *to* $N(v)$ *is* 2*–regular.*

(b) *The four edges between* Y *and* $N(v)$ *form a matching.*

(c) *No two vertices in* Y *have a common neighbour in* H.

(d) *There are at most five red edges from* Y *to* S_v.

(e) $\langle Y \rangle$ *contains exactly two edges while* $\langle X \rangle \cong \overline{K}_4$.

Proof.

(a) Every vertex in X is of degree two in F. By Lemma 5, F contains no diagonal–free 6–cycles. Hence if $N(v)$ contains two vertices of degree three, or one vertex of degree three and two vertices of degree two (in F), then it is easy to see that F contains a $K(2,2)$, which, together with v, forms a $K(2,3)$ in G, contradicting Lemma 9.

(b) This follows directly from the fact that F is 2–regular, G is 4–regular, and each vertex in Y sends one red edge to $N(v)$.

(c) If any two vertices in Y have a common neighbour in H, then by Lemma 4, these vertices have a common neighbour in $N(v)$, contradicting (b).

(d) If there are at least six red edges from Y to S_v, then some vertex in S_v is adjacent to at least two vertices in Y, contradicting (c).

(e) By (c), the graph $\langle Y \rangle$ has at most two edges. If $\langle Y \rangle$ has exactly two edges, then Y sends eight red edges to $X \cup S_v$, so that by (d), Y sends at least three red edges to X. Since $\langle X \cup Y \rangle$ contains exactly five edges, it follows that there are exactly three edges from Y to X and that $\langle X \rangle \cong \overline{K}_4$. If $\langle Y \rangle$ has exactly one edge, it follows similarly that there are at least five edges between X and Y, which is impossible. Similarly, $\langle Y \rangle$ is not isomorphic to $\overline{K}_4$. ∎

We now present our main result.

THEOREM 3. $s(3,7) = 18$.

Proof. Let G be a (3,7,18)–graph and v an arbitrary vertex of G. We count the number of 5–cycles through v. Clearly, any such 5–cycle contains exactly two vertices in N(v) and no vertices in S_v since S_v sends no red edges to N(v). Hence every 5–cycle through v contains exactly one edge in $\langle X \cup Y \rangle$. Further, any edge in $\langle Y \rangle$ lies on exactly one such cycle whereas every edge from X to Y lies on exactly two such cycles. Therefore there are exactly eight 5–cycles through v. However, this holds for every vertex v of G so that G contains precisely $(8 \times 18)/5$ 5–cycles, which is impossible since $(8 \times 18)/5$ is not an integer. Consequently, no (3,7,18)–graph exists, so that $s(3,7) \leq 18$.

An example of a (3,7,17)–graph G which shows that $s(3,7) \geq 18$ is the graph G with $V = \{0,1,\cdots,16\}$, where vertices i and j are adjacent if and only if $i - j = \pm 1$ or $\pm 4 \pmod{17}$. That $IR(\overline{G}) = 2$ and $IR(G) = 6$ can easily be verified by computer — *c.f.* [4]. ■

4. THE ALGORITHM

In this section we describe the computer algorithm which is used in the proof of Theorem 1. This algorithm is based on one which can be found in [8].

Algorithm (*Constructing all the* (3,n,p)*–graphs* G *in which a vertex* v *has degree* $r \geq 2$ *with the vertices in* N(v) *having degrees* $d_1 \geq d_2 \geq \cdots \geq d_r \geq 2$)

Input The set $\{H_1,H_2,\cdots,H_k\}$ of all (3,n–1,p–r–1)–graphs.

Output All (3,n,p)–graphs as specified in the algorithm heading, if any exist.

Method

1. Replace d_j by $d_j - 1$ for $j = 1,2,\cdots,r$.
2. Let $i = 0$.
3. Replace i by $i + 1$. If $i = k + 1$, then stop.
4. Generate a list of all independent sets of cardinality d_j in H_i for

$j = 1,2,\cdots,r$ — name these sets $S^j_1, S^j_2, \cdots, S^j_{\ell_j}$. If $\ell_j = 0$ for some j, go to step 3.

