An Exploration of AdS_3 Gravity as a Holographic Theory

by

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B.A., Concordia College, 2017

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Physics & Astronomy

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ABSTRACT

The holographic principle is a supposed property of quantum gravity which states that the description of a volume of space can be encoded on a lower-dimensional boundary to the region. One such example is that of the Bekenstein-Hawking entropy of a black hole, which states the remarkable fact that the entropy of a black hole is simply its surface area. The most successful framework for holography is the AdS/CFT correspondence, which provides a non-perturbative formulation of string theory (and other theories of quantum gravity such as JT gravity). The purpose of this thesis is two-fold. The first is to give an overview of the AdS/CFT correspondence, including two separate historically significant applications of the correspondence. The latter application involves that of the AdS_3/CFT_2 correspondence, which ties into the second purpose of this thesis. $(2+1)$-dimensional gravity with negative cosmological constant (also known as"pure AdS_3 gravity") is exactly soluble system as a classical system, and although it is still yet undetermined if AdS_3 quantum gravity is UV-complete there are still various computations one can make. In particular, this thesis considers coupling pure AdS_3 gravity to a large number of light matter fields dual to relevant operators. This model has eternal traversable wormholes supported by suitable boundary conditions on the matter fields. Within this construction, the field theory of boundary gravitons is used to compute the spectrum of gravitational fluctuations and in principle the gravitational contribution to scattering processes in the wormhole.

Contents

List of Tables

List of Figures

ACKNOWLEDGEMENTS

This master's degree was an eventful one: I switched from astronomy to theoretical physics and I moved countries to switch institutions all the while trying to survive a pandemic. Throughout it all, I've encountered a great many number of people which helped to make this journey a rewarding endeavour.

I'd like to thank all of my old friends I met at San Francisco State University as well as my new friends at the University of Victoria for providing a sense of comradery within Academia. I'd also like to thank **Drs. Akash Jain**, **Adam Ritz**, **Maarten Golterman**, and **Joseph Barranco** for teaching incredible theory courses and for being great mentors throughout my graduate school career.
I would also like to give special thanks to:

Dr. Kristan Jensen for giving me the opportunity to thrive at UVic (in the master's program, and beyond) by being an incredibly supportive advisor. If it weren't for you, I literally wouldn't be here.

Jonathan, David, Huy, Donald, and Arum for helping me stay sane from the extreme isolation during the initial wave of the pandemic in 2020 as we all forged ahead with our graduate studies in physics. Our nightly discord sessions playing Left 4 Dead 2, *et al.* are always a fun time, and I will never forget our foray into Visage (the scariest game ever made).

James, Bryn, Casey, Erick, Mikey, and Megan for keeping in touch all throughout the pandemic via our Discord server; it's the best online community I've ever been apart of.

Lori Shaw For helping me literally cross the boarder into Canada when United Airlines refused to believe I had authorization.

And a very special thanks to:

Susan Tang For the endless love and support over these past few years. This has been quite the adventure, hasn't it?

My Parents For always being my biggest supporters, even during my most cynical days.

Chapter 1

Introduction

This thesis studies models of quantum AdS_3 gravity, including both for itself and in its relation to the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. The AdS/CFT correspondence has generated significant progress in the field of quantum gravity because it provides a non-perturbative formulation of string theory through a dual description in terms of conformal field theory. The duality[1] states

$$\mathrm{AdS}_{d+1} \;= \mathrm{CFT}_d, \tag{1.1}$$

where d is the number of spacetime dimensions, and the left-hand-side represents a consistent theory of quantum gravity on AdS_{d+1}. More specifically, string theories whose spacetime metric asymptotes towards AdS near their boundaries have been argued to be equivalent to a certain conformal field theory (or vice-versa) [1, 2]. In fact, this relation contains all the central concepts of known fundamental physics: Maxwell's equations and their non-Abelian extension, the Dirac and Klein-Gordon equations, quantum mechanics, quantum field theory and general relativity. Furthermore, it contains insight into supersymmetry (supergravity and supersymmetric gauge theory), string theory, and extra dimensions.

One such theory of quantum gravity having applications in $\mathrm{NAdS}_2/\mathrm{NCFT}_1$[2] which has been studied over the years is known as Jackiw-Teiltelboim gravity [3]. As this theory is relatively simple[3], computations can be made in it without invoking pertur-

[1] The term "duality" will be defined and discussed in section 2.1.1.

[2] The "N" stands for "near" AdS_2 gravity and "nearly" CFT_1; this idea will be briefly discussed in chapter 4.

[3] It has been shown that JT and AdS_3 gravity theories have no propagating mode of bulk gravitation (only edge modes).

bation theory or the AdS/CFT correspondence[4]. One particularly interesting construction in JT gravity was that of an eternal traversable wormhole as done in [4]. A massive particle can travel from one end of the spacetime to another through the wormhole along a time-like path. Unlike most wormhole configurations, this one does not violate causality.

A natural area to study following the formation of JT gravity is in increasing the number of spatial dimensions by one. Does a consistent theory of quantum AdS_3 gravity exist[5]? And even if it doesn't, what kind of insight can be gained from an effective field theory formulation? As a starting point, classical AdS_3 gravity has been shown to be exactly soluable as a Chern-Simons theory by Edward Witten in [5]. Furthermore, although it is known that pure (quantum) AdS_3 gravity is not dual to a single two-dimensional CFT [6], it may have a holographic dual to a statistical ensemble[6]. As one final question of particular interest to this thesis, can an eternal traversable wormhole be constructed in AdS_3 gravity similar to what was done in [4] using the formalism developed in [7][7]? This possibility will be explored in [8], and will also be presented in this thesis.

Each chapter of this thesis can largely be read as a self-contained essay on each topic, although the overarching ideas behind the AdS/CFT correspondence and quantum gravity will build upon each other as presented in chapter-order[8]. The chapters are as follows:

- **Chapter 2** will give an overview of the AdS/CFT correspondence, and introduce the AdS/CFT "dictionary" which maps the matter fields on the gravity side with the operators on the CFT side. The remainder of the chapter will then provide a qualitative overview of the historically significant application of the correspondence: Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and its dual description four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory.

