PERRON-FROBENIUS THEORY FOR A GENERALIZED EIGENPROBLEM

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ABSTRACT

Motivated by economic models, the generalized eigenvalue problem $Ax = \lambda Bx$ is investigated under the conditions that A is nonnegative and irreducible, there is a nonnegative vector u such that $Bu > Au$, and $b_{ij} \le a_{ij}$ for all $i \ne j$. The last two conditions are equivalent to $B - A$ being a nonsingular M-matrix. The focus is on generalizations of the Perron-Frobenius theory, the classical theory being recovered when B is the identity matrix. These generalizations include identification of a generalized eigenvalue $\rho(A,B)$ in the interval (0, 1) with a positive eigenvector, characterizations and easily computable bounds for $\rho(A,B)$, and localization results for all generalized eigenvalues. Dropping the condition that A is irreducible, necessary and sufficient conditions for the problem to have a solution with $x > 0$ are formulated in terms of basic and final classes, which are natural extensions of these concepts in the classical theory.

1. INTRODUCTION

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We say that A is nonnegative, and write $A \ge 0$ if $a_{ij} \ge 0$ for all i, j. The matrix A is called positive, $A > 0$, if $a_{ij} > 0$ for all i, j. If $A = [a_{ij}]$, $B = [b_{ij}]$ are $n \times n$ matrices, then $A \ge B$ $(A > B)$

means that $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$ for all i, j. We say that $A \leq B$ $(A < B)$ if and only if $-A \geq -B$ $(-A > -B)$. Similar notation applies to vectors. The identity matrix of the appropriate order is denoted by I. The transpose of the matrix A is denoted by A^T.

An $n \times n$ matrix A is said to be reducible [5, p. 360] if A is the 1×1 zero matrix or if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

where B, D are (nonvacuous) square matrices. If A is not reducible, then it is called irreducible.

Let A, B be (real) $n \times n$ matrices and consider the equation

$$Ax = \lambda Bx. \tag{1.1}$$

If A is an irreducible, nonnegative matrix and if $B = I$, then the analysis of (1.1) reduces to the spectral theory of irreducible, nonnegative matrices. In particular, by the Perron-Frobenius Theorem, there exists a vector $x > 0$ such that

$$Ax = \rho(A)x,$$

where $\rho(A)$ is the spectral radius, or the Perron root, of A. Furthermore, if $v \geq 0$, $v \neq 0$ and $Av = \lambda v$, then $\lambda = \rho(A)$ and $v = \alpha x$ for some $\alpha > 0$. We assume familiarity with the basic aspects of the Perron-Frobenius theory, see, for example, [1, 3, 5].

When A and B are complex matrices, the generalized eigenvalue problem (1.1) has been much studied; see, for example [4, p. 394], [12, Ch. 6]. Economists have considered (1.1) in the context of Sraffa's model for the "joint production of commodities by means of commodities", see Sraffa [11, p. 53] and the essays by Manara, Pasinetti and Schefold in [10]. In this context, A and B are assumed to be nonnegative. Fujimoto [2], motivated by a nonlinear extension of the Perron-Frobenius theory in [8], considered (1.1) in the presence of the following conditions:

(C1) $A \geq 0$

(C2) A is irreducible

(C3) there exists a vector $u \geq 0$ such that $Bu > Au$

(C4) for all $i \neq j$, $b_{ij} \leq a_{ij}$.

Economic interpretation of these conditions is given in [2]. In view of the interpretation, the assumption that $B \geq 0$ is also made in [2] although we do not find it necessary to impose it here.

The main purpose of the present paper is to investigate (1.1) from the point of view of linear algebra, particularly the Perron-Frobenius theory. We now give a summary of the main results obtained in this paper.

It has been shown in [2] that if (C1)-(C4) are satisfied, then (1.1) has a solution $\lambda \in (0, 1)$ and $x > 0$. In Section 2 we give a more direct proof of this result than that contained in [2], imposing conditions weaker than (C1)-(C4). The solution $\lambda \in (0, 1)$

is unique in the sense that if $Av = \lambda' Bv$, $v \geq 0$, $v \neq 0$, then $\lambda' = \lambda$. We denote this unique λ by $\rho(A, B)$. Thus $\rho(A, I) = \rho(A)$, the spectral radius of A, and therefore $\rho(A, B)$ can be thought of as a generalization of the spectral radius of a nonnegative, irreducible matrix.