5. Generate a list of all independent sets of cardinality $n - 2$ in H_i — name these sets $S_1, S_2, \cdots, S_q$.

Comment We say that the pair of sets $(S^{j_1}_{t_1}, S^{j_2}_{t_2})$ is *good* if and only if the graph $H_i - (S^{j_1}_{t_1} \cup S^{j_2}_{t_2})$ has no independent set of cardinality $n - 2$.

6. For $j = 1,2,\cdots,r$, create an $(\ell_j \times q)$–matrix $n(j)$ as follows:

   Entry (a,b) in $n(j)$ is set to one if and only if $S^j_a \cap S_b = \emptyset$, 0 otherwise. (Of course, if $q = 0$, it is not possible to create these matrices.)

Comment Note that the following statements are equivalent:

(i) The pair of sets $S^{j_1}_{t_1}, S^{j_2}_{t_2})$ is good.

(ii) The graph $H_i - (S^{j_1}_{t_1} \cup S^{j_2}_{t_2})$ has no independent set of cardinality $n - 2$.

(iii) Each of the sets S_a has nonempty intersection with at least one of the sets $S^{j_1}_{t_1}$ or $S^{j_2}_{t_2}$.

(iv) The t_1–*th* row vector of $n(j_1)$ and the t_2–*th* row vector of $n(j_2)$ are orthogonal.

(Of course, if $q = 0$, every such pair is good.)

7. For each pair (a,b), $a < b$, $a,b \in \{1,2,\cdots,r\}$, create an $(\ell_a \times \ell_b)$–matrix $m(a,b)$ as follows:

   Entry (e,f) in $m(a,b)$ is set to one if and only if the pair (S^a_e, S^b_f) is a good pair, 0 otherwise.

   Furthermore, if $d_a = d_b$, entry (e,f) in $m(a,b)$ is set to zero for each $f = 1,2,\cdots,\ell_a$ and each $e = 1,2,\cdots,f - 1$.

Comment We will now determine all possible r–tuples of sets $(S^1_{t_1}, S^2_{t_2}, \cdots, S^r_{t_r})$ where each pair of sets is a good pair — call such an r–tuple a *good* r–tuple. For each good r–tuple, let $N(v) = \{w_1, w_2, \cdots, w_r\}$ and, for $j = 1,2,\cdots,r$, let $E_j = \{w_j x \mid x \in S^j_{t_j}\}$. Also, let $E_{r+1} = \{vw_j \mid j = 1,2,\cdots,r\}$. Form a new graph G as follows: let $V(G) = V(H_i) \cup N[v]$ and let $E(G) = E(H_i) \cup \bigcup_{j=1}^{r+1} E_j$. This graph is then tested for a (3,n)–graph.

8. Let $c = 2$, let $t_1 = 1$ and let $t_2 = 0$.
9. If $c = 0$, then go to step 3.

   Increment t_c by one until either $t_c > \ell_c$ or $m(c-1,c)(t_{c-1},t_c) = 1$.

Comment If $t_c > \ell_c$, then the (c–1)–tuple $(S^1_{t_1}, S^2_{t_2}, \cdots, S^{c-1}_{t_{c-1}})$ can never be part of a good r–tuple and, thus, we backtrack.

10. If $b_c > \ell_c$, then decrement c by 1 until either $t_c < \ell_c$ or $c = 0$. If $c = 1$, then replace t_c by $t_c + 1$, replace c by 2 and replace t_c by 0. Go to step 9.
11. If $m(c-1,c)(t_{c-1},t_c) = 1$, then the c–tuple $(S^1_{t_1}, S^2_{t_2}, \cdots, S^c_{t_c})$ can be part of a good r–tuple if all the pairs $(S^j_{t_j}, S^c_{t_c})$ for $j = 1,2,\cdots,c-1$ are also good. If not, we go to step 9. If this is indeed the case, we consider two cases:

    if $c = r$, we have found a good r–tuple, we form our new graph G as discussed above, test it for a (3,n)–graph and proceed to step 9;

    if $c < r$, we move forward in our algorithm by replacing c by $c + 1$, t_c by 0 and going to step 9.

