# *PARTIAL DOMINATING FUNCTIONS FOR TREES AND CYCLES*

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# PARTIAL DOMINATING FUNCTIONS FOR TREES AND CYCLES

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## Abstract

Let $G = (V, E)$ be a graph and $Q \subseteq V$. A function $f : V \rightarrow [0, 1]$ is a *Q-dominating function (Q-DF) of G* if $\Sigma_{u \in N[v]} f(u)$ is at least one if $v \in Q$, but strictly less than one if $v \in V - Q$ ($N[v]$ denotes the closed neighbourhood of vertex $v$). In the special case where $Q = V$, the $Q$-dominating functions of $G$ are precisely the well-studied dominating functions of $G$. The concept of a $Q - DF$ arises naturally from the theory of weighted majority games.

This paper completely solves the problems of existence of $Q - DF$'s in trees and cycles.

## 1. Introduction

A *dominating function* $(DF)$ of a graph $G = (V, E)$ is a function $f : V \to [0,1]$ such that for each $v \in V$,

$$f[v] = \sum_{u \in N[v]} f(u) \geq 1,$$

where $N[v]$ denotes the closed neighbourhood of $v$, that is,

$$N[v] = \{v\} \cup \{x \in V : x \text{ is adjacent to } v\}.$$

The theory and applications of dominating functions have received considerable attention in the literature and the reader is referred to [3] for an extensive bibliography.

In many applications it is not necessary to demand that $f[v]$ is at least one for *all* vertices $v \in V$ and the purpose of this paper is to generalize the concept of a dominating function of a graph to that of a partial dominating function (also called a $Q$-dominating function), which is defined as follows.

Let $G = (V, E)$ be a graph and $Q \subseteq V$. A function $f : V \to [0,1]$ is a *$Q$-dominating function* $(Q - DF)$ of $G$ if

$$\left. \begin{aligned} f[v] = \sum_{u \in N[v]} f(u) &\geq 1 \text{ for } v \in Q, \\ &< 1 \text{ for } v \in P = V - Q. \end{aligned} \right\} \qquad (1)$$

The concept of a $Q - DF$ arises naturally from the theory of weighted majority games [11-14]. In this context $V$ is the set of *players* in a game and $f$ assigns *voting weight* to each player. A subset $S$ of $V$ is called a *coalition.* A coalition $S$ *wins* a vote, or is called *winning*, if and only if

$$\sum_{u \in S} f(u) \geq 1,$$

where value 1 on the right hand side of the inequality is referred to as the *quota* for the game. Such games have been studied extensively in the analysis of voting situations in local and national governing bodies in many countries [1, 6].

With political applications in mind, various authors have considered restricted weighted majority games, *i.e.* games in which only a subset of all possible $2^{|V|}$ coalitions are permissible. For example restrictions may be imposed on the size of coalitions ([4]), on movements between political parties, on the formation of new parties ([5]) or because a party might be unwilling to enter coalitions with ideologically remote parties.

The idea of using graphs to restrict or particularize games is very natural and is described, for example in [7, 9]. A game is modelled by a graph $G = (V, E)$ where $E$ is some relation (often political affinity) on the set $V$ of players and the set of permissible coalitions is defined by properties of $G$.

We propose a new way of defining such a restricted game, namely that the coalition $S$ is permissible if and only if $S \subseteq N[v]$ for some $v \in V$. Then a $Q - DF$ of $G$ may be regarded as a description or realization of a game in which $Q$ is precisely the set of all players whose closed neighbourhoods are winning coalitions.

Given the voting weight of each player in such a graph-restricted weighted majority game it is a simple matter to determine those players whose closed neighbourhoods are winning coalitions. We are concerned with the reverse problem (which has practical significance), namely to determine for a given graph $G = (V, E)$ and $Q \subseteq V$, whether it is possible to assign voting weights to $V$ so that the closed neighbourhood of $v \in V$ is winning if and only if $v \in Q$. This problem may also be stated: *Given* $G = (V, E)$ *and* $Q \subseteq V$, *does there exist a* $Q - DF$ *for* $G$?

In Section 2, we give a complete solution of this existence problem for the cycles $C_n$ and in Section 3 an algorithmic solution for trees is presented.

## 2. Existence of $Q - DF$'s in Cycles

Let $C_n$ have vertex sequence $v_1, \ldots, v_n$ and for $i = 0, 1, 2$ define $A_i = \{v_j \mid j \equiv i \pmod 3\}$. The question of existence of $Q - DF$'s in cycles is completely answered by the following result.

**Theorem 1.** *For $Q \subseteq V(C_n)$, $C_n$ has no $Q - DF$ if and only if $n = 0 \pmod 3$ and for some $\{i, j\} \subseteq \{0, 1, 2\}$, $A_i \subseteq Q$ and $A_j \subseteq P = V(C_n) - Q$.*

**Proof** (sufficiency). Let $n = 3k$. If $k = 1$, hypothesis implies the existence of $u \in P$ and $v \in Q$. But $C_3$ is complete and $N[u] = N[v]$. If $f$ is a $Q - DF$ then $1 > f[u] = f[v] \geq 1$, a contradiction. Hence assume $k \geq 2$ and suppose without losing generality $A_1 \subseteq Q$ and $A_2 \subseteq P$.