In Section 3 we obtain max-min and min-max characterizations of $\rho(A, B)$ that extend corresponding well-known characterizations of $\rho(A)$. We also obtain certain inclusion regions for all values of λ that could arise in a solution of (1.1).

In Section 4 we obtain some inequalities for $\rho(A, B)$. It is shown that if $A_1 \geq A_2$, then $\rho(A_1, B) \geq \rho(A_2, B)$ whereas if $B_1 \geq B_2$, then $\rho(A, B_1) \leq \rho(A, B_2)$. We also show that if A_{11} and B_{11} are corresponding principal submatrices of A and B, respectively, then $\rho(A, B) \geq \rho(A_{11}, B_{11})$.

In Section 5 we consider the situation where A is not necessarily irreducible, i.e., when A, B satisfy (C1), (C3) and (C4), and define $\rho(A, B)$ in this case. We then define the notions of a basic class and a final class, which are natural extensions of the same concepts defined for a reducible, nonnegative matrix. We show that if (1.1) has a solution $x > 0$, then a class is basic precisely when it is final, and give a converse under a further restriction on B. This generalizes a result in [3, p. 77], (see also [1, p. 40]).

2. PRELIMINARIES

An $n \times n$ matrix M is called a (nonsingular) M-matrix if $m_{ij} \le 0$ for all $i \ne j$ and if there exists a vector $v \ge 0$ such that $Mv > 0$. For several equivalent definitions of an M-matrix see [1, Ch. 6].

LEMMA 2.1 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4). *Then* $(B - A)^{-1} A$ *exists, is nonnegative and irreducible.*

Proof Conditions (C3), (C4) are equivalent to the assertion that $B - A$ is an M-matrix. Then $B - A$ is nonsingular and $(B - A)^{-1} \ge 0$. Since $A \ge 0$ it follows that $(B - A)^{-1} A \ge 0$. Since $B - A$ is an M-matrix, every principal submatrix of $B - A$ is an M-matrix and therefore the $(n - 1) \times (n - 1)$ principal minors of $B - A$ are all positive. Therefore the diagonal entries of $(B - A)^{-1}$ are all positive. Let $D = [d_{ij}]$ be the diagonal matrix with d_{ii} being the i-th diagonal entry of $(B - A)^{-1}$. Then $(B - A)^{-1} A \ge DA$, and since A is irreducible, it follows that $(B - A)^{-1} A$ is irreducible. □

The equivalence noted at the beginning of this proof implies that, without loss of generality, condition (C3) can be replaced by the condition $Be > Ae$, where e is the vector of all ones. However, we use condition (C3) in what follows.

THEOREM 2.2 *Let* A, B be $n \times n$ *matrices such that* $B - A$ *is nonsingular and suppose* $(B-A)^{-1}A$ *is nonnegative and irreducible. Then there exists* $\lambda \in (0, 1)$ *and a vector* $x > 0$ *such that* $Ax = \lambda Bx$. *Furthermore, if* $Av = \lambda' Bv$ *and* $v \geq 0$, $v \neq 0$ *then* $\lambda' = \lambda$ *and* $v = \alpha x$ *for some* $\alpha > 0$.

Proof Since $(B-A)^{-1}A \geq 0$ and is irreducible, by the Perron-Frobenius theorem there exist $\mu > 0$ and a vector $x > 0$ such that

$$(B-A)^{-1}Ax = \mu x.$$

Thus $Ax = (B-A)\mu x = \mu Bx - \mu Ax$. Therefore

$$Ax = \frac{\mu}{1+\mu} Bx.$$

If we set $\lambda = \dfrac{\mu}{1+\mu}$, then $\lambda \in (0, 1)$ and $Ax = \lambda Bx$. This proves the first part of the theorem. Note that $\lambda' \neq 1$ (since $B - A$ is nonsingular). As $\mu = \lambda/(1-\lambda)$, the second part follows by the uniqueness assertion in the Perron-Frobenius theorem stated in Section 1. □

$(I-A)^{-1}A$ is nonnegative and irreducible, $\rho(A,I) = \rho(A)$. The following result is contained in [2] with the additional assumption that $B \geq 0$ (which we do not impose). The result, which is an easy consequence of Lemma 2.1 and Theorem 2.2, is stated for future reference.

