COLLISIONS HAVE POSITIVE MEASURE

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Abstract

Newton, Clairaut, Maneff, and others, have considered a model of the universe given by a force of the type $\nu/r^2 + \mu/r^3$, where ν, μ are constants and r is the distance between particles. Unlike the classical model, this one explains the phenomena of the solar system with the same accuracy as relativity. We prove that the set of initial data leading to collisions in the 3-body problem with Maneff's gravitational law contains an open set, so it has positive Lebesgue measure. The same is true for the set of singularities of the n-body problem. The asymptotic behavior of collision orbits is also discussed.

1. Introduction

Article 9 in Book I of Newton's *Principia* contains an intriguing result which is stated in Corollary 2 to Proposition 43 (see also [W] p. 83). Willing to understand the motion of the Moon, Newton considers a central force problem which leads to the motion of a particle on an ellipse. He proves that if the particle moves such that the ellipse performs a rotation (in its own plane and around one focal point), then a force of the type μ/r^3, where μ is a constant and r is the distance to the focus, has to be added to the central force. In particular, if the central force is given by the inverse square law ν/r^2, the above effect is obtained only with a force of the type $\nu/r^2 + \mu/r^3$ (see also [C], vol. 4, p. 517, point 11). Moreover, in the manuscripts known as the Portsmouth Collection (unpublished during his life time), Newton tried to explain the motion of Moon's perigee with the help of the above model. His attempts to use the inverse square force alone were offering a value only half of that given by observations. In 1749 Clairaut found a good explanation for the motion of the Moon's perigee within the inverse square force model, but only after being himself on the point of substituting this law with the one of the form $\nu/r^2 + \mu/r^3$ (see [M], p. 363). At the end of the last century Simon Newcomb and his collaborators were able to offer theoretical explanations, within 2" of arc, for all gravitational phenomena of the solar system (with one exception), in the frame of the inverse square law model. The unexplained phenomenon was the perihelion advance of Mercury and of the other inner planets, being one of the question marks leading to the discovery of relativity.

In spite of its grandious success, relativity fails to be of much use in studying the most natural mathematical question of astronomy: the motion in the n-body problem. Difficulties arise especially when one is searching what happens in the neighborhood of collisions. Therefore, attempts to modify the Newtonian gravitational law, without leaving the field of classical mechanics, existed before and after Einstein. Most of the proposed models have failed to explain other phenomena [H]. There exist today physical tests, as light deflection or time delay, which must be passed by any gravitational model.

In the twenties, Maneff considered a gravitational law (probably in ignorance of the attempts made by Newton and Clairaut) which is essentially the one given by $\nu/r^2 + \mu/r^3$. This law explains, with a very good approximation (as good as the one offered by relativity), the perihelion advance of the inner planets as well as the motion of the Moon [H], [M1], [M2], [M3], [M4]. The constant μ being taken very small, the perturbative force becomes relevant only in the neighborhood of collisions. Otherwise Maneff's law approximates the inverse square model, the influence of the inverse cubic correction being negligible (on reasonably long but finite intervals of time) when the particles do not perform close encounters. Since pathological behavior is likely to occur in the neighborhood of singularities, a study of solutions leading to collisions, or passing close to them, is a natural way to start to understand this gravitational law.

Since Maneff's potential is *quasihomogeneous*, i.e. equal to the sum of two homogeneous functions (of degree -1 and -2, respectively), an attempt to understand triple collision orbits of 3-body problems with quasihomogeneous potentials, was made in [DI]. We have seen there that Maneff's law is the only one, among quasihomogeneous laws, which fails to have the property that triple collision orbits tend to form asymptotically central configurations. This result was obtained by studying the invariant set given by the one-dimensional (rectilinear) problem, and proving the existence of rectilinear solutions that reach a triple collision after infinitely many elastic binary collisions. Another distinctive feature concerns the flow on the collision manifold for the one-dimensional 3-body problem of Maneff's model. This is the only flow, among quasihomogeneous ones, which is not *gradient-like.* Unfortunately the technique adopted there seemes to be inappropriate for higher dimensional problems, being restricted to the one-dimensional universe.