Suppose contrary to the result that $C_n$ has a $Q - DF$ $f$. Then from (1) for each $i = 0, \ldots, k - 1$, noting that addition of subscripts is modulo $n$ (*i.e.* $v_0 = v_{3k}$),

$$f[v_{3i+1}] = f(v_{3i}) + f(v_{3i+1}) + f(v_{3i+2}) \geq 1$$

and

$$f[v_{3i+2}] = f(v_{3i+1}) + f(v_{3i+2}) + f(v_{3i+3}) < 1.$$

Hence

$$f(v_{3i+3}) < f(v_{3i})$$

and we have

$$f(v_0) > f(v_3) > f(v_6) > \cdots > f(v_{n-3}) > f(v_0)$$

a contradiction.

The remainder of this section is devoted to the proof of the converse. For any $Q \subseteq V(C_n)$ which does not satisfy the conditions of Theorem 1, a $Q - DF$ for $C_n$ is exhibited. A variety of special cases must be considered separately, so that the necessity proof of Theorem 1 is accomplished by a sequence of propositions.

A basic tool which will be used several times, is the following simple method of assigning function values to the vertices of a path in $C_n$ so that (1) holds for all vertices, except perhaps the two endvertices and the function values on vertices $v_i, v_j$ whose subscripts are congruent modulo $n$, differ by an arbitrarily small amount.

**Lemma 2.** *Let $\eta$ be a path in $C_n$ with vertex sequence $v_1, \ldots, v_k$, where $k \geq 3$ and let $\alpha_0, \alpha_1, \alpha_2 \in (0,1)$ satisfy $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Then for any $\epsilon > 0$, there exists $f : V(\eta) \to (0,1)$ such that*

(a) $f(v_1) = \alpha_1$, $f(v_2) = \alpha_2$,

(b) *$f$ satisfies condition (1) for all $v \in \{v_2, \ldots, v_{k-1}\}$, and*

(c) *for each $i = 0, 1, 2$, if $j \equiv i \pmod 3$, then $|f(v_j) - \alpha_i| < \epsilon$.*

**Proof.** We first define the integer-valued function $x$ on $V(\eta)$ as follows. Let $x(v_j)$ be abbreviated to $x_j$.

Set $x_1 = x_2 = 0$ and for $j \geq 3$ define $x_j$ recursively by

$$x_j + x_{j-1} + x_{j-2} = \begin{cases} -1 & \text{for } v_{j-1} \in P \\ 0 & \text{for } v_{j-1} \in Q. \end{cases}$$

Then for $j \geq 3$

$$x_j + x_{j-1} + x_{j-2} = 0 \text{ or } -1$$

and

$$x_{j+1} + x_j + x_{j-1} = 0 \text{ or } -1.$$

Hence for $j \geq 3$

$$|x_{j+1} - x_{j-2}| \leq 1.$$

Now let $i \in \{0, 1, 2\}$ and $j \equiv i \pmod 3$. It follows from the above that

$$|x_j - x_i| \leq |x_j - x_{j-3}| + |x_{j-3} - x_{j-6}| + \cdots + |x_{i+3} - x_i| \leq \frac{j}{3}.$$

Hence $|x_j| \leq |x_j - x_i| + |x_i| \leq \frac{j}{3} + 1$. Define the real-valued function $f$ on $V(\eta)$ by

$$f(v_j) = \alpha_i + \lambda x_j\,,$$

where $\lambda (> 0)$ is sufficiently small that for $j \in \{1, \ldots, k\}$

$$|\lambda x_j| = \lambda |x_j| \leq \lambda \left(\frac{n}{3} + 1\right) < \epsilon \tag{2}$$

and

$$|\lambda x_j| < \min\{\alpha_0, 1-\alpha_0, \alpha_1, 1-\alpha_1, \alpha_2, 1-\alpha_2\}. \tag{3}$$

By (3), $f(v_j) \in (0,1)$ and by (2), condition (c) is satisfied.

Further, condition (a) is immediate and for $j \in \{2, \ldots, k-1\}$

$$\begin{aligned} f[v_j] &= f(v_{j-1}) + f(v_j) + f(v_{j+1}) \\ &= (\alpha_0 + \alpha_1 + \alpha_2) + \lambda(x_{j-1} + x_j + x_{j+1}) \\ &= 1 + \lambda(x_{j-1} + x_j + x_{j+1}). \end{aligned}$$

By definition of $x$

$$f[v_j] = \begin{cases} 1 \text{ if } v_j \in Q \\ 1-\lambda \text{ if } v_j \in P \end{cases}$$

and condition (b) is satisfied. ∎

**Proposition 3.** *Let $C_n$ contain a path of at least three vertices all of which* (i) *are contained in $Q$ or* (ii) *are contained in $P$. Then $C_n$ has a $Q-DF$.*

**Proof.** Let the vertex sequence of $C_n$ be $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ where $m \geq 3$. Let $\eta$ be the path formed by $v_{m+1}, \ldots, v_n$.

**(i)** Let $\{v_1, \ldots, v_m\}$ induce a maximal path whose vertices are elements of $Q$. If $m \geq n-2$, let $f \mid \{v_2, \ldots, v_{m-1}\} = 1$ and $f(u) = 0$ on all other vertices $u$. Then $f$ is a $Q-DF$.