THEOREM 2.3 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4). *Then there exist* $\lambda \in (0, 1)$ *and a vector* $x > 0$ *such that* $Ax = \lambda Bx$. *Furthermore, if* $Av = \lambda' Bv$ *for* $\lambda' \geq 0$ *and* $v \geq 0$, $v \neq 0$, *then* $\lambda' = \lambda$ *and* $v = \alpha x$ *for some* $\alpha > 0$.

REMARK 2.4 *If* A, B *satisfy* (C1)-(C4), *then so do* A^T, B^T *and therefore there exist* $\tilde{\lambda} > 0$ *and a vector* $y > 0$ *such that*

$$y^T A = \tilde{\lambda} y^T B. \tag{2.1}$$

Post-multiplying (2.1) *by* $x > 0$ *that satisfies* $Ax = \rho(A, B)Bx$, *we conclude that* $\tilde{\lambda} = \rho(A, B)$.

3. BOUNDS AND INCLUSION REGIONS

The following observation will frequently be useful.

REMARK 3.1 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4) *and suppose* D *is an* $n \times n$ *diagonal matrix with positive diagonal entries. Then it is easily verified that the matrices* $D^{-1}AD$, $D^{-1}BD$ *also satisfy* (C1)-(C4) *and furthermore,*

$$\rho(A, B) = \rho(D^{-1}AD, D^{-1}BD).$$

If A is an $n \times n$ matrix then we set $r_i(A) = \sum_{j=1}^{n} a_{ij}$, the i-th row-sum of A, $i = 1, 2, ..., n$. In the next result we obtain lower and upper bounds for $\rho(A, B)$ in terms of the row-sums of A, B. If A is an $n \times n$ nonnegative irreducible matrix, we may assume that (C3) is satisfied with $B = I$ by replacing A by αA for sufficiently small $\alpha > 0$. Then the next result reduces to the well known fact that $\rho(A)$ lies between the maximum and minimum row sums of A; see, for example, [5, p. 492].

THEOREM 3.2 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4) *and suppose* $r_i(B) > 0$, $i = 1, 2, ..., n$. *Let* $q_i = \dfrac{r_i(A)}{r_i(B)}$, $i = 1, 2, ..., n$ *and let* $q_m = \min_i q_i$, $q_M = \max_i q_i$. *Then*

$$q_m \le \rho(A, B) \le q_M. \tag{3.1}$$

Furthermore, equality holds in either of the inequalities in (3.1) *if and only if* $q_m = q_M$.

Proof Let $C = [c_{ij}]$ be the $n \times n$ matrix defined as

$$c_{ij} = q_m \frac{a_{ij}}{q_i}, \quad i, j = 1, 2, ..., n.$$

Then $C \leq A$. Also, $Ce = q_m Be$ where e denotes the column vector of all ones. By Remark 2.4 there exists $y > 0$ such that $y^T A = \rho(A, B) y^T B$. In view of these observations we have

$$q_m y^T Be = y^T Ce \leq y^T Ae = \rho(A, B) y^T Be.$$

Since $Be > 0$ and $y > 0$, then $y^T Be > 0$ and it follows that $q_m \leq \rho(A, B)$. It is clear from the proof that equality holds in this inequality if and only if $Ce = Ae$, and this is the case precisely when $q_m = q_M$. The proof of the second inequality is similar. □

COROLLARY 3.3 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4). *Then*

(i) $$\rho(A, B) = \max_{v:\ v \geq 0,\ Bv > 0} \ \min_i \frac{(Av)_i}{(Bv)_i}$$

(ii) $$\rho(A, B) = \min_{v:\ v \geq 0,\ Bv > 0} \ \max_i \frac{(Av)_i}{(Bv)_i}.$$

Proof First let $v > 0$ such that $Bv > 0$. Let D be the $n \times n$ diagonal matrix with $v_1, \ldots, v_n$ along the diagonal. By Remark 3.1 and Theorem 3.2 we have

$$\rho(A, B) = \rho\left(D^{-1}AD, D^{-1}BD\right) \geq \min_i \frac{r_i\left(D^{-1}AD\right)}{r_i\left(D^{-1}BD\right)} = \min_i \frac{(Av)_i}{(Bv)_i}.$$

Therefore

$$\rho(A, B) \geq \min_i \frac{(Av)_i}{(Bv)_i}. \tag{3.2}$$

If $v \geq 0$ such that $Bv > 0$, then (3.2) can be proved using a continuity argument. If we set $v = x$ where $x > 0$ satisfies $Ax = \rho(A, B)Bx$ (see Theorem 2.3), then equality holds in (3.2) and thus (i) is proved. The proof of (ii) is similar. □

Setting $B = I$, and if necessary replacing A by αA, in Corollary 3.3 we obtain the max-min and min-max characterizations of the Perron root of an irreducible nonnegative matrix (see, for example, [5, p. 493]).