In [D2] we have studied the rich dynamics of the invariant set given by the planar isosceles 3-body problem with Maneff's law. The above discussed properties were seen to be also true in this invariant set. A study of the center manifolds and the regularization problem was also performed, and the network of homoclinic and heteroclinic orbits was discussed. An anisotropic Maneff model has been shown to give rise to drastic changes of the collision manifold and of the flow on it, by proving the existence of a subcritical pitchfork bifurcation of the equilibria.

The goal of this paper is to study triple collision solutions of Maneff's 3-body problem. We see first that the only singularities of the solutions are due to collisions. The main result is that for every negative energy level, the set of initial data leading to collisions has positive Lebesgue measure. The idea of the proof is to blow up the triple collision singularity using McGehee transformations, to paste instead a manifold, and study the behavior of the flow in the neighborhood of this manifold. The proof also works in the general n-body problem, implying that the set of initial data leading to singularities has positive Lebesgue measure for every negative energy level. Since it is known that in the calssical case the set of initial conditions leading to collisions is negligible, this shows a big difference between the Newtonian and the Maneff model.

We are next interested in the asymptotic behavior of orbits leading to a triple collision. To give a partial answer to this problem we first describe the set of *central configurations* by proving that this set has 5 classes of elements. There are 3 collinear and 2 triangular classes of central configurations, and their shape depends on the values m_1, m_2, m_3 of the masses. The triangular ones, for example, have the sides proportional to $(m_1+m_2)^{1/4}, (m_2+m_3)^{1/4}$ and $(m_1+m_3)^{1/4}$, so they never degenerate to a segment. Obviously, the equilateral triangles occur only if $m_1 = m_2 = m_3$. We then see that the flow of the transformed vector field has 10 rest points, and they correspond to central configurations. Orbits approaching 5 of these rest points are triple-collision orbits, and those leaving the 5 others are triple-ejection orbits. We finally conjecture the existence of orbits which tend to a triple collision outside the class of central configurations and such that the position vectors do not have limiting positions.

2. Singularities

Consider 3 point masses $m_i > 0, i = 1,2,3$, in the Euclidean space $\Re^3$, having coordinates $\mathbf{q_i} = (q_i^1, q_i^2, q_i^3)$, $i = 1,2,3$, in an absolute reference system. Let $\mathbf{q} = (\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) \in \Re^9$ be the *configuration* of the system of particles and define the *quasihomogeneous* potential

$W = U + V$, where

$$U: \Re^9 \setminus \Delta \to \Re_+, \quad U(\mathbf{q}) = G \cdot \sum_{1 \le i < j \le 3} m_i m_j q_{ij}^{-1},$$

$$V: \Re^9 \setminus \Delta \to \Re_+, \quad V(\mathbf{q}) = \frac{3G^2}{2c^2} \cdot \sum_{1 \le i < j \le 3} m_i m_j (m_i + m_j) q_{ij}^{-2}$$

are homogeneous functions of degree -1 and -2 respectively, $q_{ij} = |\mathbf{q_i} - \mathbf{q_j}|$ is the Euclidean distance between particles i and j, Δ denotes the collision set

$$\Delta = \bigcup_{1 \le i < j \le 3} \{\mathbf{q} | \mathbf{q_i} = \mathbf{q_j}\},$$

while G and c are real constants representing the gravitational constant and the speed of light. The equations of motion are given by the system

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}), \end{cases} \tag{2.1}$$

where $M = \text{diag}\ (m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$, $\nabla = (\partial_1, \partial_2, \partial_3)$ is the gradient operator and $\mathbf{p} = M\dot{\mathbf{q}}$ denotes the *momentum* of the system.