Therefore assume $n \geq 6$ and $m \leq n-3$, *i.e.* $|V(\eta)| \geq 3$. Define $f : V(\eta) \to (0,1)$ by Lemma 2 using $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$ and $\epsilon = \frac{1}{12}$. Then we define $g : V(C_n) \to [0,1]$ by

$$\begin{aligned} g(u) &= f(u) \quad & u \in V(\eta) \\ &= 0 & u \in \{v_1, v_m\} \\ &= 1 & u \in \{v_2, \ldots, v_{m-1}\}. \end{aligned}$$

By Lemma 2 and the definition of $g$, it is easily verified that $g$ is a $Q-DF$ for $C_n$. Note that by maximality of $v_1, \ldots, v_m$, $\{v_{m+1}, v_n\} \subseteq P$. By definition $g[v_{m+1}] = 0 + \frac{1}{3} + \frac{1}{3} < 1$

and

$$
\begin{aligned}
g[v_n] &= g(v_{n-1}) + g(v_n) + g(v_1) \\
&< \frac{1}{3} + \epsilon + \frac{1}{3} + \epsilon + 0 \text{ (by Lemma 2)} \\
&= \frac{5}{6} < 1.
\end{aligned}
$$

**(ii)** Let $\{v_1, \ldots, v_m\}$ induce a maximal path whose vertices are elements of $P$. If $m \geq n-2$, it is easy to define $Q-DF$'s with values in $\{0, \frac{1}{2}\}$. Therefore assume $n \geq 6$ and $m \leq n-3$. Define $f$ as in Case (i) and $g$ by

$$
\begin{aligned}
g(u) &= f(u) \quad u \in V(\eta) \\
&= \frac{1}{3} \quad \text{if } u = v_m \\
&= 1 - f(v_n) - f(v_{n-1}) \quad \text{if } u = v_1 \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

Then $g$ is a $Q-DF$ for $C_n$. ∎

From this point onwards, we assume that the sets $Q$ do not satisfy the hypothesis of Proposition 3. Thus the membership of vertices of $C_n$ (vertices of subpaths of $C_n$) in $Q$ and $P$ may be represented by a $PQ$-*sequence*, which is a circular (linear) sequence with alternate entries from the sets $\{P_1, P_2\}$ and $\{Q_1, Q_2\}$. The entry $P_2$, for example, means that two consecutive vertices of $C_n$ are in $P$. As a further illustration let $n = 10$ and $Q = \{v_1, v_2, v_5, v_7, v_8\}$. Then $Q_2P_2Q_1P_1Q_2P_2$ is a $PQ$-sequence for $C_n$. Our next result solves the existence problem when $Q_2$ is absent from $PQ$-sequences.

**Proposition 4.** *If $Q$ does not satisfy the hypothesis of Theorem* 1 *and $Q_2$ does not appear in a $PQ$-sequence, then $C_n$ has a $Q-DF$.*

**Proof.** If a $PQ$-sequence contains only $Q_1$ and $P_2$, then $Q$ satisfies the hypothesis of Theorem 1. If it contains only $Q_1$ and $P_1$, define $f$ by $f \mid Q = 0$ and $f \mid P = \frac{1}{2}$. Then $f$ is the required $Q-DF$.

Therefore assume that a $PQ$-sequence contains $Q_1$ and at least one of each of $P_1, P_2$. Then $C_n$ may be partitioned into paths of two types T1 and T2. The first (type T1) are maximal paths with $PQ$-sequences of the form $Q_1P_2Q_1P_2 \cdots P_2Q_1$. These paths include all $P_2$'s from the $PQ$-sequence of $C_n$. When all T1 paths are deleted from $C_n$, there remains a non-empty set of type T2 paths whose $PQ$-sequences are of the form $P_1Q_1P_1Q_1 \cdots Q_1P_1$. (It is possible that a T2 path is a single vertex and has $P_1$ as its $PQ$-sequence.)

Each type T1 path has a number of vertices congruent to 1 (mod 3). Let the maximum number of vertices in a T1 path be $3k+1$. If a T1 path has $3j+1$ $(1 \leq j \leq k)$ vertices and has vertex sequence $v_1, \ldots, v_{3j+1}$, define $f$ on the path as follows:

$$f(v_1) = f(v_{3j+1}) = \frac{1}{k+5},$$

$$f(v_{3i+1}) = \frac{k+1-j}{k+5} \qquad i = 1, \ldots, j-1,$$

$$f(v_{3i+2}) = \frac{i+2}{k+5} \qquad i = 0, \ldots, j-1$$

$$\text{and } f(v_{3i}) = \frac{j+2-i}{k+5} \qquad i = 1, \ldots, j\,.$$

If $u$ is a vertex of a T2 path, let

$$f(u) = \begin{cases} \frac{k+2}{k+5} & u \in P \\ 0 & u \in Q. \end{cases}$$

We show that $f$ is a $Q-DF$ for $C_n$. For $v \in Q$ in a T2 path

$$f[v] = \frac{2(k+2)}{k+5} > 1.$$

For $v \in P$ where $v$ is a non-endvertex of a T2 path

$$f[v] = \frac{k+2}{k+5} < 1.$$

For $v \in P$ and $v$ is an endvertex of a T2 path

$$f[v] = \begin{cases} \frac{k+2}{k+5} + \frac{1}{k+5} & \text{(T2 path not a single vertex)} \\ \frac{k+2}{k+5} + \frac{1}{k+5} + \frac{1}{k+5} & \text{(T2 path is a single vertex).} \end{cases}$$

In each case $f[v] < 1$.