The next result is a generalization of an inequality contained in [1, p. 37]; see also [9, p. 111]. We remark that this is our only result that requires the restriction that $B \geq 0$.

THEOREM 3.4 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4) *and suppose* $B \geq 0$. *Let* $x > 0$ *with* $Ax = \rho(A,B)Bx$. *Let* $q_i = \dfrac{r_i(A)}{r_i(B)}$, $i = 1, 2, \ldots, n$ *and let*

$x_m = \min_i x_i$, $x_M = \max_i x_i$, $q_m = \min_i q_i$, $q_M = \max_i q_i$. *Then*

$$\left(\frac{x_M}{x_m}\right)^2 \geq \frac{q_M}{q_m}. \tag{3.3}$$

Proof Note that $r_i(B) > 0$ since $B \geq 0$ and $B - A$ is a nonsingular M-matrix. We have, for $i = 1, 2, \ldots, n$,

$$\rho(A, B)\, x_M r_i(B) \geq \rho(A, B) \sum_{j=1}^{n} b_{ij}\, x_j = \sum_{j=1}^{n} a_{ij}\, x_j \geq r_i(A)\, x_m.$$

Therefore,

$$\frac{x_M}{x_m} \geq \frac{q_i}{\rho(A, B)}, \quad i = 1, 2, \ldots, n.$$

Hence, in particular,

$$\frac{x_M}{x_m} \geq \frac{q_M}{\rho(A, B)}. \tag{3.4}$$

Similarly, for $i = 1, 2, \ldots, n$,

$$x_M\, r_i(A) \geq \sum_{j=1}^{n} a_{ij}\, x_j = \rho(A, B) \sum_{j=1}^{n} b_{ij}\, x_j \geq \rho(A, B)\, x_m r_i(B).$$

Hence

$$\frac{x_M}{x_m} \geq \frac{\rho(A, B)}{q_i}, \quad i = 1, 2, \ldots, n,$$

and, in particular

$$\frac{x_M}{x_m} \geq \frac{\rho(A, B)}{q_m}. \tag{3.5}$$

Inequality (3.3) follows immediately from (3.4), (3.5). □

We now consider the location of the generalized eigenvalues, i.e., finite values of λ such that $Av = \lambda Bv$ for some $v \neq 0$. Note that here λ may be real or complex and vector v may have real or complex components. Also note that the remaining results in this section hold under the weaker condition that $(B-A)^{-1}A \geq 0$ and irreducible.

LEMMA 3.5 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4). *Then the number of generalized eigenvalues is at least* 1 *and at most* n.

Proof By Theorem 2.3, there is a generalized eigenvalue $\rho(A, B)$. Since $B - A$ is nonsingular, $\det(\lambda B - A)$ is a nonzero polynomial of degree at most n and has at most n roots. □

Note that there are n generalized eigenvalues if and only if B is nonsingular. However, if $B = J$, the matrix with each entry equal to 1, and $A = B - I$, then $\rho(A, B) = (n-1)/n$ is the only generalized eigenvalue.

THEOREM 3.6 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4). *If* $\rho = \rho(A, B) \neq \frac{1}{2}$, *then let* C *be the circle defined as*

$$C = \left\{(x, y): \left(x + \frac{\rho^2}{1 - 2\rho}\right)^2 + y^2 = \frac{\rho^2(1-\rho)^2}{(1-2\rho)^2}\right\}.$$

Then all generalized eigenvalues $\lambda = x + iy$ *are located in the complex plane as follows.*

(i) *If* $\rho = \frac{1}{2}$ *then all generalized eigenvalues* λ *have real part* $\leq \frac{1}{2}$.

(ii) *If* $\rho < \frac{1}{2}$ $\left(\rho > \frac{1}{2}\right)$ *then all generalized eigenvalues lie on or inside (outside)* C.