Analogous to the Newtonian case, there exist 10 uniform first integrals. The integrals of the *momentum* and *center of mass* imply that the set $\mathbf{Q} \times \mathbf{P}$ is invariant for the equations (2.1), where

$$\mathbf{Q} = \{\mathbf{q} | \sum m_i \mathbf{q_i} = \mathbf{0}\} \text{ and } \mathbf{P} = \{\mathbf{p} | \sum \mathbf{p_i} = \mathbf{0}\}.$$

From now on we will restrict the equations of motion to the above invariant set, which physically means that the motion is regarded with respect to the center of mass of the 3 particles. We will also use the integral of energy

$$T(\mathbf{p}(t)) - W(\mathbf{q}(t)) = h,$$

where $T: \Re^9 \to [0, \infty)$, $T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{3} m_i^{-1}|\mathbf{p_i}|^2$ is the *kinetic energy* and h is the *energy constant*, and the integrals of the angular momentum

$$\sum \mathbf{q}_i \times \mathbf{p}_i = \mathbf{c},$$

where $\mathbf{c}$ is a constant vector obtained by integration, called the *angular momentum constant*.

Standard results of the differential equations theory ensure, for given initial data $(\mathbf{q}, \mathbf{p})(0) \in (\Re^9 \setminus \Delta) \times \Re^9$, the existence and uniqueness of an analytic solution $(\mathbf{q}, \mathbf{p})$ of the equations (2.1), defined on a maximal interval $[0, t^*)$, $0 < t^* \le \infty$. Analogously one can work with intervals of the form $(t^*, 0]$. In case t^* is finite, the solution is said to experience a *singularity*. If $\mathbf{q}$ has a limit as $t \to t^*$, then we call the singularity a *collision*.

Otherwise it will be called a *noncollision singularity*. It is known that in the Newtonian 3-body problem, the only possible singularities are due to collisions. This is also true for Maneff's law.

Theorem 2.1. *In the 3-body problem with Maneff's gravitational law, all singularities of the solutions are due to collisions between particles.*

Proof. As in the Newtonian case one can show that t^* is a singularity if and only if $lim\ \rho(\mathbf{q}(t)) = 0$, or equivalently $lim\ inf\ \rho(\mathbf{q}(t)) = 0$, when $t \to t^*$, where $\rho(\mathbf{q}) = min\{|\mathbf{q}_i - \mathbf{q}_j|, i, j = 1, 2, 3, i \neq j\}$ (see [D1], pp. 9-11).

Notice further that if $J(\mathbf{q}) = (1/2)\sum m_i|\mathbf{q}_i|^2$ is the *moment of inertia* of the system of particles, using Euler's relation for homeogeneous functions, it follows that $\ddot{J} = 2T - U - 2V$. From the energy integral we obtain $\ddot{J} = U + 2h$. Now, if t^* is a singularity, then $\rho(\mathbf{q}(t)) \to 0$, consequently $U(\mathbf{q}(t)) \to \infty$, and therefore $\ddot{J}(\mathbf{q}(t)) \to \infty$. Hence $\dot{J}$ is an incresing function in a neighborhood of t^*, so $J(\mathbf{q}(t)) \to J^*$, where $J^* \geq 0$ is finite or infinite.

We prove now that if $\rho(\mathbf{q}(t)) \to 0$ when $t \to t^*$, then at least one of the mutual distances q_{12}, q_{13}, q_{23} tends to 0. Suppose this is not the case, so that the superior limit of q_{ij}, for all $i, j = 1, 2, 3$, with $i \neq j$, is strictly positive. Then $\rho(\mathbf{q}(t)) \to 0$ only if at least two mutual distances, say q_{13} and q_{23}, interchange the role of being the minimum distance when t approaches t^*. Let t_ν be the subsequence of time instants where this role is interchanged, so $t_\nu \to t^*$ when $\nu \to \infty$. Thus $q_{13}(t_\nu) = q_{23}(t_\nu) \to 0$ when $\nu \to \infty$, and by the triangle inequality, $q_{12}(t_\nu) \to 0$. Since J can be written as $\frac{1}{2m}\sum m_i m_j q_{ij}$, where $m = m_1 + m_2 + m_3$, it follows that $J(\mathbf{q}(t_\nu)) \to 0$ as $\nu \to \infty$, but as J has always a limit, this limit is 0. Therefore all mutual distances approach 0. This contradicts the above assumption. The proof is now complete.