5. THE (3,4)–GRAPHS

In this section we construct all the (3,4,p)–graphs, for $p = 5,6,7$. The (3,4,p)–graphs for $p = 5$ and $p = 6$ appear in Figure 1 and 2 respectively — they were found by examining the graphs of order 5 and 6 which appear in [9]. As before, we use red (blue) edges to denote edges of G $(\overline{G})$.

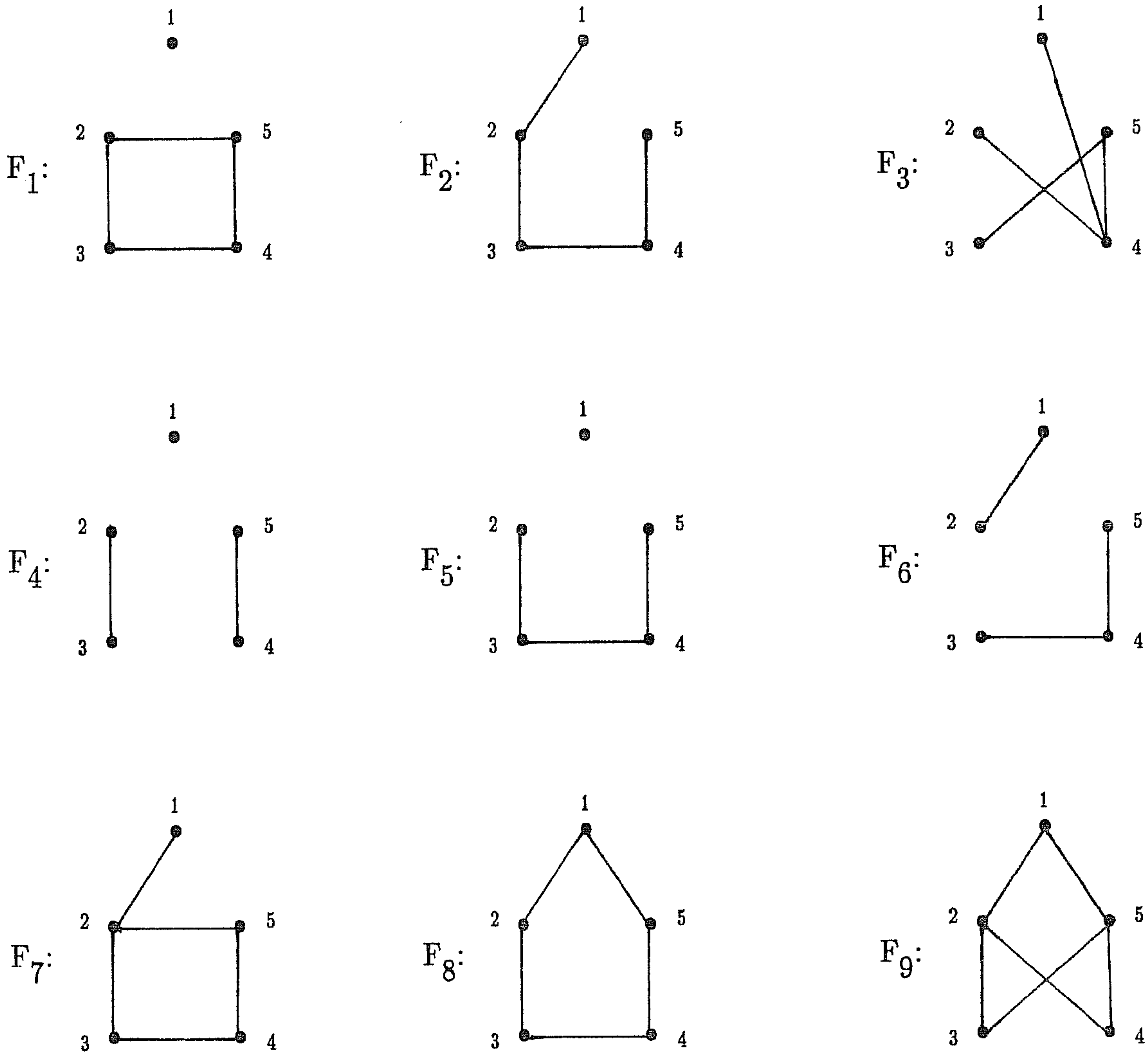

Figure 1. The (3,4,5)–graphs.