- **Chapter 3** "zooms-in" on the AdS/CFT correspondence by reducing the number of spacetime dimensions by two. Specifically, the focus of this chapter is to provide a derivation of the Cardy formula from the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence,

[4]As will be mentioned later, its holographic dual isn't actually a single CFT.

[5]AdS_3 is specifically considered here over Minkowski or de Sitter spacetimes due to the corresponding applications in the AdS/CFT.

[6]Analogous to the dual theory of JT gravity mentioned in the previous footnote.

[7]The work Juan Maldacena and Xiao-Liang Qi did in [4] also utilized concepts developed in[7].

[8]And occasionally, equations and discussions from previous chapters will be cited as "jumping-off points" for various intermediate steps.

and with it verify that at high temperature the leading-order entropy of a BTZ blackhole (equivalent to global AdS_3 after a coordinate transformation) exactly agrees with the Bekenstein-Hawking entropy of a black hole. Along the way, it is discussed how Virasoro symmetry is present in any two-dimensional CFT, and by the AdS/CFT correspondence is therefore also present in the theory of gravity. As such, the physical consequences of the resulting Virasoro symmetry will also be discussed.

- **Chapter 4** walks through the construction of a quantum theory of AdS_3 on a cylinder by writing the classical theory in terms of Chern-Simons variables, and then quantizing in a method analogous to a WZW model [9, 5]. The final AdS_3 action is called the "Alekseev-Shatashvili" action, as it exactly agrees with the quantization of the coadjoint orbit of the Virasoro group $Diff(S^1)/PSL(2;\mathbb{R})$ found in [10].

- **Chapter 5** considers a quantum theory of AdS_3 on spacetimes with topology of an annulus times time [11]. Unlike ordinary global AdS_3 on a cylinder (one boundary) this spacetime has two boundaries, and in Euclidean signature the leading order[9] bulk geometry connecting these two boundaries is a Euclidean wormhole. Following [8], new work is done with the wormhole by analytically continuing to Lorentzian signature while simultaneously making this wormhole "eternally traversable" by introducing a double-trace deformation[10] into the action. Then, the spectrum of fluctuations of this eternal traversable wormhole are computed to study the free-propagator of boundary gravitons.

- **Chapter 6** concludes and summarizes the thesis with musings on potential avenues for future projects.

Conventions present in this thesis are as follows. Natural units will often be assumed $c = k_B = G = \bar{h} = 1$, although Newton's constant G will occasionally be made explicit. The usual spacetime indices will be Greek letters, and the Minkowski metric $\eta_{\mu\nu}$ will abide by the metric signature $(-, +..., +)$. In chapter 4.1, the dreibein will contain both spacetime indices and flat indices; these flat indices will be carried by the Minknowski metric η_{AB}, and will be denoted by capital Latin letters.

[9] The leading order term is with respect to a genus expansion; this will be discussed in chapter 5.

[10] First done for any number of dimensions in [7], but the approach in this thesis was primarily inspired by [4].

Chapter 2

Some Review of the AdS/CFT Correspondence

2.1 Generalities

This section will begin with the background context for the two sides of the correspondence, and will primarily follow the approach laid out by [2]. Afterwards, the rest of this chapter will be spent exploring one historically significant example of the correspondence: Type IIB String Theory on $\mathrm{AdS}_5 \times S^5$ and its duality which is a four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory. The approach taken here will combine the pedagogy of [2] with additional details from [12].

2.1.1 What is the AdS/CFT Correspondence?

The AdS/CFT correspondence, also known as "gauge/gravity" duality, is a prime example of "holography[1]," and describes the equivalence between theories of quantum gravity with an *asymptotically* AdS_{d+1} spacetime and an ordinary CFT_d. The main types of observables one can consider within these two theories are correlation functions and thermodynamic quantities. As a way to express this equivalency mathematically, the thermodynamic quantities of both theories must be equal, which corresponds to equating the partition functions

$$Z_{Gravity}(\beta) = Z_{CFT_d}(\beta). \tag{2.1}$$

[1] The gravitational theory is "holographic" since it lives in one higher dimension.

Here, the partition function for the CFT is defined in the canonical form

$$Z_{CFT_d}(\beta) = tr\left(e^{-\beta H_{CFT_d}}\right), \quad \beta = \frac{1}{T}, \tag{2.2}$$

and the Euclidean gravity partition function is written as a path integral

$$Z_{Gravity}(\beta) = \int D[g]D[\phi]e^{-S_E}, \tag{2.3}$$

where the measures indicate integrating over both spacetime metrics g and matter fields ϕ, and S_E is the full quantum gravity action[2] in Euclidean signature in which time is compact with periodicity β on the boundary

$$y \sim y + \beta, \quad y \equiv it. \tag{2.4}$$

In order for the equivalence between the partition functions to be exact, all quantum corrections must be included in the gravity path integral. This equivalence between the theories is called a duality. As a result of duality, a physical quantity which is computed in one theory can also be computed in the corresponding dual theory. In this way, the power of duality allows for computations for one theory to be made in terms of the other. This is useful because calculations made on one side of the correspondence often map to incalculable quantities on the other side.

Although it can be useful to think of the CFT as "living on the conformal boundary[3]," it is not quite accurate to say that the CFT actually lives on the boundary. The reasons of which are two-fold. First, the two theories should not be imagined to be present at the same time; either one has a CFT or an AdS spacetime, and never both at once. Second, the CFT is dual to the entire gravity theory, so in a sense the CFT really does also "live everywhere." Nevertheless, the theory of gravity is often referred to as "the bulk theory," and the CFT theory is often referred to as "the boundary theory."

[2]The full quantum theory of gravity may be more complicated and contain more compact bulk dimensions than those of AdS_{d+1}.