Consider further the McGehee transformations [M]:

$$\begin{cases} r = (\mathbf{q}^T M \mathbf{q})^{\frac{1}{2}} \\ \mathbf{s} = r^{-1}\mathbf{q} \\ y = \mathbf{p}^T \mathbf{s} \\ \mathbf{x} = \mathbf{p} - yM\mathbf{s}. \end{cases} \tag{2.2}$$

Notice that $\mathbf{s}^T M \mathbf{s} = 1$ and $\mathbf{x}^T \mathbf{s} = 0$. Compose (2.2) with the transformations

$$\begin{cases} v = ry \\ \mathbf{u} = r\mathbf{x} \end{cases} \tag{2.3}$$

and define, along a solution, the time-rescaling transformation

$$d\tau = r^{-2} dt. \tag{2.4}$$

Under the transformations (2.2), (2.3), (2.4), which are analytic diffeomorphisms, the equations of motion (2.1) become

$$\begin{cases} r' = rv \\ v' = v^2 + \mathbf{u}^T M^{-1}\mathbf{u} - rU(\mathbf{s}) - 2V(\mathbf{s}) \\ \mathbf{s}' = M^{-1}\mathbf{u} \\ \mathbf{u}' = -(\mathbf{u}^T M^{-1}\mathbf{u})M\mathbf{s} + \nabla V(\mathbf{s}) + 2V(\mathbf{s})M\mathbf{s} + r[U(\mathbf{s})M\mathbf{s} + \nabla U(\mathbf{s})], \end{cases} \tag{2.5}$$

where, by abuse, we've maintained the same notations for the new variables. The prime denotes differentiation with respect to the new (fictitious) time variable τ. In case of a collision orbit the singularity time instant t^* is transformed into the fictitious time instant τ^*, which may be finite or infinite.

The integral of energy takes the form

$$\frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) - rU(\mathbf{s}) - V(\mathbf{s}) = r^2 h, \tag{2.6}$$

and the integrals of the angular momentum become

$$\sum \mathbf{s}_i \times \mathbf{u}_i = \mathbf{c}. \tag{2.7}$$

Also notice that the sets $\{(r, \mathbf{s}, v, \mathbf{u}) | r = 0\}$ and $\{(r, \mathbf{s}, v, \mathbf{u}) | r > 0\}$ are invariant manifolds for the equations (2.6). We call the set

$$C = \{(r, \mathbf{s}, v, \mathbf{u}) | r = 0 \text{ and equation (2.6) holds}\}$$

the *triple-collision manifold*. Notice that C is pasted to the phase space to replace the triple-collision singularity, and though fictitious, the behavior of the flow on it gives information about orbits coming close to C. We will further exploit this information.

3. Collision Solutions

In this section we prove the main result of the paper from which follows that the set of initial data leading to binary and triple collisions has positive Lebesgue measure. For this let us first write the equations (2.5) in a more convenient form. Using the energy relation (2.6) we obtain

$$\begin{cases} r' = rv \\ v' = r^2 h + rU(\mathbf{s}) \\ \mathbf{s}' = M^{-1}\mathbf{u} \\ \mathbf{u}' = -(\mathbf{u}^T M^{-1}\mathbf{u})M\mathbf{s} + \nabla V(\mathbf{s}) + 2V(\mathbf{s})M\mathbf{s} + r[U(\mathbf{s})M\mathbf{s} + \nabla U(\mathbf{s})]. \end{cases} \tag{3.1}$$

The system is restricted to the invariant set given by the following equations:

$$\sum m_i \mathbf{s}_i = \mathbf{0}, \tag{3.2}$$

$$\sum \mathbf{u}_i = \mathbf{0}, \tag{3.3}$$

$$\mathbf{s}^T M \mathbf{s} = 1, \tag{3.4}$$

$$\mathbf{s}^T \mathbf{u} = 0, \tag{3.5}$$

and has the relations derived from the energy and angular momentum integrals:

$$\frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) - rU(\mathbf{s}) - V(\mathbf{s}) = r^2 h, \tag{3.6}$$

$$\sum \mathbf{s}_i \times \mathbf{u}_i = \mathbf{c}. \tag{3.7}$$

For h and $\mathbf{c}$ fixed, the dimension of the phase space reduces therefore to 8. We further consider the equations (3.1) to be restricted to this space. The main result can now be stated and proved.