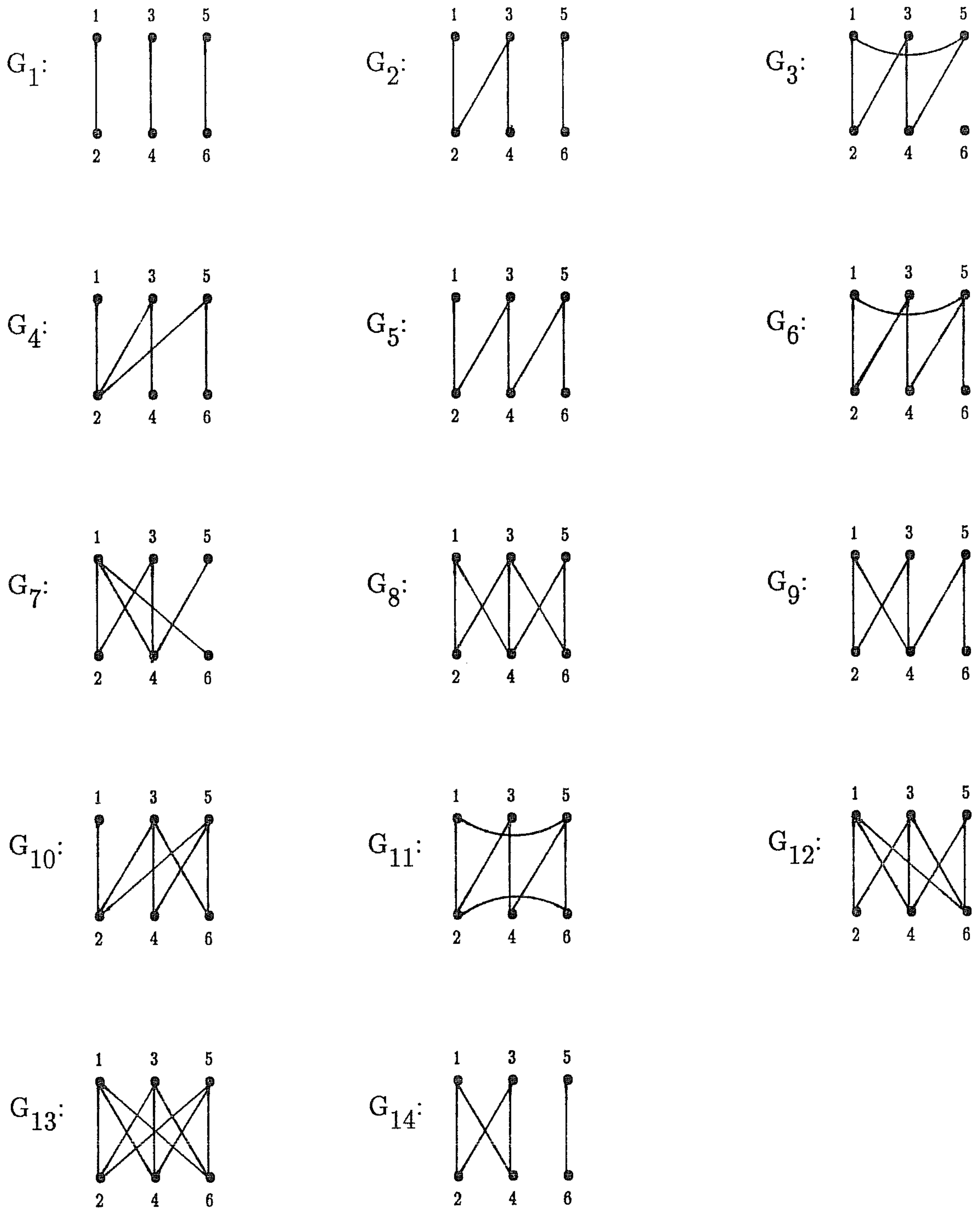

Figure 2. The (3,4,6)–graphs.

Claim: *The graphs which appear in Figure* 3 *are the only* (3,4,7)*–graphs.*

Proof (of Claim). Suppose G is a (3,4,7)–graph. By Lemma 1, we have $1 \leq \delta(G) \leq \Delta(G) \leq 3$. If $\deg(v) = 1$ for some $v \in V$, then $\langle\overline{N[v]}\rangle = H \cong C_5$ and we have graphs H_2, H_3 and H_8 of Figure 3. If G is 2–regular, then $G \cong C_7 = H_1$. Now suppose that $2 \leq \delta(G) \leq \Delta(G) = 3$, say $\deg v = 3$ for $v \in V$. Let $N(v) = \{x,y,z\}$ and $\overline{N[v]} = \{a,b,c\}$ and note that $H \cong P_3$ or $H \cong K_2 \cup K_1$ (where again, $H = \langle\overline{N[v]}\rangle$). Suppose firstly that $H \cong P_3$ with vertex sequence (say) (a,b,c). Since $IR(G) \leq 3$, every vertex in $\overline{N[v]}$ sends a red edge to a vertex in $N(v)$. By Lemma 4, vertices a and c have a common neighbour, say x, in $N(v)$. Assume that yb is also red. Since $\deg(b) = 3$ and $\delta(G) \geq 2$, vertex z sends a red edge to a or c — assume without loss of generality that az is red. Thus, if cz is blue (red), we obtain the graph H_4 (H_5) in Figure 3. Now suppose that $H \cong K_2 \cup K_1$ with c the isolated vertex of H. Again, each vertex in $\overline{N[v]}$ sends a red edge to a vertex of $N(v)$ and we may assume that ax and by are red. Now c sends red edges to at least two vertices in $N(v)$ and hence we may assume that cy is red. If cz is blue, then cx is red and we may assume without losing generality that az is red. This yields a graph isomorphic to H_4. Let cz be red. If all remaining edges are blue, we obtain H_6. Other possible red edges are az, bz (but not both) and cx. Of these, the red subset $\{az\}$ $(\{bz\}, \{cx\}, \{az,cx\}, \{bz,cx\})$ yields a graph isomorphic to H_4 (H_7,H_7,H_5,H_5). ■