Let $v$ be an element of a T1 path with vertex sequence $v_1, \ldots, v_{3j+1}$. If $v \in \{v_1, v_{3j+1}\} \subseteq Q$, then

$$f[v] = \frac{k+2}{k+5} + \frac{1}{k+5} + \frac{2}{k+5} = 1.$$

For $v = v_{3i+1}$ where $j \in \{1, \ldots, j-1\}$, $v \in Q$ and

$$f[v] = f(v_{3i}) + f(v_{3i+1}) + f(v_{3i+2})$$
$$= \frac{j+2-i}{k+5} + \frac{k+1-j}{k+5} + \frac{i+2}{k+5} = 1.$$

Let $v = v_{3i}$ where $i \in \{1, \ldots, j-1\}$. Then $v \in P$ and

$$f[v] = \frac{1}{k+5} \quad ((j+2-i) + (k+1-j) + ((i-1)+2))$$
$$= \frac{k+4}{k+5} < 1.$$

if $v \in \{v_2, v_{3j}\}$ $(\subseteq P)$, then

$$f[v] = \frac{1}{k+5} + \frac{2}{k+5} + \frac{j+1}{k+5} \leq \frac{k+4}{k+5} < 1.$$

Finally, if $v = v_{3i+2}$, where $i \in \{1, \ldots, j-1\}$, then $v \in P$ and

$$f[v] = \frac{1}{k+5} \quad ((k+1-j) + (i+2) + (j+2-(i+1)))$$
$$= \frac{k+4}{k+5} < 1.$$

Therefore $f$ is a $Q-DF$ for $C_n$ as asserted. ∎

We next deal with the case when the $PQ$-sequence contains no $Q_1$.

**Proposition 5.** *If $Q$ does not satisfy the condition of Theorem 1 and $Q_1$ is absent from a $PQ$-sequence, then $C_n$ has a $Q$-DF.*

**Proof.** If $P_2$ is also absent, then $Q$ satisfies the condition of Theorem 1. If $P_1$ is absent (*i.e.* we have $Q_2P_2Q_2P_2 \cdots Q_2P_2$) define $f$ by:

$$f(u) = \begin{cases} 0 & u \in P \\ \frac{1}{2} & u \in Q. \end{cases}$$

Then $f$ is the required $Q$-DF.

Hence we assume $P_1$ and $P_2$ both appear in a $PQ$-sequence. Then $C_n$ may be partitioned into paths of two types, namely S1 with $PQ$-sequence of the form $Q_2P_1Q_2P_1 \cdots P_1Q_2$ and type S2 with $PQ$-sequence $P_2Q_2P_2 \cdots Q_2P_2$.

For $u$ on any S2 path define $f$ by:

$$f(u) = \begin{cases} 0 & u \in P \\ \frac{1}{2} & u \in Q. \end{cases}$$

Consider the S1 path with vertex sequence $v_1, \ldots, v_{3j+2}$. Define $f$ on the vertices of this path by:

$$\begin{aligned} f(v_{3i}) &= 0 & i = 1, \ldots, j \quad (i.e. \text{ vertices of } P \\ f(v_{3i+1}) &= \frac{1}{2^{i+1}} & i = 0, \ldots, j \\ f(v_{3i+2}) &= 1 - \frac{1}{2^{i+1}} & i = 0, \ldots, j \end{aligned}$$

It is easily checked that $f$ is a *Q-DF* for $C_n$. ∎

**Proposition 6.** *If $P_2, Q_2, P_2$ are consecutive entries in a PQ-sequence for $C_n$, then $C_n$ has a Q-DF.*

**Proof.** If $n = 6$, the result is true by Proposition 3. If $n \geq 7$, let the vertex sequence of $C_n$ by $v_1, \ldots, v_n$ where (satisfying hypothesis) $\{v_{n-4}, v_{n-3}, v_n, v_1\} \subseteq P$ and $\{v_{n-1}, v_{n-2}\} \subseteq Q$. Let path $\eta$ have vertex sequence $v_1, \ldots, v_{n-4}$. Define function $f$ on $V(\eta)$ by Lemma 2 using $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$ and $\epsilon = \frac{1}{12}$. Then define $g : V(C_n) \to [0, 1)$ by

$$g(u) = \begin{cases} f(u) & u \in V(\eta) \\ 0 & u \in \{v_{n-3}, v_n\} \\ \frac{1}{2} & u \in \{v_{n-2}, v_{n-1}\}. \end{cases}$$

Then $g[v_{n-2}] = g[v_{n-1}] = 1$, and by Lemma 2 $g[v_n] = \frac{5}{6} < 1$, $g[v_1] = \frac{2}{3} < 1$, $g[v_{n-4}] < \frac{2}{3} + 2\epsilon = \frac{5}{6} < 1$ and $g[v_{n-3}] \leq \frac{1}{2} + \frac{1}{3} + \epsilon = \frac{11}{12} < 1$. All other vertices satisfy (1) by Lemma 2 and hence $g$ is a *Q-DF*. ∎

**Proposition 7.** *If $Q$ does not satisfy the conditions of Theorem 1 and either $P_1$ or $P_2$ does not appear in a PQ-sequence for $C_n$, then $C_n$ has a Q-DF.*

**Proof.** Suppose first that $P_1$ is absent from a $PQ$-sequence for $C_n$. The cases when $Q_1$ or $Q_2$ are absent, are covered by Propositions 4 and 5. When both are present, $C_n$ has a *Q-DF* by Proposition 6.