[3]The term "conformal boundary" refers to the AdS boundary [2]

2.1.2 What is Anti-de Sitter Spacetime?

Anti-de Sitter space is the maximally symmetric[4] Lorentzian manifold[5] which is a solution of the Einstein equations with negative cosmological constant; it can be imagined as a $d+1$-dimensional hyperboloid embedded in a $d+2$ spacetime. This is visualized with AdS_2 in figure 2.1.2. The line element of AdS_{d+1} in global coordinates

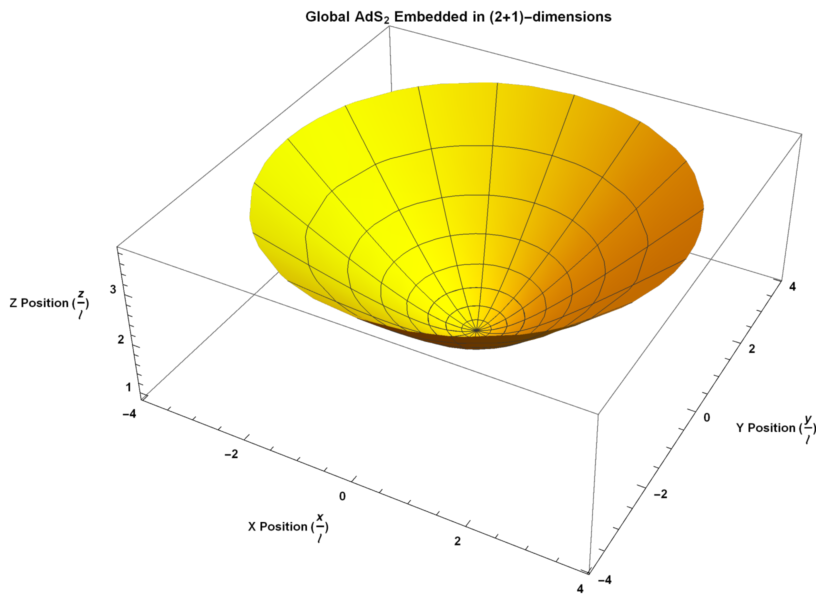

Figure 2.1: This is AdS_2 spacetime represented as a hyperboloid embedded in $(2+1)$-dimensions. The surface of this "bowl" is a visual representation of the AdS_2 spacetime. The angular parameter winds around the bowl, while time extends into (and out of) the bowl; l is the AdS_2 radius.

is given by

$$ds^2_{d+1} = l^2 \left(-cosh^2(\rho)dt^2 + d\rho^2 + sinh^2(\rho)d\Omega^2_{d-1}\right), \tag{2.5}$$

where l is the AdS_{d+1} radius. To understand how the conformal boundary behaves, one can construct the Penrose diagram by extracting the $cosh^2(\rho)$ factor and define a new coordinate [2]

$$d\sigma = \frac{d\rho}{cosh(\rho)}, \quad \sigma = 2\,tan^{-1}\left(tanh(\rho/2)\right). \tag{2.6}$$

[4]The term "maximally symmetric" indicates that the space is both homogeneous and isotropic everywhere.

[5]A Lorentzian manifold is a pseudo-Riemanian Manifold whose signature is $[-,+,+,+]$. A Riemanain manifold is a type smooth manifold with an inner product on the tangent space, and a smooth manifold allows for directional differentiation along tangent vectors. A manifold is a topological space which behaves as Euclidean locally.

As ρ goes from 0 to ∞, σ goes from the finite distance 0 to $\pi/2$. In this way, global AdS_{d+1} is topologically a cylinder with radial coordinate ρ (reparameterized as σ), and the time coordinate forming the length of the cylinder. This is visualized with global AdS_3 in figure 2.1.2.

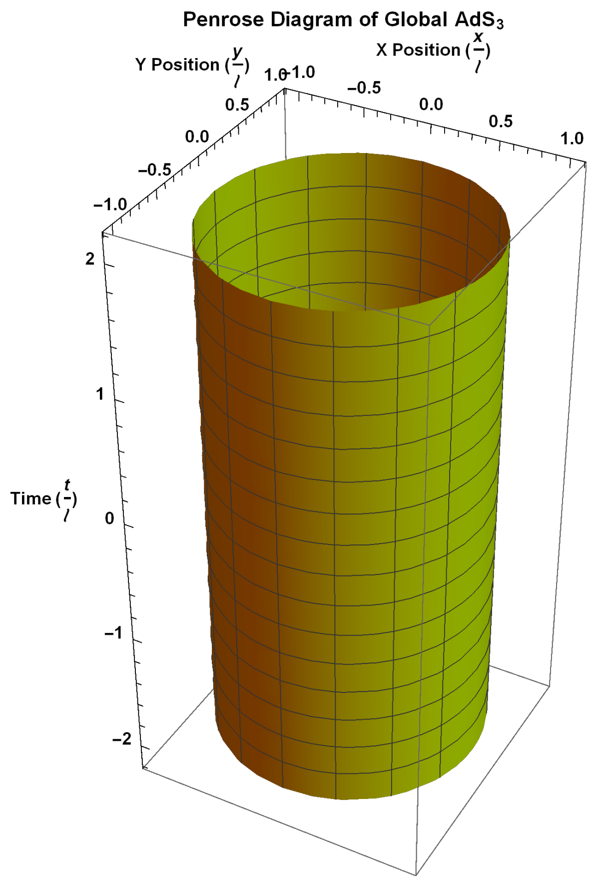

Figure 2.2: This is global AdS_3 visualized using Penrose coordinates. The angular coordinate forms a circle, the radial coordinate $\rho \to \sigma$ is the radius of the circle, and time t extends along the edge of the circle to infinity. Together, this creates a cylinder.