Proof Let $\mu^* = \frac{\rho}{1-\rho}$. As observed from the proof of Theorem 2.2, any eigenvalue μ of $(B-A)^{-1}A$ satisfies $|\mu| \leq \mu^*$. Thus any generalized eigenvalue λ satisfies $\left|\frac{\lambda}{1-\lambda}\right| \leq \mu^*$. Let $\lambda = x + iy$. Then

$$x^2 + y^2 \leq \mu^{*2}\left((x-1)^2 + y^2\right). \tag{3.6}$$

First suppose $\rho = \frac{1}{2}$. Then $\mu^* = 1$ and (3.6) gives $x \leq \frac{1}{2}$. This proves (i). If

$\rho < \frac{1}{2}$ then $\mu^* < 1$ and (3.6) gives

$$\left(x + \frac{\mu^{*2}}{1 - \mu^{*2}}\right)^2 + y^2 \le \frac{\mu^{*2}}{(1 - \mu^{*2})^2}$$

and this inequality upon simplification using $\mu^* = \frac{\rho}{1 - \rho}$ shows that λ lies on or inside C. The other part in (ii) is proved similarly. □

The following example illustrates that the inclusion region (ii) of Theorem 3.6 when $\rho(A,B) > 1/2$ is unbounded.

EXAMPLE 3.7

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2(1+\varepsilon^2) & 3-\varepsilon^2 & 0 \end{bmatrix}$ with $3 > \varepsilon^2$, and let $B = I + A$. Then

(C1)-(C4) are satisfied, $\rho(A, B) = 2/3$, and the other generalized eigenvalues are $1 \pm i/\varepsilon$, which has unbounded imaginary part as ε approaches zero.

As suggested by a referee, it is possible to give more precise information on the location of the generalized eigenvalues by using a result of Dmitriev, Dynkin and Karpelevich, see Kellogg and Stephens [7, Th. 1]. This shows that when A, B are

$n \times n$ matrices satisfying (C1)-(C4), then any generalized eigenvalue $\lambda = x + iy$ satisfies

$$\left(x - \frac{(2\mu^* + 1)}{2(\mu^* + 1)}\right)^2 + \left(y - \frac{\tan \pi/n}{2(\mu^* + 1)}\right)^2 \geq \frac{\sec^2 \pi/n}{4(\mu^* + 1)^2},$$

where $\mu^* = \dfrac{\rho}{1 - \rho}$ and $\rho = \rho(A,B)$. We omit the proof since it only involves a routine verification.

In contrast to the Perron root, $\rho(A, B)$ is not necessarily the generalized eigenvalue with maximum modulus unless $B^{-1}A$ is nonnegative. If $\rho(A, B) < 1/2$, then, by Theorem 3.6, $\rho(A, B) = \max \mathrm{Re}(\lambda)$, where λ is any generalized eigenvalue. Moreover, in this case, B is nonsingular. This can be seen as follows.

THEOREM 3.8 *Let* A, B *satisfy* (C1)-(C4) *and suppose* B *is singular. Then* $\rho(A,B) \geq 1/2$.

Proof If B is singular, then there exists a nonzero vector z such that $Bz = 0$. Then $(B - A)z = -Az$, and hence $(B - A)^{-1} Az = -z$, showing that -1 is an eigenvalue of $(B - A)^{-1} A$. Thus $\rho\left((B - A)^{-1} A\right) \geq 1$, and so $\rho(A,B) \geq 1/2$. □

4. PERTURBATION INEQUALITIES

We will use the next two known results, see, for example, [6, p. 196, 210].

LEMMA 4.1 *Let* M_1, M_2 *be* $n \times n$ *M-matrices such that* $M_1 \geq M_2$. *Then* $M_2^{-1} \geq M_1^{-1}$.

LEMMA 4.2 *Let* M *be an M-matrix, let* $N = M^{-1}$ *and suppose* M, N *are conformally partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ \\ M_{21} & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ \\ N_{21} & N_{22} \end{bmatrix}$$

where the diagonal blocks are square. Then $N_{11} \geq M_{11}^{-1}$.

We also need the following result, see, for example, [1, p. 27].

LEMMA 4.3 *If* X, Y *are* $n \times n$ *matrices satisfying* $0 \leq X \leq Y$, *then* $\rho(X) \leq \rho(Y)$. *Furthermore, if* $X \neq Y$ *and if* X *is irreducible, then* $\rho(X) < \rho(Y)$.

THEOREM 4.4 *(i) Let* A_1, A_2, B *be* $n \times n$ *matrices such that* A_1, B *as well as* A_2, B *satisfy* (C1)-(C4) *and suppose* $A_1 \geq A_2$. *Then* $\rho(A_1, B) \geq \rho(A_2, B)$.