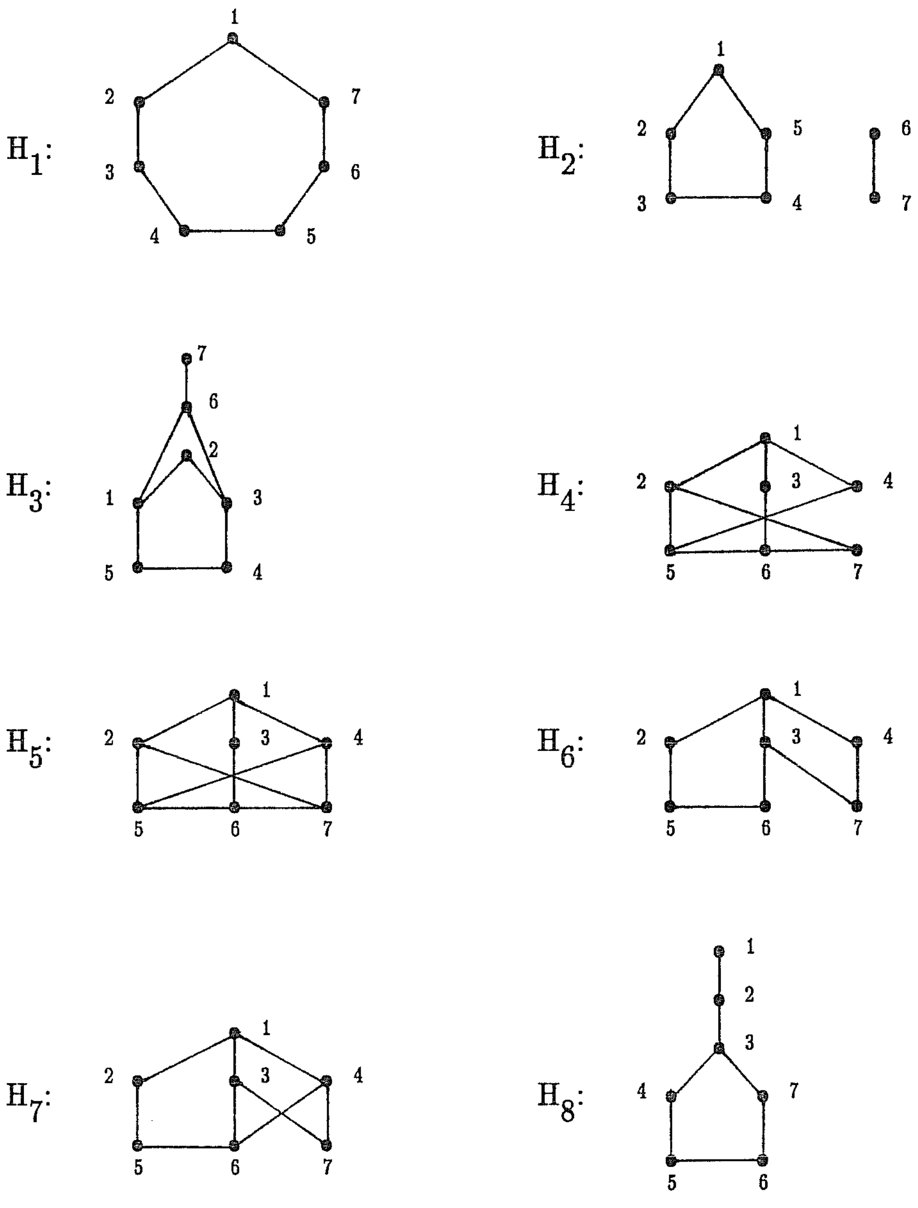

Figure 3. The (3,4,7)–graphs.

6. THE (3,5,10)–GRAPHS

In this section we construct all the (3,5,10)–graphs using the algorithm of Section 4 and the (3,4)–graphs of Section 5.

If G is a (3,5,10)–graph, its vertices have degree two, three or four. If G is a (3,5,10)–graph with a vertex v of degree two, then $\langle\overline{N[v]}\rangle$ is a (3,4,7)–graph and the possible degrees in G of the vertices adjacent to v are: 4,4; 4,3; 4,2; 3,3; 3,2 and 2,2. These graphs appear in Table I. As an illustration of the notation used in this table, we take the graph 1 of Table I. The vertex v appears as vertex 10. The middle columns show that the neighbours of 10 are 8 and 9 and that these vertices are adjacent to the vertices in {1,3,5,10} and {2,10} respectively. Then $\langle\overline{N[10]}\rangle$ is given in the last column. In this case, $\langle\overline{N[10]}\rangle$ is H_1 of Figure 3.