Unlike in flatspace, a massless particle can travel from one end of the AdS conformal boundary to the other in a finite amount of time t. Near the boundary $\rho \to \infty$, the line element can be written in Poincaré coordinates

$$ds^2_{d+1} = \frac{l^2}{z^2}\left(-dt^2 + dz^2 + d\mathbf{x}^2\right), \tag{2.7}$$

where $\mathbf{x} = (x^0, x^1, ..., x^{d-1})$ and $z = 0$ marks the exact location of the boundary. This metric is also a solution to Einstein's field equations, and covers a sub-region of global AdS$_{d+1}$. This spacetime is known as the Poincaré patch, and is an example of an *asymptotically* AdS spacetime. A spacetime is said to be "asymptotically AdS" if its metric approaches the Poincaré patch metric for $\rho \to \infty$. Another example of an asymptotically AdS spacetime is given by the metric for an AdS black hole; a black hole whose metric asymptotes to AdS at spatial infinity[6]. Generally, if a spacetime manifold whose boundary conditions behave the same as AdS gravity, then the metric asymptotes to a AdS gravity near the boundary.

To return to the gravity side of (2.1), let's start breaking down the Euclidean path integral. In QFT, a path integral is computed by first fixing the spacetime manifold $\mathcal{M}$, and then integrating over all fields defined on $\mathcal{M}$. In quantum gravity, the geometry itself must also be integrated over. As such, performing gravity path integral calculations is a highly non-trivial process, and in general one instead expands this thermal partition function as a series of saddle points

$$Z_{Gravity}(\beta) \approx exp\left(-S_E^{(0)}[g_{AdS}, \phi] + S^{(1)} + ...\right), \tag{2.8}$$

where the semi-classical limit in which g and ϕ are solutions to the classical equations of motion contains only the leading order $S_E^{(0)}$ term which depends on the asymptotic AdS$_{d+1}$ metric g_{AdS}, and not the full bulk theory. The $S^{(1)}$ corresponds to 1-loop corrections, and higher orders correspond with higher loop contributions. Note that if one is interested in the gravity partition function (the Euclidean path integral) such as this, the metric g_{AdS} must itself also be in Euclidean signature.

In the semi-classical limit, various canoncial ensemble thermodynamic quantities can be easily computed. One such quantity is the total energy

$$E = -\partial_\beta log\, Z_{Gravity}(\beta), \tag{2.9}$$

although since temperature will often be fixed it is more useful to work with Helmoltz free energy

$$\beta F = -log\, Z_{Gravity}(\beta) \approx S_E^{(0)}[g_{AdS}, \phi]. \tag{2.10}$$

From the definition of Helmoltz free energy $F \equiv E - T\mathcal{S}$, the entropy can also be computed

$$\mathcal{S} = \beta(E - F),$$

[6]An example of an AdS black hole is a BTZ black hole, which will be discussed in chapter 3.

or in substituting in (2.10) and (2.9)

$$\mathcal{S} = (1 - \beta\partial_\beta)\, log\, Z_{Gravity}(\beta). \tag{2.11}$$

And from (2.10), the approximate entropy is therefore

$$\mathcal{S} \approx (\beta\partial_\beta - 1) S_E[g_{AdS}, \phi]. \tag{2.12}$$

This entropy would indeed agree with the area law found by Bekenstein and Hawking, $\mathcal{S} = \frac{A}{4G}$ [13], where A is the area of the Lorentzian black hole obtained from the analytic continuation to real time of g_{AdS}. This will be discussed in section 3 when applying the AdS/CFT correspondence to the AdS_3/CFT_2 case.

Regardless if one is interested in the semi-classical limit or the full quantum theory, the gravity action in the path integral needs to be specified. As will be discussed later, a fundamental component to the AdS/CFT correspondence is how the bulk fields behave near the spacetime boundary, and that the metric for any given theory of quantum gravity must be asymptotically AdS. AdS gravity itself has a boundary as well, and to see how the presence of a boundary boundary affects the action, consider the Einstein-Hilbert action with negative cosmological constant

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}(R + \frac{2}{l}) + S_{bdy}, \tag{2.13}$$

or in Euclidean signature

$$S_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{g}(R + \frac{2}{l}) + S_{E,bdy}, \tag{2.14}$$

where l is the AdS radius and the boundary term S_{bdy} is necessary for AdS gravity due to the presence of a boundary. Without this additional term the variational principle would not be well defined. Its purpose can be inferred by the need to specify boundary conditions and to regulate the gravity action. That is, although this action can provide a classical partition function, there are infinities to regulate in an analogous fashion to adding counter-terms in ordinary QFT. In this case, the boundary term $S_{E,bdy}$ acts as a "counter-term" with some cut-off distance $r = r_0$, and is given by

$$S_{E,GH} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{h}\, tr(K), \tag{2.15}$$

which is known as the Gibbons-Hawking-York boundary term[7] [14]. Here, K_{ij} is the extrinsic curvature of the boundary spacetime $\partial\mathcal{M}$ and h_{ij} is the corresponding induced metric[8]

$$tr(K) = h^{ij}K_{ij}, \quad K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij}, \tag{2.16}$$

where $\mathcal{L}_n$ is the Lie derivative with respect to the normal[9]. As stated, this Gibbons-Hawing-York boundary term is necessary such that the variational principle is well defined; the associated classical equations of motion must produce a stationary action. In attempting to vary only the bulk part of the action with respect to the metric, this (schematically) looks like

$$\delta S_{E,EH} = \int_{\mathcal{M}} (EOMs)\delta g + \int_{\partial\mathcal{M}} \left(A(g, \partial(g))\delta g + B(g, \partial g)\partial\delta g\right), \tag{2.17}$$

where the first term vanishes when the equations of motion (EOMs) are satisfied, and the second term vanishes by requiring that the metric not vary out at the boundary $\delta g\big|_{\partial\mathcal{M}} = 0$. Yet, $\partial\delta g\big|_{\partial\mathcal{M}} \neq 0$, and hence the necessity for the S_{GH} boundary term becomes apparent. Stated without derivation, the variation of the full action will behave as [2]

$$\delta S = \int_{\mathcal{M}} (EOMs)\delta g + \frac{1}{2}\int_{\partial\mathcal{M}} \sqrt{h}\, T^{\mu\nu}\delta g_{\mu\nu} = 0, \tag{2.18}$$

where $T^{\mu\nu}$ is the boundary stress-energy tensor. This idea of the boundary stress tensor will be explored further when AdS_3/CFT_2 is discussed in more detail later in section 3.1. In the meantime, there is still the right-hand-side (the CFT side) of (2.1).