Furthermore, the inequality is strict if $A_1 \neq A_2$.

(ii) Let A, B_1, B_2 *be* $n \times n$ *matrices such that* A, B_1 *as well as* A, B_2 *satisfy* (C1)-(C4) *and suppose* $B_1 \geq B_2$. *Then* $\rho(A, B_1) \leq \rho(A, B_2)$. *Furthermore, the inequality is strict if* $B_1 \neq B_2$.

Proof (i) If $A_1 \geq A_2$ then $B - A_1 \leq B - A_2$ and by Lemma 4.1, $(B - A_1)^{-1} \geq (B - A_2)^{-1}$. Since $A_1 \geq 0$, $A_2 \geq 0$, it follows that

$$(B - A_1)^{-1} A_1 \geq (B - A_2)^{-1} A_2 \tag{4.2}$$

and therefore by Lemma 4.3,

$$\rho\left((B - A_1)^{-1} A_1\right) \geq \rho\left((B - A_2)^{-1} A_2\right). \tag{4.3}$$

If $A_1 \neq A_2$ then $B - A_1 \neq B - A_2$ and therefore $(B - A_1)^{-1} \geq (B - A_2)^{-1}$ with strict inequality for at least one entry. Since A_2 is irreducible it does not have a zero row and therefore it follows that equality does not hold in (4.2). By Lemma 2.1, $(B - A_2)^{-1} A_2$ is irreducible and therefore by Lemma 4.3, (4.3) must be strict. The result follows by using the relation (see the proof of Theorem 2.2)

$$\rho(A_i, B) = \frac{\rho\left((B - A_i)^{-1} A_i\right)}{1 + \rho\left((B - A_i)^{-1} A_i\right)}, \quad i = 1, 2.$$

The proof of (ii) is similar. □

THEOREM 4.5 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1)-(C4) *and suppose we have the conformal partition*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where the diagonal blocks are square and nonvacuous.

Then

$$\rho(A,B) > \rho(A_{11}, B_{11}).$$

Proof Let $C = (B-A)^{-1}$ and partition C conformally as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

Then $C_{11}A_{11} + C_{12}A_{21}$ is a principal submatrix of CA and CA is irreducible by Lemma 2.1. It follows (see [1, p. 28]) that

$$\rho\left((B-A)^{-1}A\right) > \rho\left(C_{11}A_{11} + C_{12}A_{21}\right). \qquad (4.4)$$

Since $C_{12} \geq 0$, $A_{21} \geq 0$ and since $C_{11} \geq (B_{11} - A_{11})^{-1}$ by Lemma 4.2, it follows from (4.4) that

$$\rho\left((B-A)^{-1}A\right) > \rho\left((B_{11}-A_{11})^{-1}A_{11}\right)$$

and therefore

$$\rho(A,B) > \rho(A_{11},B_{11}). \quad \square$$

5. BASIC CLASSES AND FINAL CLASSES

We need the following result in the sequel.

LEMMA 5.1 *Let* A, B be $n \times n$ *matrices satisfying* (C1)-(C4). *Then for* $\rho(A,B) < \lambda \le 1$, $\lambda B - A$ *is an M-matrix. Furthermore,* $\lambda B - A$ *is irreducible if* $\rho(A,B) < \lambda < 1$ *and, in particular,* $(\lambda B - A)^{-1} > 0$.

Proof Fix λ, $\rho(A,B) < \lambda \le 1$, and assume $i \ne j$. If $b_{ij} \le 0$ then clearly, $\lambda b_{ij} - a_{ij} \le 0$. If $b_{ij} > 0$ then $\lambda b_{ij} \le b_{ij} \le a_{ij}$ and again $\lambda b_{ij} - a_{ij} \le 0$. Note that if $\rho(A,B) < \lambda < 1$ and if $a_{ij} > 0$, then $\lambda b_{ij} - a_{ij} < 0$ and thus $\lambda B - A$ is irreducible.