TABLE I

The (3,5,10)–graphs in which a vertex v has degree two

Graph	*Neighbours of vertices in*	N(10)	$\langle\overline{N[10]}\rangle$
1	8–1,3,5,10	9–2,10	H_1
2	8–1,3,10	9–2,4,10	H_1
3	8–1,3,10	9–8,10	H_1
4	8–1,3,5,10	9–1,4,6,10	H_1
5	8–1,3,6,10	9–1,3,7,10	H_2
6	8–6,10	9–7,10	H_2
7	8–2,6,10	9–7,10	H_3
8	8–2,4,6,10	9–1,4,7,10	H_3
9	8–1,3,7,10	9–2,6,10	H_3
10	8–1,4,7,10	9–4,6,10	H_3
11	8–2,4,6,10	9–7,10	H_3
12	8–1,4,7,10	9–5,10	H_3
13	8–2,6,10	9–1,7,10	H_3
14	8–4,6,10	9–3,7,10	H_3
15	8–1,3,7,10	9–2,10	H_3
16	8–2,4,10	9–2,7,10	H_3
17	8–2,7,10	9–2,7,10	H_3
18	8–2,4,6,10	9–2,7,10	H_3
19	8–1,3,7,10	9–2,7,10	H_3
20	8–1,6,10	9–3,10	H_4

21	8–2,4,10	9–7,10	H_4
22	8–2,3,4,10	9–3,5,10	H_4
23	8–2,4,6,10	9–5,7,10	H_4
24	8–2,3,4,10	9–3,7,10	H_4
25	8–3,4,7,10	9–3,4,10	H_4
26	8–3,4,10	9–3,7,10	H_4
27	8–2,3,4,10	9–3,4,7,10	H_4
28	8–2,3,4,10	9–3,5,7,10	H_4
29	8–3,4,7,10	9–3,4,7,10	H_4
30	8–2,4,6,10	9–7,10	H_4
31	8–2,3,10	9–3,5,10	H_4
32	8–3,4,7,10	9–3,10	H_4
33	8–2,3,10	9–3,7,10	H_4
34	8–2,4,10	9–5,7,10	H_4
35	8–2,4,6,10	9–5,10	H_5
36	8–2,4,10	9–1,5,10	H_5
37	8–2,4,10	9–5,7,10	H_5
38	8–2,3,10	9–3,5,10	H_5
39	8–2,3,4,10	9–3,5,10	H_5
40	8–2,3,4,10	9–3,5,7,10	H_5
41	8–2,4,6,10	9–5,7,10	H_5
42	8–2,4,10	9–5,10	H_5
43	8–2,3,10	9–1,10	H_5
44	8–1,6,10	9–3,10	H_5
45	8–2,3,4,10	9–1,10	H_5
46	8–2,4,10	9–7,10	H_6
47	8–2,3,4,10	9–7,10	H_6
48	8–3,4,5,10	9–6,10	H_6
49	8–4,10	9–7,10	H_6
50	8–2,3,4,10	9–2,6,7,10	H_6
51	8–2,4,6,10	9–2,6,7,10	H_6
52	8–2,3,4,10	9–1,10	H_7
53	8–1,6,7,10	9–3,10	H_7
54	8–3,4,10	9–7,10	H_7
55	8–2,3,4,10	9–2,6,7,10	H_7
56	8–2,3,10	9–1,7,10	H_7

We now construct all the (3,5,10)–graphs in which a vertex v has degree three with the additional assumption that the minimum degree in such a graph is three. If G is a (3,5,10)–graph and v is a vertex of degree three in G, then the possible degrees of the vertices adjacent to v are: 4,4,4; 4,4,3; 4,3,3; 3,3,3. These graphs appear in Table II.