2.1.3 What is a Conformal Field Theory?

A Word on Conformal Transformations

A transformation which is one-to-one and onto while also preserving distances is called an isometry, and the continuous isometries of a given metric are determined by computing the Killing vectors. Conformal transformations are "almost" continuous isometries, since they are one-to-one and onto mappings which preserve angles but

[7]There are other counter-terms which appear in theories of gravity, but the GHY counter-term is the only one which is relevant for this thesis.

[8]The induced metric is undefined at the boundary, and so the integral must be regulated to some cut-off region. It's at this constant (radial) slice where the induced metric describes the boundary.

[9]In practice, one can write the AdS metric in Gaussian normal coordinates, and then just have the derivative be the ordinary derivative with respect to the normal coordinate to the boundary [15].

not distance. For example, starting with Minkowski spacetime

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.19}$$

and transforming x into a function of y, $x \to x(y)$, the conformal transformation of Minkowski spacetime is defined as

$$\eta^{\mu\nu} dx^{mu} dx^\nu \to e^{2\Omega(y)} \eta_{\mu\nu} dy^\mu dy^\nu, \tag{2.20}$$

where now the metric gets mapped to a function multiplied by the original metric. A classic example of conformal mapping is when a three-dimensional projection of the globe is mapped to a two-dimensional surface in which the angle is preserved, but distance is not. The mapping between the globe and the map is therefore not an isometry. The key takeaway here is that in not preserving distances, a conformal transformation is one that does not leave the metric invariant.

Conformal Field Theories

A conformal field theory (CFT) is a UV complete quantum field theory[10] (QFT) which is invariant under conformal transformations. Conformal invariance is a symmetry under *global* transformations, which include scaling transformations (in addition to the usual rotational and translational symmetries) generated by the dilatation operator D[11]. What this implies for observables is that correlation functions behave nicely under a coordinate rescaling $x \to \lambda x$. Explicitly, correlation functions between operators obey [2]

$$\langle O_1(x_1) O_2(x_2) ... O_n(x_n) \rangle = \lambda^{\Delta_1 + \Delta_2 + ... \Delta_n} \langle O_1(\lambda x_1) ... O_n(\lambda x_n) \rangle, \tag{2.21}$$

where Δ_i is a scaling dimension for the given operator O_i. Following scaling and translation invariance, the two-point functions can be expressed as

$$\langle O_1(x_1) O_2(x_2) \rangle \propto \frac{1}{|x_1 - x_2|^{2\Delta}}. \tag{2.22}$$

[10] A non-UV complete QFT is known as an "effective field theory," and is only usable up to some specific energy scale.

[11] In the special case of $d = 2$ dimensions, the conformal transformations are labeled by functions and holomorphic reparameterizations. These are said to be *local* transformations.

As a quick example, consider a massless (real) scalar field theory in $d = 3+1$ dimensions. The two-point correlation function is [16]

$$\langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{1}{2k^2}e^{-ik\cdot(x-y)} = \frac{1}{|x-y|^2},$$

where in this example $\Delta = 1$. Note however that if this was a massive scalar field, the mass m would change the correlation function and the resulting right-hand-side would no longer behave in this simple way. It turns out that this is a general quality of a CFT; CFTs only have dimensionless parameters, and so there can be no mass terms in the CFT Lagrangian [2]. By contrast, most bulk fields on the gravity side of the AdS/CFT correspondence do have mass. With that being said, there may still be QFTs with no mass terms that are still not necessarily conformal field theories[12]; A CFT may also include interactions. The main type of CFT which will be considered are conformal gauge theories, which are conformal field theories with Lagrangians in the form of a gauge theory. There are examples in which the form of the gauge theory is Yang-Mills, and others which are a Wess-Zumino-Witten model; the former will be discussed in section 2.2 and the latter in 4.

2.1.4 The AdS/CFT Correspondence "Dictionary"

It can be shown that scattering problems in gravity map to correlation functions in CFT, which implies that the boundary value of a bulk field for gravity $\phi(\rho \to \infty, x)$ acts as a source for a CFT operator $O(x)$ [2]. This allows for the two theories to be connected, where each field $\phi_i(\rho, x)$ in the gravitational theory needs to have a corresponding operator $O_i(x)$ in the CFT; the mass m of ϕ determines the dimension of O. This relationship is generalized in the AdS/CFT correspondence by

$$Z_{Gravity}\left[\phi_0^i(x); \partial\mathcal{M}\right] = \left\langle exp\left(-\sum_i \int d^dx\phi_0^i(x)O_i(x)\right)\right\rangle_{CFT\,on\,\partial\mathcal{M}} \tag{2.23}$$

where ϕ_0^i is the value of the bulk field *near*[13] the boundary, and the index i runs over all bulk fields and operators of the CFT. Specifically, the boundary value ϕ_0^i acts as a source for operator O_i, while the mass of the generic field ϕ_i fixes the dimension of O_i. This equation is called the GKPW dictionary as it acts as the "dictionary" which

[12] A standard example would be QCD with vanishing quark masses.

[13] The bulk field is evaluated at the boundary in the case of $\Delta = 0$. This can be seen in (2.25).

maps bulk fields to CFT operators and vice-versa[14]. As an aside, note that CFTs are, by definition, UV complete. Thus, (2.23) is a non-perturbative formulation of a UV complete theory of quantum gravity.