Now suppose a $PQ$-sequence has no $P_2$ and (using Propositions 4 and 5) $Q_1, Q_2$ are both present. Then $C_n$ may be partitioned into two types of paths. The first type R1 is a maximal path with $PQ$-sequence of the form $P_1Q_2P_1Q_2\cdots Q_2P_1$ and when these are removed, the remainder is a set of type R2 paths having $PQ$-sequence of the form $Q_1P_1,\ldots,P_1Q_1$ (R2 paths could be single vertices of $Q$).

If $u$ is a vertex of an R2 path, let

$$f(u)=\begin{cases}\frac{1}{2} & u\in P\\ 0 & u\in Q.\end{cases}$$

Let an R1 path have vertex sequence $v_1,\ldots,v_{3j+1}$. Define

$$\begin{aligned} f(v_{3i+1}) &= \tfrac{1}{2} & i&=0,\ldots,j \ (i.e.\text{ vertices of } P)\\ f(v_{3i+2}) &= \tfrac{1}{2^{i+1}} & i&=0,\ldots,j-1\\ f(v_{3i}) &= \tfrac{1}{2}-\tfrac{1}{2^{i+1}} & i&=1,\ldots,j\,. \end{aligned}$$

Then $f$ is a $Q$-$DF$ for $C_n$. ∎

It remains to consider $PQ$-sequences of $C_n$ which contain all of the entries $P_1$,$P_2$,$Q_1$,$Q_2$ (this implies $n\geq 6$). The congruence classes of $n$ (modulo 3) are treated separately. The next result, a simple application of Lemma 2, disposes of the case $n\equiv 2 \pmod 3$.

**Proposition 8.** *If $n\equiv 2 \pmod 3$ and $P_2$ appears in a $PQ$-sequence for $C_n$, then $C_n$ has a $Q$-$DF$.*

**Proof.** Let $C_n$ have vertex sequence $v_1,\ldots,v_n$ where $\{v_1,v_n\}\subseteq P$. Define $f$ on $v_1,\ldots,v_n$ by Lemma 2 using $\alpha_0=\frac{5}{12}$, $\alpha_1=\frac{1}{4}$, $\alpha_2=\frac{1}{3}$ and $\epsilon=\frac{1}{24}$. By Lemma 2

$$\left|f(v_n)-\frac{1}{3}\right|<\epsilon$$

and

$$\left|f(v_{n-1})-\frac{1}{4}\right|<\epsilon.$$

Hence

$$\begin{aligned} f[v_1] &= f(v_1)+f(v_2)+f(v_n)\\ &=\frac{1}{4}+\frac{1}{3}+\left(f(v_n)-\frac{1}{3}\right)+\frac{1}{3}\\ &<\frac{11}{12}+\epsilon=\frac{23}{24}<1. \end{aligned}$$

Further

$$\begin{aligned} f[v_n] &= f(v_{n-1}) + f(v_n) + f(v_1) \\ &= \left(f(v_{n-1}) - \frac{1}{4}\right) + \frac{1}{4} + \left(f(v_n) - \frac{1}{3}\right) + \frac{1}{3} + \frac{1}{4} \\ &< \frac{10}{12} + 2\epsilon = \frac{11}{12} < 1. \end{aligned}$$

By Lemma 2, $f$ is a *Q-DF* for $C_n$. ∎

**Proposition 9.** *If* $n \equiv 1 \pmod 3$ *and each of* $P_1, P_2, Q_1, Q_2$ *is present in a PQ-sequence for* $C_n$*, then* $C_n$ *has a Q-DF.*

**Proof.** A $Q_1$ is present in the $PQ$-sequence of $C_n$. By considering the two cases in which (a) $P_2$ is adjacent to this $Q_1$, (b) this $Q_1$ is adjacent to $P_1$'s in the sequence, with suitable labelling of vertices $C_n$ has vertex sequence $v_1, \ldots, v_n$ and either

(a) $\{v_{n-3}, v_n\} \subseteq Q$ and $\{v_1, v_{n-1}, v_{n-2}\} \subseteq P$ or

(b) for some $j$, $\{v_j, v_{j+2}, v_{j+4}\} \subseteq Q$ and $\{v_{j+1}, v_{j+3}\} \subseteq P$. These two cases are treated separately.

**Case (a)** Define $f$ on the path $\eta$ with vertex sequence $v_1, \ldots, v_{n-2}$ using Lemma 2 with $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{8}, \alpha_0 = \frac{5}{8}$ and $\epsilon = \frac{1}{16}$. Further let $f(v_{n-1}) = \frac{1}{4}$ and $f(v_n) = \frac{1}{2}$. It is easy to apply Lemma 2 to show that $f$ is a *Q-DF*. The details are omitted.