Theorem 3.1 *For every negative value of the energy constant, $h < 0$, the set of initial data leading to binary or triple collisions contains an open set, so it has positive Lebesgue measure.*

Proof. Take a solution defined on $[0, \tau^*)$, and suppose that $U(\mathbf{s})$ is unbounded when $\tau \to \tau^*$. Then on one hand τ^* is finite. On the other hand $U(\mathbf{q}) = r^{-1}U(\mathbf{s})$ is unbounded, so this means that τ^* was derived, through the transformations (2.4), from a singularity which, by Theorem 2.1, is necessarily a collision. In fact it is either a binary collision, or a triple collision having the property that two of the vectors $\mathbf{s}_i$ and $\mathbf{s}_j$ coincide asymptotically. The binary collision occurs if r stays positive when $\tau \to \tau^*$, while the triple collision appears if $r(\tau) \to 0$. Since this type of triple collision implies the unboundedness of $U(\mathbf{s})$, we will call it *triple collision of singular type*. A triple collision for which $U(\mathbf{s})$ remains bounded will be called *triple collision of regular type*.

Let us suppose now that $U(\mathbf{s})$ is bounded. Then $V(\mathbf{s}), \nabla V(\mathbf{s})$ and $\nabla U(\mathbf{s})$ are also bounded, so $\tau = \infty$. Using the second equation in (3.1), we can find a constant $K > 0$ such that if $h < 0$, then $v' < Kr$. Multiplying this inequality with $v < 0$, we obtain $vv' > Krv = Kr'$, the last equality following from the first equation in (3.1). This can be written as $\frac{1}{2K}(v^2)' = r'$, and integrating it between 0 and τ we obtain the inequality

$$r(\tau) < \frac{1}{2K}v^2(\tau) + K_1, \quad \text{for all } \tau > 0, \tag{3.8}$$

where $K_1 = r(0) - \frac{1}{2K}v^2(0)$.

We will now show that for any $h < 0$ there exists a positive constant $\mu = \mu(h) > 0$ such that any solution having initial data inside the set

$$S_\mu = \{(r, v, \mathbf{s}, \mathbf{u}) \mid K_1 < 0, v < 0, r < \mu\}$$

(which is obviously an open set, so it has positive Lebesgue measure, see Figure 1), leads to a binary or to a triple collision. Let $(r, v, \mathbf{s}, \mathbf{u})(0)$ be an element of S_μ and suppose that the corresponding solution is defined on $[0, \tau^*)$, τ^* finite or infinite. If $U(\mathbf{s})$ is unbounded when $\tau \to \tau^*$ then the solution leads to a collision. If $U(\mathbf{s})$ is bounded then relation (3.8) takes place. Now, since $\mathbf{s}^T M \mathbf{s} = 1$, it follows that there exists a constant $\nu > 0$ such that $U(\mathbf{s}(\tau)) > \nu$ for all τ in $[0, \infty)$. Then for every $h < 0$ we can find a positive constant $\mu = \mu(h) > 0$ such that if $r < \mu$, then $v' = r(rh + U(\mathbf{s})) > \mu r$. Therefore $v' > 0$ for all $\tau > 0$, so v is an increasing function of τ in S_μ. This is indeed the case since $r' = rv$ and as $v(0)$ has been taken negative, r is a decreasing function. Notice also that v cannot become positive or zero without making r to tend to zero, which means that in such a situation the solution experiences a triple collision. Thus each S_μ is a positively invariant set.