Let $x > 0$ be such that $Ax = \rho(A,B)Bx$. Then

$$(\lambda B - A)x = (\lambda - \rho(A,B))Bx > 0,$$

where the strict inequality is due to the fact that $Bx > 0$. It follows that $\lambda B - A$ is an M-matrix. If $\lambda < 1$ then, as noted earlier, $\lambda B - A$ is irreducible and it follows (see [1, p. 141]) that $(\lambda B - A)^{-1} > 0$. □

Suppose A, B are $n \times n$ matrices satisfying (C1), (C3) and (C4); thus A is not necessarily irreducible. Motivated by the proof of Theorem 2.2 we define $\rho(A, B)$ as

$$\rho(A, B) = \frac{\rho\left((B - A)^{-1}A\right)}{1 + \rho\left((B - A)^{-1}A\right)}.$$

If $A \geq 0$ then recall (see [1, p. 39]) that there exists a permutation matrix P such that PAP^T is in Frobenius Normal Form,

$$PAP^T = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} \tag{5.1}$$

where each diagonal block A_{ii} is square and is either irreducible or a 1×1 zero matrix. For convenience, we assume in what follows that each A_{ii} is irreducible.

When (5.1) holds we associate classes $1, 2, \ldots, k$ with A. Class i has access to class j if and only if $A_{ij} \neq 0$.

Let PBP^T be conformally partitioned as

$$PBT^T = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{bmatrix}. \tag{5.2}$$

Since $B - A$ is an M-matrix, so is $B_{ii} - A_{ii}$, $i = 1, 2, ..., k$ and therefore A_{ii}, B_{ii} satisfy (C1)-(C4). Also, Theorem 4.5 and a continuity argument show that

$$\rho(A, B) \geq \max_i \rho(A_{ii}, B_{ii}). \tag{5.3}$$

We now introduce the following definitions. We say that class i is basic if

$$\rho(A_{ii}, B_{ii}) = \rho(A, B).$$

Class i is called final if for all $k \neq i$,

$$A_{ik} = B_{ik} = 0.$$

Observe that when $B = I$, these definitions reduce to the usual definitions of basic and final classes (see [1, p. 40]).

The main purpose of this section is to obtain the following generalization of a result in [3, p. 77], see also [1, p. 40].

THEOREM 5.2 *Let* A, B *be* $n \times n$ *matrices satisfying* (C1), (C3), (C4) *and let* P *be a permutation matrix so that* (5.1), (5.2) *hold. If there exists* $x > 0$ *such that*

$$Ax = \rho(A,B)Bx \tag{5.4}$$

then the basic classes and the final classes are the same. Conversely, if $B_{ij} = 0$ *for* $i < j$, *and the set of basic classes coincides with that of the final classes, then there exists* $x > 0$ *such that* (5.4) *holds.*

Proof Suppose there exists $x > 0$ such that (5.4) holds. We assume without loss of generality that the permutation matrix P is the identity matrix. Partition x conformally so that

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \rho(A,B) \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(k)} \end{bmatrix}.$$

Thus, for $i = 1, 2, \ldots, k$,

$$A_{i1}x^{(1)} + \ldots + A_{ii}x^{(i)} = \rho(A,B)\left(B_{i1}x^{(1)} + \ldots + B_{ik}x^{(k)}\right). \tag{5.5}$$

Since $B_{ij} \le A_{ij}$, $j \ne i$ and since $A_{ij} = 0$, $j > i$, it follows from (5.5) that

$$A_{i1}x^{(1)} + \ldots + A_{ii}x^{(i)} \le \rho(A,B)\left(A_{i1}x^{(1)} + \ldots + A_{i,i-1}x^{(i-1)} + B_{ii}x^{(i)}\right). \tag{5.6}$$

Hence

$$(1 - \rho(A,B))\left(A_{i1}x^{(1)} + \dots + A_{i,i-1}x^{(i-1)}\right) + A_{ii}x^{(i)} \le \rho(A,B)B_{ii}x^{(i)} \tag{5.7}$$

and therefore,

$$A_{ii}x^{(i)} \le \rho(A,B)B_{ii}x^{(i)}. \tag{5.8}$$

Fix $i \in \{1,2,\dots,k\}$ and suppose class i is not final. Then either $A_{ij} \ne 0$ for some $j < i$ or $B_{ij} \ne 0$ for some $j \ne i$. Since $x > 0$ it follows that the inequalities in (5.6), (5.7) cannot both be replaced by equalities and therefore the inequality in (5.8) must be strict in at least one component. There exists $y > 0$ such that

$$y^T A_{ii} = \rho\left(A_{ii}, B_{ii}\right) y^T B_{ii}. \tag{5.9}$$

From (5.8), (5.9) we have

$$\rho\left(A_{ii}, B_{ii}\right) y^T B_{ii}x^{(i)} = y^T A_{ii}x^{(i)} < \rho(A,B) y^T B_{ii}x^{(i)}.$$

Since $y^T B_{ii}x^{(i)} > 0$ it follows that $\rho\left(A_{ii}, B_{ii}\right) < \rho(A,B)$ and thus class i is not basic. Therefore we have shown that every basic class is final. Conversely, if class i is final then it follows immediately from (5.5) that $\rho\left(A_{ii}, B_{ii}\right) = \rho(A,B)$ and therefore class i is basic. This completes the proof of the first part of the theorem.