**Case (b)** Since $P_2$ is present in the $PQ$-sequence, we label so that $\{v_1, v_n\} \subseteq P$ and $2 \leq j \leq n-5$.

Let paths $\eta_1, \eta_2$ have vertex sequences $v_1, \ldots, v_j$ and $v_n, v_{n-1}, \ldots, v_{j+4}$ respectively. If $j$ is such that $|V(\eta_1)|$ or $|V(\eta_2)| = 2$, then $C_n$ has a *Q-DF* by case (a) and hence we may assume that each of $\eta_1, \eta_2$ has at least three vertices. Define $f$ on $V(\eta_1), V(\eta_2)$ using Lemma 2 with $\alpha_0 = \frac{5}{12}, \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{3}$ and $\epsilon = \frac{1}{24}$. Let $f(v_{j+1}) = f(v_{j+3}) = \frac{1}{2}$ and $f(v_{j+2}) = 0$.

Observe that

$$f[v_1] = f[v_n] = \frac{1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{5}{6} < 1.$$

Further

$$\begin{aligned} f[v_j] &= (f(v_{j=1}) + f(v_j)) + f(v_{j+1}) \\ &> \left(\frac{1}{4} + \frac{1}{3} - 2\epsilon\right) + \frac{1}{2} \qquad \text{(By Lemma 2)} \\ &= \frac{13}{12} - 2\epsilon = 1. \end{aligned}$$

A similar argument shows $f[v_{j+4}] \geq 1$.

Moreover if $v \in \{v_{j+1}, v_{j+3}\}$, $f[v] \leq \frac{1}{2} + \frac{5}{12} + \epsilon < 1$ and $f[v_{j+2}] = 1$. It follows that $f$ is a $Q$-*DF* for $C_n$. ∎

**Proposition 10.** *Let $n \equiv 0 \pmod 3$ and $C_n$ have a PQ-sequence which includes all entries $P_1, P_2, Q_1, Q_2$. If $Q$ does not satisfy the condition of Theorem 1, then $C_n$ has a $Q$-DF.*

**Proof.** Since $Q$ does not satisfy the Theorem 1 condition, one of two situations must arise:

1) For $i = 0, 1, 2,$ $\quad A_i \not\subseteq Q$, or
2) With suitable labelling $A_1 \subseteq Q$, $A_2 \cap Q \neq \emptyset$ and $A_0 \cap Q \neq \emptyset$. These situations are handled separately.

**Case 1** Since a $P_2$ occurs, we may label vertices with $\{v_1, v_n\} \subseteq P$. Since $n \equiv 0 \pmod 3$ the case 1 condition ensures the existence of $j$ such that $v_{3j+2} \in P$ and $v_{3j+2} \notin \{v_2, v_{n-1}\}$ (otherwise there are three consecutive vertices of $C_n$ in $P$). Hence each of the paths $\eta_1$ with vertex sequence $v_1, \ldots, v_{3j+2}$ and $\eta_2$ with sequence $v_{3j+2}, \ldots v_n$, has at least five vertices. Define $g$ on $V(\eta_1)$ using Lemma 2 with $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{3}, \alpha_0 = \frac{5}{12}$ and $\epsilon = \frac{1}{12}$. In order to apply Lemma 2 to $\eta_2$, re-name its vertices $v_{3j+2}, \ldots, v_n$ to be $w_1, \ldots w_{n-3j-1}$. Now define $h$ on $V(\eta_2)$ using Lemma 2 with $\alpha_1 = g(v_{3j+2})$, $\alpha_2 = \frac{1}{4}$, $\alpha_0 = 1 - \alpha_1 - \alpha_2$ and $\epsilon = \frac{1}{24}$. We emphasize $g(v_{3j+2}) = h(v_{3j+2}) = h(w_1)$. Finally define $f$ on $V(C_n)$ by

$$f(u) = \begin{cases} g(u) & u \in V(\eta_1) \\ h(u) & u \in V(\eta_2). \end{cases}$$

By Lemma 2 (b) all $v \in V(C_n) - \{v_1, v_{3j+2}, v_n\}$ satisfy (1).

By Lemma 2(c) $\left|g(v_{3j+1})-\frac{1}{4}\right|<\frac{1}{12}$ and $\left|g(v_{3j+2})-\frac{1}{3}\right|<\frac{1}{12}$. Hence

$$\begin{aligned} f[v_{3j+2}] &= g(v_{3j+1})+g(v_{3j+2})+h(w_2) \\ &< \left(\frac{1}{4}+\frac{1}{12}\right)+\left(\frac{1}{3}+\frac{1}{12}\right)+\frac{1}{4}=1. \end{aligned}$$

Also by Lemma 2(c)

$$|h(w_{n-3j-2})-g(v_{3j+2})|<\frac{1}{24}$$

and

$$\left|h(w_{n-3j-1})-\frac{1}{4}\right|<\frac{1}{24}.$$

Therefore

$$\begin{aligned} f[v_1] &= g(v_1)+g(v_2)+h(w_{n-3j-1}) \\ &< \frac{1}{4}+\frac{1}{3}+\left(\frac{1}{4}+\frac{1}{24}\right)<1. \end{aligned}$$