Relation (3.8) shows that for a given initial condition, and consequently for a fixed constant K_1 (which implies $v(0) = -\sqrt{2\mu r(0) - 2\mu K_1}$), if $v(\tau)$ tends to the value $v^* =$

$-\sqrt{-2\mu K_1} > v(0)$, then necessarily $r(\tau)$ approaches 0, and consequently a triple collision occurs. Of course, r may tend to 0 even before v reaches the value v^*, implying a triple collision. The only undesired situation would appear if v tends to a value $v^{**} < v^*$ when $\tau \to \infty$. In such a case we cannot draw, directly from (3.8), the conclusion that the orbit ends in a triple collision. We prove now that in spite of this, r still approaches 0. Indeed, since $v(\tau) \to v^{**} < v^*$ as $\tau \to \infty$ and v is an increasing function, it follows that $v'(\tau) \to 0$ for $\tau \to \infty$. Otherwise v could not tend to v^{**}. But the solution is inside some set S_μ, so $v' > \mu r$, which implies $r(\tau) \to 0$, as $\tau \to \infty$. Other possibilities do not occur.

We have thus proved that for any $h < 0$, there is an invariant set $S_{\mu(h)}$ having positive Lebesgue measure, such that any solution in $S_{\mu(h)}$ leads to a binary or a triple collision. This completes the proof.

Remark 1. From the physical point of view the above proof shows that for negative energy values, if all particles come close enough together, then either a binary or a triple collision occurs.

Remark 2. The above proof can be easily adapted to any number of particles, and it shows that in the n-body problem the set of initial data leading to solutions with singularities has positive Lebesgue measure. It has not been searched yet whether there exist noncollision singularities for Maneff's n-body problem if $n \geq 4$.

4. Central Configurations

The goal of this section is to describe the set of central configurations of the 3-body problem for an inverse cubic force which is directly proportional with the product of the masses times the sum of the masses. The reason for doing that will become clear in the next section. Consider for the time being a potential function of the form

$$\Gamma\colon \Re^9 \to \Re_+, \quad \Gamma(\mathbf{q}) = K \cdot \sum_{1\leq i<j\leq 3} \gamma(m_i, m_j) q_{ij}^{-d}, \tag{4.1}$$

where $\gamma(m_i, m_j) = \gamma(m_j, m_i)$, $K > 0$ is a constant and for our purpose is enough to take $d \geq 1$.

A *central configuration* for the particles $m_1, m_2, m_3 > 0$ is a solution $\mathbf{q}_0$ of the equations

$$\nabla\Gamma(\mathbf{q}) = \sigma M \mathbf{q}, \tag{4.2}$$

where $\sigma \neq 0$ is a constant and M is the matrix defined in the previous section. Since homothetic transformations and rotations of the geometric configuration given by $\mathbf{q}_0$ are also central configurations, factorize the set of central configurations to the equivalence relations given by homotheties and rotations. Thus, by a central configuration we will understand a representative of one such class.

Central configurations play an important role in the study of the classical n-body problem, $n \geq 3$. It is known that in the Newtonian 3-body case there exist exactly five classes of central configurations. Three of them correspond to collinear configurations (one for each ordering of three particles on a non-oriented line) and two correspond to equilateral configurations (one for each possible orientation of a triangle in the plane). We have proved in [DI] the following generalization of that result.

Theorem 4.1. *In case*

$$(\gamma(m_i,m_j)m_k)^{\frac{1}{a+2}} < (\gamma(m_i,m_k)m_j)^{\frac{1}{a+2}} + (\gamma(m_k,m_j)m_i)^{\frac{1}{a+2}} \tag{4.3}$$

for all choices of mutually distinct indices $i,j,k \in \{1,2,3\}$, the set of central configurations corresponding to the potential Γ in (4.1) is formed by three collinear configurations and two triangular configurations. Otherwise, it is formed only by the three collinear configurations. Moreover, the case $\gamma(m_i,m_j) = km_im_j$, where $k \neq 0$ is a constant, is the only one giving rise to equilateral configurations.

From this result we can see that the occurence of equilateral central configurations in the Newtonian case (independent on the value of the masses), is a consequence of the fact that the classical law is directly proportional with the product of masses. In case of laws of the form (4.1) that do not fulfill this property, equilateral central configurations appear only for equal values of the masses. Applying Theorem 4.1 to a potential of the form

$$V:\Re^9 \setminus \Delta \to \Re_+, \quad V(\mathbf{q}) = \frac{3G^2}{2c^2} \cdot \sum_{1\le i<j\le 3} m_im_j(m_i+m_j)q_{ij}^{-2}, \tag{4.4}$$

we obtain the following result.