To break this down, first consider the left-hand-side of (2.23). This is the previously discussed Euclidean gravity path integral, where the metric (and action) corresponds to the full theory of gravity being considered. The AdS/CFT correspondence works by considering how the bulk fields fluctuate near the boundary, in which the near-boundary geometry is the asymptotic AdS space (although the fluctuations are determined by the bulk theory). In this case, let the Euclidean line element for the asymptotic AdS_{d+1} be given in Poincaré coordinates,

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + dx^2\right), \tag{2.24}$$

where $z = 0$ is exactly on the boundary, and x includes all spatial coordinates as well as Euclidean time $x^0 = y \equiv -it$. Before attempting to evaluate the path integral, recall that AdS_{d+1} spacetime has a boundary. Therefore, the bulk fields must obey boundary conditions in order for the path integral to be well defined, although the specifics depend on their spin. As an example, if the bulk fields were scalars, the boundary conditions would have them obey

$$\phi_i(z,x) = z^{d-\Delta}\phi_0^i(x) + ..., \tag{2.25}$$

where "..." are subleading terms as $z \to 0$. The mass of said bulk field is related to the conformal (CFT scaling) dimension by [2, 12]

$$m^2 = \Delta(\Delta - d), \quad \Delta = \frac{d}{2}\left(1 + \sqrt{1 + \frac{4m^2l^2}{d^2}}\right). \tag{2.26}$$

There are also boundary conditions on the metric, which are dictated both by the metric itself as well as its topology. This feature is evident by $Z_{Gravity}$ having the explicit dependence on the boundary manifold $\partial\mathcal{M}$. As mentioned in section 2.1.2, a spacetime which is asymptotically AdS specifically is one which has the same boundary conditions, but there are infinitely many metrics which contain such boundary conditions. To be clear, the fields ϕ_0^i satisfy boundary conditions dictated by the asymptotic AdS gravity spacetime, but said asymptotic AdS boundary conditions

[14] Named for Gubser, Klebanov, Polyakov, and Witten.

themselves are dictated by the specific metric and topology of the bulk theory.

The right-hand-side of (2.23) is the generating functional of correlators in a CFT, which will be denoted as $Z_{CFT}[\phi_0]$.[15] As stated, in this equation the bulk fields $\phi_0^i(x)$ act as sources to the CFT operators $O_i(x)$. The connected correlation functions are computed by taking the functional derivative of $Z_{CFT}[\phi_0]$ evaluated at the boundary (where the operators "live")

$$\langle O_1(x_1)...O_n(x_n)\rangle_{CFT} = (-1)^n \frac{\delta^n}{\delta\phi_0^1(x_1)...\delta\phi_0^n(x_n)} Log\ (Z_{CFT})\,[\phi_0]\bigg|_{\phi_0^i=0}, \tag{2.27}$$

where the relationship between gravity and the CFT is evident by the fact that the correlation functions depend on $Z_{CFT}[\phi_0^i]$[16].

To summarize, a bulk field in the gravitational theory $\phi_i(z,x)$ is dual to an operator $O_i(x)$ in the CFT, and therefore each boundary value of $\phi_i = \phi_0^i$ corresponds to a source for a local operator in the CFT $O_i(x)$; the spin of ϕ_0^i is equal to the spin of $O_i(x)$. As per the example in (2.26), the mass of scalar field $\phi_i(x)$ fixes the scaling dimension of $O_i(x)$. These statements remain true even when the bulk field carries spin, but there are two examples of massless fields worth discussing separately. In particular, for spin = 1 and spin = 2 fields [2]:

- *Spin = 1 (Vector):* If the given theory of gravity has a spin-1 vector field A_μ, then the dual CFT has a spin-1 operator J_μ. If A_μ is massless, then the conformal dimension $\Delta_J = d - 1$ and J_μ is a conserved current. Otherwise, $\Delta_J > d - 1$ and the current is not conserved.

- *Spin = 2 (Graviton):* Every theory of gravity has a massless spin-2 particle called the graviton. This is dual to the stress tensor $T_{\mu\nu}$ of the CFT. The fact that the graviton is massless corresponds to the fact that $T_{\mu\nu}$ is a conserved current, and also fixes the scaling dimension to $\Delta_T = d$.

In both cases, it is apparent that the spin of field ϕ is directly related to conserved Noether currents in the CFT, and in the vector case only the *massless* particles contain the symmetry. These cases illustrate an important feature of the AdS/CFT correspondence: *gauge symmetries* of the (full) theory of gravity correspond to *global*

[15] Specifically, this is the generating functional corresponding with connected correlation functions. The total correlation function includes both connected and disconnected correlations.

[16] For an example computation utilizing the "dictionary" (2.23) to find correlation functions corresponding to a massive scalar field theory, see [17].

symmetries of the CFT. This feature will be an important one when attempting to write a theory of AdS_3 gravity as a Chern-Simons gauge theory in chapter 4, and some preliminary ideas to this will be introduced in chapter 3. For the remainder of this chapter, a qualitative overview will be given on the following application of the AdS/CFT correspondence: Type IIB string theory asymptotic to $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory. This is a historically significant application of the correspondence, as it was the basis for Maldacena's seminal work in realizing the AdS/CFT correspondence as one of the most successful realizations of the holographic principle[17] [19].

2.2 The AdS_5/CFT_4 Correspondence

The AdS/CFT correspondence "dictionary" given in (2.23) should hold for any theory of gravity and corresponding CFT. That is, given a (quantum) theory of gravity, (2.23) can be used to define the corresponding CFT, or vice-versa. However, aside from certain examples, the correspondence is often well defined and useful only in certain limits. To illustrate this, consider the well-known example of the AdS_5/CFT_4 correspondence:

$$\text{Type IIB Strings on } AdS_5 \times S^5 \quad = \quad (\mathcal{N} = 4) \text{ supersymmetric Yang-Mills.} \tag{2.28}$$

The exact details of the equivalence are beyond the scope of this thesis since an understanding of supersymmetric gauge theory and string theory (and supergravity) is necessary to make computations on either side of the correspondence. Therefore, a more qualitative assessment of each side will be discussed instead; the discussion presented will primarily follow [2] and section five of [12]. The purpose of this example will simply be to motivate the (historical and practical) significance of this equivalence, and thereby giving greater context to the rest of the thesis as the model of quantum gravity simplifies to just AdS_3 in later sections and chapters. The key idea is that a given theory of quantum gravity and a CFT are two seemingly unrelated theories, and yet the correspondence instead implies that they are equivalent. To proceed with the example, consider each side of the correspondence individually.