Finally

$$\begin{aligned} f[v_n] &= h(w_{n-3j-2})+h(w_{n-3j-1})+g(v_1) \\ &= (h(w_{n-3j-2})-g(v_{3j+2}))+\left(g(v_{3j+2})-\frac{1}{3}\right)+\frac{1}{3} \\ &+\left(h(w_{n-3j-1})-\frac{1}{4}\right)+\frac{1}{4}+g(v_1) \\ &< \frac{1}{24}+\frac{1}{12}+\frac{1}{3}+\frac{1}{24}+\frac{1}{4}+\frac{1}{4}=1. \end{aligned}$$

We conclude that $f$ is a $Q$-$DF$ for $C_n$.

**Case 2.** Let $n=3k$ and consider the sequences $s_i=v_{3i+1}, v_{3i+2}, v_{3i+3}$, where $i=0,\ldots,k-1$. For example $s_i$ will be said to have type $qpp$ if $v_{3i+1}\in Q$ and $\{v_{3i+2}, v_{3i+3}\}\subseteq P$.

Since each $v_{3i+1}\in Q$ and $A_2\cap Q\neq\emptyset$, some $s_i$ has type $qqp$. No $s_i$ has type $qqq$, $A_0\cap Q\neq\emptyset$ and therefore some $s_i$ has type $qpq$. Also, since $P_2$ appears in a $PQ$-sequence, some $s_i$ has type $qpp$. Moreover each $s_i$ has one of these three types.

If $s_i$ has type $qpq$, then $s_{i+1}$ does not have type $qqp$ (otherwise there are three consecutive elements of $Q$).

If $s_i$ has type $qqp$ and $s_{i+1}$ has type $qpq$, then $C_n$ has a *Q-DF* (define $f$ in precisely the same way as in Case (b) in the proof of Proposition 9).

It may be assumed that $s_i, s_j$ of types $qqp$ and $qpq$ are always separated by at least one $s_m$ of type $qpp$. (This implies $n \geq 12$.)

It follows that with suitable re-labelling of vertices, we may assume that $s_{k-1}, s_0$ have types $qpq$, $qpp$ respectively and that for some $j < k-1$, $s_j$ has type $qpp$ while for each $\ell$ satisfying $j < \ell \leq k-1$, $s_\ell$ has type $qpq$. Further for some $i$ in $\{1, \ldots, j-1\}$, $s_i$ has type $qqp$. The situation is illustrated in Fig. 1.

Define $f$ on the path with vertex sequence $v_1, \ldots, v_{3i+2}$ using Lemma 2 with $\alpha_1 = \frac{5}{12}$, $\alpha_2 = \frac{1}{3}$, $\alpha_0 = \frac{1}{4}$ and $\epsilon = \epsilon_1$. Next define $f$ on the path with vertex sequence $v_{3i+2}, \ldots, v_{3j+2}$ using Lemma 2 with $\alpha_1 = f(v_{3i+2})$ (already defined), $\alpha_2 = \frac{5}{12}$, $\alpha_0 = 1 - \alpha_1 - \alpha_2$ and $\epsilon = \epsilon_2$. Finally define $f$ on the path with vertex sequence $v_{3j+2}, \ldots, v_n$ using Lemma 2 and $\alpha_1 = f(v_{3j+2})$ (already defined) $\alpha_2 = \frac{1}{3}$, $\alpha_0 = 1 - \alpha_1 - \alpha_2$ and $\epsilon = \epsilon_3$.

| | $s_0$ | | | | $s_i$ | | | | $s_j$ | | | | $s_{k-1}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | $p$ | $p$ | $\ldots$ | $q$ | $q$ | $p$ | $\ldots$ | $q$ | $p$ | $p$ | $\ldots$ | $q$ | $p$ | $q$ |
| $v_1$ | $v_2$ | $v_3$ | | $v_{3i+1}$ | $v_{3i+2}$ | $v_{3i+3}$ | | $v_{3j+1}$ | $v_{3j+2}$ | $v_{3j+3}$ | | $v_{n-2}$ | $v_{n-1}$ | $v_n$ |
| $\frac{5}{12}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | | $\frac{5}{12}$ | $\frac{1}{3}$ | $\frac{5}{12}$ | | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

**Fig. 1: The sequences $s_\ell$ in Case 2.**

By choosing $\epsilon_1, \epsilon_2, \epsilon_3$ sufficiently small, the values of $f$ may be made arbitrarily close to those given in Fig. 1 and yield a *Q-DF* for $C_n$ (the obvious calculations are omitted).

This completes the proofs of Proposition 10 and Theorem 1. ∎

## 3. Existence of $Q$-$DF$'s in Trees

The following proposition gives a simple necessary condition for the existence of a $Q$-$DF$ in a graph $G$.