Theorem 4.2. *The set of central configurations corresponding to the potential V is formed by three collinear and two triangular configurations. The triangular configurations always exist, and the sides of the triangles are proportional with the values*

$$a = (m_2+m_3)^{1/4}, \; b = (m_1+m_3)^{1/4}, \; c = (m_1+m_2)^{1/4}.$$

Proof. The existence of the central configurations follows directly from the statement of Theorem 4.1., by comparing (4.1) with (4.4). The values a, b and c are easy to compute from the proof of Theorem 4.1 (see [DI]).

5. Asymptotic Behavior

The goal of this section is to describe possible asymptotic behavior of triple-collision solutions of the planar 3-body problem with Maneff's gravitational law. Let us begin by looking for rest points of the flow.

Proposition 5.1. *The equations* (3.1) *have* 10 *equilibrium solutions. They are of the form $(r,v,\mathbf{s},\mathbf{u})$, where $r=0$, $\mathbf{s}=\mathbf{s}_0$ is a central configuration for V, $v = \pm\sqrt{2V(\mathbf{s}_0)}$, and $\mathbf{u}=\mathbf{0}$.*

Proof. The only part that is not obvious concerns the central configuration $\mathbf{s} = \mathbf{s}_0$. It follows from the equations (3.1) that in order to have an equilibrium solution, the relations $\nabla V(\mathbf{s}) = 2V(\mathbf{s})M\mathbf{s}$ have to be fulfilled. But using transformations (2.2)–(2.4), one can see that these are equivalent to relations (4.2), so $\mathbf{s}_0$ has to be a central configuration. Since v can take a positive or a negative value, there will be 10 restpoints of the flow. This completes the proof.

The next step is to compute the eigenvalues corresponding to the 10 central configurations. Since all equilibria belong to the collision manifold (because $r = 0$), the computation of eigenvalues will tell us if there exist solutions tending to a triple collision.

Proposition 5.2. *For an equilibrium* $(r, \mathbf{s}, v, \mathbf{u}) = (0, \mathbf{s}_0, 2V(\mathbf{s}_0), \mathbf{0})$ *of the equations* (3.1), *the eigenvalues attached to it are* $\lambda_r = v_0$, $\lambda_v = 0$, *while the other* 12 *eigenvalues are of the form* $\lambda = \pm\sqrt{m_i\mu_{ij}}$, $i = 1, 2, 3$, $j = 1, 2$, *where* μ_{ij} *are the* 6 *eigenvalues of the symmetric matrix* $\nabla^2 V(\mathbf{s}_0) - 2V(\mathbf{s}_0)M$.

Proof. After computing the eigenvalues corresponding to r and v, the equation to solve (having the unknown λ), is given by

$$\begin{vmatrix} -\lambda I & M^{-1} \\ \nabla^2 V(\mathbf{s}_0) - 2V(\mathbf{s}_0)M & -\lambda I \end{vmatrix} = 0,$$

where I is the 6×6 unit matrix. Notice that $\nabla^2 V(\mathbf{s}_0) - 2V(\mathbf{s}_0)M$ is a 6×6 symmetric matrix, having consequently only real eigenvalues μ_{ij}, $i = 1, 2, 3$, $j = 1, 2$. Suppose now that $\mathbf{w}$ is a 6-dimensional vector satisfying

$$(\nabla^2 V(\mathbf{s}_0) - 2V(\mathbf{s}_0)M)\mathbf{w} = \mu_{ij}\mathbf{w}, i = 1, 2, 3, \ j = 1, 2.$$

Then

$$\begin{pmatrix} \mathbf{0} & M^{-1} \\ \nabla^2 V(\mathbf{s}_0) - 2V(\mathbf{s}_0)M & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \lambda\mathbf{w} \end{pmatrix} = \begin{pmatrix} M^{-1}\lambda\mathbf{w} \\ \mu_{ij}\mathbf{w} \end{pmatrix}.$$

Consequently λ is an eigenvalue of the matrix attached to equations (3.1) if

$$m_i^{-1}\lambda^2 = \mu_{ij}, \ i = 1, 2, 3, \ j = 1, 2,$$

which implies the relations from the statement of the theorem. This completes the proof.