[17] The holographic principle is the more general notion that the volume of space which can be thought of as encoded on a lower dimensional boundary. This idea was given a string theory interpretation by Leonard Susskind [18].

2.2.1 Type IIB String Side

Consider Type IIB string theory[18,19,20] with N D3 branes stacked on top of each other at coordinate $r = 0$ for the gravity theory. On the left-hand-side of (2.28), the string theory has three adjustable scales: the AdS length l, the Planck length l_P, and the string length l_s. At low enough energies, the gravity action is in Einstein-Hilbert form where higher curvature corrections are suppressed by the string scale

$$S_{IIB} \sim \frac{1}{G_N} \int \sqrt{g} \left(R + 2\Lambda + L_{matter} + l_s^4 R^4 + ... \right) \tag{2.29}$$

where $[G_N] = 2 - D = -8$ (D is the total number of the string theory spacetime dimensions), and is not renormalizable[21]. The 10 dimensional metric g is that of an extremal black brane, and is given by

$$ds^2 = \left(1 + \frac{r_3^4}{r^4} \right)^{-1/2} (-dt^2 + d\mathbf{x}^2) + \left(1 + \frac{r_3^4}{r^4} \right)^{1/2} (dr^2 + r^2 d\Omega_5^2), \tag{2.30}$$

where $\mathbf{x}$ is a coordinate in $\mathbb{R}^4$, and r_3 is the radius of a D3 brane given by

$$r_3^4 = 4\pi g_s N (\alpha')^2. \tag{2.31}$$

Here, α' is the square of the string length $\alpha' = l_s^2$, g_s is the string coupling constant, and N is the number of D3 branes stacked on top of each other[22]. To explain this latter concept, a black brane is like a black hole, but the horizon is a plane instead of a sphere. In string theory, this solution is the geometry corresponding to a stack of Q_3 D3 branes. For the extremal black brane, Q_3 is the conserved charge of this solution related to r_3. A D3 brane is a type of Dirichlet membrane (D-brane), and are a class of extended objects upon which open strings can end with Dirichlet boundary conditions. D-branes are classified by their spatial dimension, and so in this case a D3 brane is extended along three spatial dimensions. In extremal black holes[23], AdS

[18] Type II string theory accounts for two of five consistent superstring theories in 10 dimensions. Type IIA is the non-chiral theory, and Type IIB is the chiral theory.

[19] The five types of 10-dimensional superstring theories are: Type I, Type IIA, IIB, HO and HE.

[20] M-theory is an 11-dimensional theory which unifies these five theories into one.

[21] It is not renormalizable for any $D > 3$, which is why any theory of gravity with dimension greater than $d = 2 + 1$ dimensions is an effective field theory, and not UV complete [2].

[22] The N will also match the same N in the $SU(N)$ SYM theory [12].

[23] An extremal black hole is one that has the minimal mass to exist for a given charge and angular momentum. It emits no Hawking radiation, but does contain entropy [20].

appears in the near horizon region, and this is similarly true for black branes. That is, the asymptotically AdS metric appears near the boundary of the bulk spacetime, and in this case the asymptotic spacetime is $\mathrm{AdS}_5 \times S^5$. As for the black brane metric, the limit $r >> r_3$ recovers flat spacetime $\mathbb{R}^{10}$, and when $r << r_3$ the geometry is known as the "throat" since it seemingly tends to a singularity as $r << r_3$. In redefining the coordinate

$$u \equiv \frac{r_3^2}{r}, \tag{2.32}$$

and taking a large u limit will transform the metric into its asymptotic form

$$ds^2 = r_3^2 \left[\frac{1}{u^2}(-dt^2 + d\mathbf{x}^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right]. \tag{2.33}$$

That is, (2.30) becomes (2.33) near the horizon [2, 12]. In this case, the asymptotic geometry[24] describes $\mathrm{AdS}_5 \times S^5$. The 5-sphere is the $r_3^2 d\Omega_5^2$ portion, and the remaining terms correspond to AdS_5. Equation (2.31) indicates the fact that both AdS_5 and S^5 have the same radius r_3.

2.2.2 $\mathcal{N} = 4$ Supersymmetric Yang-Mills

A Yang-Mills theory is a kind of gauge theory where the gauge group is that of some Lie group G (often from the group $SU(N)$), and the Lagrangian density is of the form

$$\mathcal{L}_{YM} = tr\left(F_{\mu\nu}F^{\mu\nu}\right). \tag{2.34}$$

A supersymmetric Yang-Mills (SYM) theory is one that has its matter fields fixed by supersymmetric transformations[25,26]. It is possible to have more than one type of supersymmetry transformation. $\mathcal{N} = 4$ super Yang-Mills is special because it has the highest possible number of supersymmetires (4 total) without needing to include

[24] That is, the geometry close to the brane at $r \to 0$, or $u \to \infty$.

[25] A supersymmetry is a symmetry in which swapping bosons and fermions in a certain way leaves the predictions of the theory invariant.

[26] In supersymmetry, each particle from one class would have an associated particle in the other known as its superpartner, the spin of which differs by a half-integer.

gravity[27]. The Lagrangian for $\mathcal{N}=4$ super Yang-Mills is given by [12]

$$\mathcal{L}_{\mathcal{N}=4} = tr\Big(-\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu X^i D^\mu X^i$$

$$+\sum_{a,b,i} g_{YM}C_i^{ab}\lambda_a[X^i,\lambda_b] + \sum_{a,b,i} g_{YM}\bar{C}_{iab}\bar{\lambda}^a[X^i,\bar{\lambda}^b] + \frac{g_{YM}^2}{2}\sum_{i,j}[X^i,X^j]^2\Big), \tag{2.35}$$

where $a,b = 1,..4$ and $i = 1,...6$. Here, the C_i^{ab} and C_{iab} are the fine structure constants constants related to the Clifford Dirac matrices for $SO(6)_R \sim SU(4)_R$[28]. θ_I is the instanton angle, and g_{YM} is the gauge coupling. The dimensions of the fields are

$$[A_\mu] = [X^i] = 1, \quad [\lambda_a] = \frac{3}{2}, \tag{2.36}$$

with $[g_{YM}] = [\theta_I] = 0$. In addition to the R-symmetires from $SU(4)_R$, this theory is both Poincaré and scale invariant, which combine to form an overal conformal invariance with group $SO(4,2) \sim SU(2,2)$[29]. Incidentally, this theory is UV finite.