**Proposition 11.** *Let $f$ be a $Q$-$DF$ for $G$. Then for each vertex $v$ satisfying $N[v] \supseteq N[u]$ where $u \in Q$, $v$ is also in $Q$.*

**Proof.** $f[v] \geq f[u]$ since $N[v] \supseteq N[u]$. But $u \in Q$ implies $f[u] \geq 1$. Hence $f[v] \geq 1$, *i.e.* $v \in Q$. ∎

The condition of Proposition 11 is not always sufficient for existence of $Q$-$DF$'s. For example, the cycles $C_n$ satisfy the condition for any $Q$, but Theorem 1 shows that $Q$-$DF$'s do not always exist. However in this section we show that the necessary condition is sufficient for existence of $Q$-$DF$'s in a tree $T$. Observe that in a tree $N[u] \subseteq N[v]$ if and only if $u$ is a leaf (*i.e.* $deg(u) = 1$) and $v$ is its neighbour.

**Theorem 12.** *Let the $n$-vertex tree $T = (V, E)$ and $Q \subseteq V$ have the property that for each leaf in $Q$, its neighbour is also in $Q$. Then a $Q$-$DF$ for $T$ may be constructed in $O(n)$ time.*

**Proof.** It is easy to confirm by inspection that the theorem is true for $n = 1, 2$. Suppose therefore that $n \geq 3$.

The following algorithm constructs a $Q$-$DF$ for $T$. If a value $f(v)$ has been assigned to a vertex $v$ of the graph, we say that $v$ has been *labelled;* otherwise it is *unlabelled.* In Step 2 of the algorithm the value $\frac{1}{2}$ is an arbitrary choice from the interval $(0, 1)$.

**Algorithm Positive *Q-DF*.**

Input: *An n-vertex tree* $T = (V, E)$ *and* $Q \subseteq V$ *satisfying:*

for each leaf in $Q$, its neighbour is also in $Q$.

A designated vertex $r$ adjacent to some leaf.

$\deg(v)$ for each $v \in V$.

Output: *A Q-DF* $f : V \to (0, 1)$ *for* $T$

BEGIN

FOR each $u \in V$ DO

  IF $u \in Q$ THEN $k(u) = 1$ ELSE $k(u) = \frac{1}{2}$

END

FOR each $u \in N[r]$ DO

  $f(u) \leftarrow \frac{k(r)}{\deg(r)+1}$

END

WHILE there is an unlabelled vertex DO

  Let $w$ be a labelled vertex which has an adjacent unlabelled vertex

  Let $v$ be the labelled vertex adjacent to $w$

  $\text{par}[w] \leftarrow f(v) + f(w)$

    FOR each unlabelled vertex $u \in N[w]$ DO

      $f(u) \leftarrow \frac{k(w)(1-\text{par}[w])}{\deg(w)-1}$

    END

END

FOR each leaf $v$ of $T$ in $Q$ DO

  Let $u$ be the vertex adjacent to $v$

  $f(v) \leftarrow 1 - f(u)$

END

END Positive *Q-DF*

The function $f$ constructed by the algorithm is a *Q-DF* for $T$.

To prove this claim, first note that at any stage of the execution of the algorithm the vertices that have been assigned values induce a subtree of $T$. All vertices in the closed neighbourhood of the vertex designated $r$ in the second FOR loop and of each vertex designated $w$ in the WHILE loop have been assigned values after the execution of the respective loops. When a new vertex $w$ in the WHILE loop is considered, it must therefore be a leaf of the subtree but not a leaf of $T$. This implies that $\deg(w) \geq 2$ in Step 10 of the algorithm. Therefore, by the constructions of the algorithm, $f[v] \leq 1$ for the vertex designated $r$ in Step 4 and $f[w] \leq 1$ for each vertex designated $w$ in Step 10, where the equality will hold only if the vertex concerned is not a leaf and is a member of $Q$.

If $v$ is the vertex adjacent to $w$ in the labelled subtree, then the partial value

$$\text{par}[w] = f(v) + f(w) \leq f[v] \leq 1,$$

where the first inequality is strict if all vertices in $N[v]$ have positive values. The values assigned in Step 1 are in $(0, 1)$ and this together with the properties mentioned above and the constructions in the algorithm imply that $\text{par}[w] \in (0, 1)$ as long as the WHILE loop is executed. Therefore $f(u) \in (0, 1)$ for each vertex $u$ which is assigned a value in this loop.

Let $v$ be a leaf in $Q$ with adjacent vertex $w$, which is also in $Q$ by the conditions of the lemma. Then $f(w) + f(v) < 1$ after the execution of the WHILE loop of the algorithm, whereas $f(w) + f(v) = 1$ after the execution of the final FOR loop. The value of $v$ is increased by the execution of this final loop, but since $f(w) \in (0, 1)$, also $f(v) \in (0, 1)$.

Except possibly for leaves, the values once assigned by the algorithm are never changed. The changes in the final FOR loop provide for the possible membership of leaves in $Q$, but do not alter the membership status in $Q$ of neighbours of leaves.

The initialization in the first FOR loop requires the assignment of $n$ values. In the main body of the algorithm each vertex is visited excactly once, and at most four addition, subtraction, multiplication or division operations are required for each vertex. The final adjustment for leaves in $Q$ requires fewer than $n$ operations. Therefore the complexity of the algorithm is $O(n)$.

By the construction and the arguments above, the algorithm determines in linear time a function $f$ with the properties that

$$f : V \to (0,1),$$
$$f[v] \geq 1 \text{ for } v \in Q,$$
$$f[v] < 1 \text{ for } v \in P.$$

∎

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