Remark. Proposition 5.2 shows that the equilibrium solutions of equations (3.1) are not hyperbolic. In spite of this, since for the ten equilibria the eigenvalues corresponding to the r-variable are nonzero (five of them negative and the other five positive), it follows that five of the equilibria contain a one-dimensional stable manifold, while the other five contain a one-dimensional unstable manifold. These prove the existence of solutions leading to triple collisions (in positive or negative time) which tend to form a central configuration in the neighborhood of the collision. It is, however, hard to determine the dimension of the stable sets associated to these equilibria since the sign of the eigenvalues μ_{ij} is unknown. Some of them might be 0, increasing the nonhyperbolicity degree of the equilibria.

It is natural to ask further if there exist other solutions tending to the collision manifold, otherwise than through rest points, reaching C without asymptotic phase. In the proof of Theorem 3.1 we have distinguished between possible triple-collision solutions of *singular type* and those of *regular type*. Let us describe both these classes.

Solutions of singular type are those which run off the collision manifold in finite fictitious time, at $\tau = \tau^*$. They may or may not have a limiting position. It is clear that such solutions do not exist in the one-dimensional case when all particles move on a fixed straight line. This has been shown in [DI], and follows from the fact that before reaching the collision manifold, all possible candidates to a solution of this class end in a binary collision. It is, however, not clear if such solutions exist in the planar or spatial case.

Triple-collision solutions of regular type are globally defined in the fictitious time variable τ. They can be divided in two other subclasses: (1) and (2). Solutions in (1) are those which tend to central configurations, as we have described at the beginning of this section. Their existence is clear from Propositions 5.1 and 5.2. Solutions in (2) would be those which perform a triple collision such that the position vectors do not have limiting positions at the collision instant. We conjecture that such solutions do indeed exist.

7. Final Remarks

By tackling Maneff's gravitational model we have raised many questions and provided a few answers. Nevertheless, the existence of a large set of collision solutions is a strong argument in favor of pursuing mathematical and physical studies on this gravitational law. Due to the fact that this model brings an explanation of the perihelion advance of Mercury without to alter other achievements of celestial mechanics, the above result might make us think to revise our understanding of collision solutions and of all consequences arising from here.

References

[C] A. CAYLEY, Report on the progress of the solution of certain special problems of dynamics, in *The Collected Mathematical Papers of Arthur Cayley*, p. 513., Cambridge Univ. Press, 1891.

[D1] F.N. DIACU, *Singularities of the N-Body Problem, An Introduction to Celestial Mechanics*, Les Publication CRM, Montréal, 1992.

[D2] F.N. DIACU, On the planar isosceles problem for Maneff's gravitational law, *Journ. Math. Phys.* (to appear in vol. **34**, 12, (1993)).

[DI] F.N. DIACU & R. ILLNER, Collision/ejection dynamics for particle systems with quasihomogeneous potentials, DMS-624-IR, University of Victoria, February, 1993.

[H] Y. HAGIHARA, *Celestial Mechanics*, vol. II, part I, The MIT Press, Cambridge, MA, 1975.

[M1] G. MANEFF, La gravitation et le principe de l'égalité de l'action et de la réaction, *Comptes rendus* **178** (1924), 2159-2161.

[M2] G. MANEFF, Die Gravitation und das Prinzip von Wirkung und Gegenwirkung, *Zeitschrift für Physik* **31** (1925), 786-802.

[M3] G. MANEFF, Le principe de la moindre action et la Gravitation, *Comptes rendus* **190** (1930), 963-965.

[M4] G. MANEFF, La gravitation et l'énergie au zéro, *Comptes rendus* **190** (1930), 1374-1377.

[Mc] R. McGEHEE, Triple collision in the collinear three-body problem, *Inventiones math.* **27** (1974), 191-227.
[Mo] F.R. MOULTON, *An Introduction to Celestial Mechanics*, Dover Publ., New York, 1970.
[W] E.T. WHITTAKER, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge Univ. Press, Cambridge, 1988.