A NEW CONVOLUTION THEOREM FOR THE STIELTJES TRANSFORM AND ITS APPLICATION TO A CLASS OF SINGULAR INTEGRAL EQUATIONS

By

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Dedicated to the memory of Professor David Vernon Widder (1898–1990)

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A new convolution theorem is proved for the Stieltjes transform and is then applied in solving a certain class of singular integral equations which are related rather closely to the Riemann–Hilbert boundary value problem. Some further extensions and consequences of the convolution theorem are also considered.

1. INTRODUCTION

The convolution theorems for the Fourier, Laplace, and Mellin transforms are well–known (see, *e.g.*, [1] and [7]). Each of these results has a great potential for applications in solving convolution integral equations (*cf.* [6] and [7]), in the investigation of convolution transforms (*cf.* [4], [8], and [9]), and in the evaluation of definite integrals (*cf.* [5]). The object of the present paper is to prove a new convolution theorem for the Stieltjes transform:

(1) $$\mathscr{S}\{f(t):s\} = \int_0^\infty \frac{f(t)}{s+t}\,dt \quad (s \in D),$$

which arises naturally from the iteration of the classical Laplace transform, D being an arbitrary compact region of the complex s–plane cut along the nonpositive real axis. We also show how our convolution theorem can be applied in solving a certain class of singular integral equations which are usually investigated by reducing the problem to an equivalent Riemann–Hilbert boundary value problem.

2. THE CONVOLUTION THEOREM

Let the functions $f(t)$ and $g(t)$, defined on the interval $(0,\infty)$, satisfy the Lipschitz condition on that interval. Define a function $h(t)$ by

(2) $$h(t) = (f\otimes g)(t) := f(t)\int_0^\infty \frac{g(u)}{u-t}\,du + g(t)\int_0^\infty \frac{f(u)}{u-t}\,du.$$

Then it is known that each of the Cauchy principal integrals in (2) exists and satisfies the Lipschitz condition (*cf.*, *e.g.*, [2]). Hence the function $h(t)$ also satisfies the Lipschitz condition on the interval $(0,\infty)$.

Our main result is contained in the following

THEOREM. *Let $f(t)$ and $g(t)$, defined on the interval $(0,\infty)$, satisfy the Lipschitz condition throughout that interval.*

Then the Stieltjes transform of the convolution $(f\otimes g)(t)$, defined by (2), *exists and is given by*

$$\mathscr{S}\{(f\otimes g)(t):s\} = \mathscr{S}\{f(t):s\}\,\mathscr{S}\{g(t):s\}. \tag{3}$$

Proof. Since $h(t)$, defined by (2), does satisfy the Lipschitz condition on the interval $(0,\infty)$, under the hypothesis of the theorem, the first assertion of the theorem follows readily.

Next, by making use of the definitions (1) and (2), we have

$$\mathscr{S}\{(f\otimes g)(t):s\} = \int_0^\infty \frac{f(t)}{s+t}\left[\int_0^\infty \frac{g(u)}{u-t}\,du\right]dt + \int_0^\infty \frac{g(t)}{s+t}\left[\int_0^\infty \frac{f(u)}{u-t}\,du\right]dt. \tag{4}$$

We now invert the order of integration in the *second* integral on the right–hand side of (4), which is permissible under the hypothesis of the theorem, and we thus obtain

$$\mathscr{S}\{(f\otimes g)(t):s\} = \int_0^\infty \frac{f(t)}{s+t}\left[\int_0^\infty \frac{g(u)}{u-t}\,du\right]dt + \int_0^\infty f(u)\left[\int_0^\infty \frac{g(t)}{(s+t)(u-t)}\,dt\right]du \tag{5}$$

or, equivalently,

(6) $$\mathscr{S}\{(f\otimes g)(t):s\} = \int_0^\infty f(t)\left[\int_0^\infty \frac{g(u)}{(s+t)(u-t)}\,du\right]dt$$

$$+\int_0^\infty f(t)\left[\int_0^\infty \frac{g(u)}{(s+u)(t-u)}\,du\right]dt$$

$$= \int_0^\infty f(t)\left[\int_0^\infty \frac{g(u)}{u-t}\left\{\frac{1}{s+t} - \frac{1}{s+u}\right\}du\right]dt.$$

Simplifying this last double integral, we finally have

(7) $$\mathscr{S}\{(f\otimes g)(t):s\} = \int_0^\infty \frac{f(t)}{s+t}\,dt \int_0^\infty \frac{g(u)}{s+u}\,du$$

$$= \mathscr{S}\{f(t):s\}\,\mathscr{S}\{g(t):s\},$$

which evidently completes the proof of the theorem.