To prove the second part, note that the additional assumptions on B mean that it also is block lower triangular. Thus $\det(\lambda B - A) = \prod_{i=1}^{k} \det\left(\lambda B_{ii} - A_{ii}\right)$, so there must be at least one basic class. We assume, without loss of generality, that the classes $1, 2, \ldots, m$ are the final, basic ones. Then there exist $x^{(\ell)} > 0$, $\ell = 1, 2, \ldots, m$ such that

$$A_{\ell\ell} x^{(\ell)} = \rho(A, B) B_{\ell\ell} x^{(\ell)}, \qquad \ell = 1, 2, \ldots, m.$$

For $\ell = m+1, \ldots, k$, class ℓ is not basic and therefore, by (5.3), $\rho(A, B) > \rho(A_{\ell\ell}, B_{\ell\ell})$. Then by Lemma 5.1

$$\left(\rho(A, B) B_{\ell\ell} - A_{\ell\ell}\right)^{-1} > 0, \qquad \ell = m+1, \ldots, k.$$

For $\ell = m+1, \ldots, k$, set

$$x^{(\ell)} = \left(\rho(A, B) B_{\ell\ell} - A_{\ell\ell}\right)^{-1} \sum_{h=1}^{\ell-1} \left(A_{\ell h} - \rho(A, B) B_{\ell h}\right) x^{(h)}.$$

Then $x^{(\ell)} > 0$, $\ell = 1, 2, \ldots, k$ and the vector

$$x = \left[x^{(1)T}\ x^{(2)T} \ldots x^{(k)T}\right]^T$$

satisfies $x > 0$, $Ax = \rho(A, B) Bx$. This completes the proof. □

When $B = I$, Theorem 5.2 shows that a nonnegative matrix has a positive eigenvector if and only if its basic classes are exactly its final ones. This result is contained in [3, p. 77] and in [1, p. 40].

The condition that $B_{ij} = 0$ for $i < j$, imposed in the second part of Theorem 5.2, cannot be dispensed with. This is illustrated by the following example.

EXAMPLE 5.3 Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & -2 & 0 \\ 0 & 3 & 0 & -1 \\ -2 & 0 & 4 & 0 \\ 0 & -2 & 0 & 4 \end{bmatrix}.$$

Then A, B satisfy (C1), (C3), (C4) and $\rho(A,B) = (4+\sqrt{6})/10$. There are no basic or final classes. However, if $x \geq 0$ and satisfies $Ax = \rho(A,B)Bx$ then it can be seen that $x_1 = x_3 = 0$.

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REFERENCES

[1] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia (1994).

[2] Takao Fujimoto, A generalization of the Frobenius theorem, *Toyama University Economic Review* **23**(2) (1979), 269-274.

[3] F.R. Gantmacher, *The Theory of Matrices, Vol. II*, Chelsea, New York (1959).

[4] G.H. Golub and C.F. Van Loan, *Matrix Computations*, Johns Hopkins Univ. Press (1989).

[5] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press (1985).

[6] C.R. Johnson, Inverse M-matrices, *Linear Algebra and its Applns.* **47** (1982), 195-216.

[7] R.B. Kellogg and A.B. Stephens, Complex eigenvalues of a non-negative matrix with a specified graph, *Linear Algebra and its Applns.* **20** (1978), 179-187.

[8] M. Morishima and T. Fujimoto, The Frobenius theorem, its Solow-Samuelson extension and the Kuhn-Tucker theorem, *J. Math. Econ.* **1** (1974), 199-205.

[9] A.M. Ostrowski, On the eigenvector belonging to the maximal root of a nonnegative matrix, *Proc. Edinburgh Math. Soc.* **12** (1960/61), 107-112.

[10] Luigi L. Pasinetti (ed.), *Essays on the Theory of Joint Production*, Macmillan Press, London (1980).

[11] P. Sraffa, *Production of Commodities by Means of Commodities*, Cambridge Univ. Press (1960).

[12] G.W. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, Boston (1990).