TABLE II

The (3,5,10)–graphs with minimum degree three

in which a vertex v has degree three

Graph	*Neighbours of vertices in* N(10)			$\langle\overline{N[10]}\rangle$
57	7–1,3,5,10	8–2,6,10	9–5,6,10	G_7
58	7–1,3,5,10	8–2,4,6,10	9–5,6,10	G_7
59	7–1,3,5,10	8–2,5,6,10	9–2,6,10	G_7
60	7–2,5,6,10	8–2,5,6,10	9–3,5,10	G_7
61	7–2,5,6,10	8–3,5,6,10	9–5,6,10	G_7
62	7–1,3,5,10	8–2,4,6,10	9–2,5,6,10	G_7
63	7–1,3,5,10	8–2,5,6,10	9–2,5,6,10	G_7
64	7–2,5,6,10	8–2,5,6,10	9–3,5,6,10	G_7
65	7–1,3,5,10	8–1,4,6,10	9–4,6,10	G_{10}
66	7–1,3,5,10	8–1,4,6,10	9–1,4,6,10	G_{10}
67	7–1,3,5,10	8–1,4,6,10	9–2,4,6,10	G_{10}
68	7–1,3,5,10	8–2,4,6,10	9–2,5,10	G_{12}

The following graph is the unique 4–regular (3,5,10)–graph:

Graph	*Neighbours of vertices in* N(10)				$\langle\overline{N[10]}\rangle$
69	6–1,2,4,10	7–1,2,4,10	8–1,3,5,10	9–1,3,5,10	F_1

By now using the 69 (3,5,10)–graphs constructed above as input in the algorithm, we see that there does not exist a (3,6,14)–graph containing a vertex of degree three (*i.e.*, the output obtained is empty).

ACKNOWLEDGEMENT

Dr. Mynhardt wishes to thank the University of Victoria, whose hospitality she enjoyed while this paper was being prepared.

REFERENCES

[1] M. Behzad, G. Chartrand and L. Lesniak–Foster, *Graphs and Digraphs*, Prindle, Weber & Schmidt, Boston, 1979.

[2] R.C. Brewster, E.J. Cockayne and C.M. Mynhardt, Irredundant Ramsey numbers for graphs, *J. Graph Theory* 13(1989), 283–290.

[3] R.C. Brewster, E.J. Cockayne and C.M. Mynhardt, The irredundant Ramsey number s(3,6), *Quaestiones Math.*, to appear.

[4] E.J. Cockayne and B. Gray, A fortran program for computation of extremum dominating, independent and irredundant sets in graphs, University of Victoria Math. Dept. Internal Report DM–209–IR, 1980.

[5] E.J. Cockayne, J.H. Hattingh, J. Kok and C.M. Mynhardt, Mixed Ramsey numbers and irredundant Turán numbers for graphs. Submitted.

[6] E.J. Cockayne and C.M. Mynhardt, On the irredundant Ramsey number s(3,3,3). Submitted.

[7] E.J. Cockayne and C.M. Mynhardt, The irredundant Ramsey number s(3,3,3) = 13. Submitted

[8] C.M. Grinstead and S.M. Roberts, On the Ramsey numbers r(3,8) and r(3,9), *J. Combin. Theory* 33(1982), 27–51.

[9] F. Harary, *Graph Theory*, Addison–Wesley, Reading, 1969.

[10] J.H. Hattingh, On irredundant Ramsey numbers for graphs, *J. Graph Theory*, to appear.

E.J. Cockayne
Department of Mathematics & Statistics
University of Victoria
P.O. Box 3045
Victoria, B.C.
CANADA V8W 3P4

J.H. Hattingh
Department of Mathematics
Rand Afrikaans University
P.O. Box 524
Johannesburg
2000 SOUTH AFRICA

C.M. Mynhardt
Department of Mathematics, Applied Mathematics and Astronomy
University of South Africa
P.O. Box 392
Pretoria
0001 SOUTH AFRICA