Remarks. The convolution theorem (3), which was proven above for functions satisfying the Lipschitz condition, can easily be extended to hold true for a much wider class of functions. Suppose, for example, that

(8) $$f \in L_p(\mathbb{R}_+) \text{ and } g \in L_q(\mathbb{R}_+)$$

$$(1 < p,q < \infty;\ r^{-1} = p^{-1} + q^{-1} < 1).$$

Then we shall prove that the function h, defined by (2), is in the space $L_r(\mathbb{R}_+)$ $(1 < r < \infty)$ and (more importantly) that the convolution theorem (3) holds true. Indeed it follows readily from (8) that

(9) $$1 < r < \infty \text{ and } f\cdot g \in L_r(\mathbb{R}_+).$$

Furthermore, since the singular integral operator

$$(I^*f)(x) = \int_0^\infty \frac{f(u)}{u-x}\, du \tag{10}$$

boundedly maps the space $L_p(\mathbb{R}_+)$ $(1 < p < \infty)$ into itself (see [2]), the convolution operator in (2) would similarly map

(i) the space $L_p(\mathbb{R}_+)$ $(1 < p < \infty)$ into $L_r(\mathbb{R}_+)$ $(1 < r < \infty)$ for a given $g \in L_q(\mathbb{R}_+)$ $(1 < q < \infty)$; and
(ii) the space $L_q(\mathbb{R}_+)$ $(1 < q < \infty)$ into $L_r(\mathbb{R}_+)$ $(1 < r < \infty)$ for a given $f \in L_p(\mathbb{R}_+)$ $(1 < p < \infty)$.

Observe also that the Stieltjes transform (1) is a bounded operator which maps the space $L_p(\mathbb{R}_+)$ $(1 < p < \infty)$ into itself, and that the set of Lipschitz functions is dense in the spaces of the type $L_p(\mathbb{R}_+)$ $(1 < p < \infty)$. Hence the convolution theorem (3) holds true, under the hypothesis (8), as stated above.

An interesting consequence of the convolution theorem (3), which can indeed be proven fairly easily by appealing also to the analyticity of the Stieltjes transforms involved, is a Titchmarsh type theorem: *If the functions* f *and* g *satisfy the conditions in* (8), *and if* $f \otimes g = 0$, *then either* $f = 0$ *or* $g = 0$.

3. AN APPLICATION OF THE CONVOLUTION THEOREM

We consider the following interesting class of singular integral equations:

$$f(t) + \lambda \int_0^\infty \frac{f(u)}{u-t}\, du = g(t) \qquad (\lambda \neq 0), \tag{11}$$

where $g(t)$ is prescribed and $f(t)$ is an unknown function to be determined. The solution of the integral equation (11) was investigated earlier by reducing the problem to an equivalent Riemann–Hilbert boundary value problem (see [2] for details). In this section we shall show how the convolution theorem (3) can be applied to solve the integral equation (11).

We begin by assuming a_0 to be a (unique) root of the transcendental equation:

(12) $$\tan \pi\alpha = -\pi\lambda \qquad (0 < \mathrm{Re}(\alpha) < 1;\ \lambda \neq 0).$$

Then, in view of the well–known integral (*cf.*, *e.g.*, [3, p. 289, Entry 3.222.2]; see also [1, Vol. **II**, p. 249, Entry 15.2(28); p. 216, Entry 14.2(5)]):

(13) $$\int_0^\infty \frac{u^{\alpha-1}}{u-\zeta}\,du = \begin{cases} -\pi\zeta^{\alpha-1} \cot \pi\alpha & (\mathrm{Re}(\zeta) > 0;\ 0 < \mathrm{Re}(\alpha) < 1), \\ \pi(-\zeta)^{\alpha-1} \csc \pi\alpha & (\mathrm{Re}(\zeta) < 0;\ 0 < \mathrm{Re}(\alpha) < 1), \end{cases}$$

the integral equation (11) can be written in the form:

(14) $$f(t) \int_0^\infty \frac{u^{\alpha_0-1}}{u-t}\,du + t^{\alpha_0-1} \int_0^\infty \frac{f(u)}{u-t}\,du = -\pi t^{\alpha_0-1}\, g(t) \cot \pi\alpha_0$$

or, equivalently,

(15) $$f(t) \otimes t^{\alpha_0-1} = -\pi\, t^{\alpha_0-1}\, g(t) \cot \pi\alpha_0,$$

where we have made use of the definition (2).

Applying the convolution theorem (3), this last relationship (15) yields

(16) $$\mathscr{S}\{f(t):s\} \int_0^\infty \frac{u^{\alpha_0-1}}{u+s}\,du = -\pi\, \mathscr{S}\{t^{\alpha_0-1}\, g(t):s\} \cot \pi\alpha_0,$$

which, in view of the integral (13) again, becomes

(17) $$\mathscr{S}\{f(t):s\} = -s^{1-\alpha_0}\, \mathscr{S}\{t^{\alpha_0-1}\, g(t):s\} \cos \pi\alpha_0.$$

Finally, by appealing to the classical inversion theorem for the Stieltjes transform (see, *e.g.*, [9, p. 126, Theorem 14.1]), we obtain

(18) $$f(t) = \lim_{\epsilon \to 0+} \frac{\cos \pi\alpha_0}{2\pi i} \Big[(-t+i\epsilon)^{1-\alpha_0}\, \mathscr{S}\{u^{\alpha_0-1}\, g(u):\ -t+i\epsilon\}$$

$$- (-t-i\epsilon)^{1-\alpha_0} \mathscr{S}\{u^{\alpha_0-1}\, g(u) : -t-i\epsilon\}\Big]$$

which provides the solution of the singular integral equation (11), α_0 being a (unique) root of the transcendental equation (12).

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