Additionally, there is a discrete symmetry, which bundles the θ_I and g_{YM} together

$$\tau \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g_{YM}^2}, \tag{2.37}$$

where the theory is invariant under $\tau \to \tau + 1$ (or $\theta_I \to \theta_I + 2\pi$). The theory is also invariant under $\tau \to -1/\tau$. In combining these two symmetries, the new invariance of the theory is

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a,b,c,d \in \mathbb{Z}. \tag{2.38}$$

That is, there is a hidden $PSL(2;\mathbb{Z})$ invariance[30]. This $PSL(2;\mathbb{Z})$ symmetry forms an S-duality[31] group for the theory, since $\theta_I = 0$ changes the invariant statement to

[27]Having $\mathcal{N} > 4$ at least includes gravitinos, which require gravitons (spin = 2 massless bosons) to complete the theory. This would imply that the theory is really one of supergravity. However, one can still have a theory of supergravity with $\mathcal{N} < 4$.

[28]The subscript "R" indicates the R-symmetry, which are the symmetries which transform supercharges (the charges which generate the supersymmetries) in a supersymmetric theory into each other.

[29]This is enough to make it a CFT, but in combining this conformal symmetry with $\mathcal{N} = 4$ supercharge Poincaré symmetry (not to be confused with the previously mentioned ordinary Poincaré symmetry), their invariance produces an even larger superconformal symmetry given by the supergroup $SU(2,2|4)$ [12].

[30]This kind of transformation is called a Möbius transformation, which is a part of the Möbius group $PSL(2;\mathbb{C})$. The $PSL(2;\mathbb{Z})$ is a (discrete) subgroup of the Möbius group.

[31]The "strong-weak-duality" (S-duality) is a way of classifying a theory written in terms of strong

$g_{YM} \rightarrow -1/g_{YM}$, and effectively changes the theory from strong coupling to weak coupling. That is, $\mathcal{N}=4$ super Yang Mills is said to be self-dual[32].

In setting the θ_I aside, this gauge theory has two dimensionless parameters: N (the size of $SU(N)$) and the Yang-Mills coupling constant g_{YM}. These two parameters can be collected in the form

$$\lambda = g_{YM}^2 N, \tag{2.39}$$

which is known as the ‘t Hooft coupling [12]. It is often easier to organize a Yang-Mills gauge theory at large N as an expansion in λ and $1/N$, rather than g_{YM} and $1/N$. This is (roughly) because there are N gauge fields present in loop contributions, which changes the expansion parameter from g_{YM}^2 to λ; this will be discussed below in 2.2.3.

2.2.3 Mapping Between the Two Sides

Ignoring known coefficients, the mapping between the string theory side and the SYM side relates the ‘t Hooft coupling with the string and AdS radii [2]

$$\lambda \sim \left(\frac{l}{l_s}\right)^4, \tag{2.40}$$

and the AdS radius is directly related to the gauge group degrees of freedom

$$\frac{l^4}{G_N} \sim \left(\frac{l}{l_P}\right)^4 \sim N^2, \tag{2.41}$$

where l_P is the Planck length. The couplings of the two theories are related by [2, 12]

$$g_s = g_{YM}^2, \tag{2.42}$$

which implies a direct relationship between g_s and λ by (2.39), or g_{YM} and r_3 in (2.31). Practically, it is difficult to work with string theory quantization on a general curved manifold, and therefore it is desirable to seek out limits which allows the conjecture to become more usable (even if still non-trivial).

coupling in an equivalent way, now in terms of a weak coupling (or vice-versa).

[32] In other words, self-duality means that a theory with some couplings is dual to itself at some different values of the couplings.

2.2.4 Limits of AdS_5/CFT_4

Ultimately, the AdS_5/CFT_4 correspondence holds for any N (which is known as the "strong" version [12]), but the coupling strength of either theory will effect how one should approach computations. For example, if one wanted to keep the full string theory intact, the strategy would be to work with a CFT with weak coupling (so as to use perturbation theory on the CFT side) while simultaneously requiring a small enough N such that the string coupling g_s isn't also incredibly small[33]. However, if it was instead desirable to work with a strongly coupled CFT, there are two special limits one can consider[34] which will reduce the full Type IIB string theory into a simpler theory of gravity. These two limits are: the 't Hooft limit, and the large λ limit. In both, N is taken to be large and the CFT coupling is λ as defined in (2.39).

The 't Hooft Limit

The 't Hooft limit consists in keeping λ fixed, while simultaneously letting $N \to \infty$. In simply staring at (2.40) and (2.41), sending $N >> 1$ implies $l >> l_P$, with the freedom to choose the strength of the CFT coupling λ. In Yang-Mills theory, this limit is well defined in perturbation theory, and corresponds to a topological expansion of the field theory's Feynman diagrams [21]. On the gravity side, the 't Hooft coupling λ can be re-expressed in terms of the string coupling $g_s = \lambda/N$. Since λ is being kept fixed and $N \to \infty$, the 't Hooft limit corresponds to weak coupling string perturbation theory and recovers "classical"[35] Type IIB string theory. This form of the conjecture is "weaker" than the strong version since now the correspondence only applies for $N \to \infty$. That is [12],

$$\text{Classical IIB Strings on } \mathrm{AdS}_5 \times S^5 \quad = \quad (\mathcal{N}=4)\,\mathrm{SYM}\Big|_{N\to\infty} \tag{2.43}$$

where now the string theory side no longer requires loop corrections or higher curvature corrections. This equivalence is a historically significant result of the AdS/CFT correspondence, as it was here that Juan Maldacena first proposed the AdS/CFT correspondence in [19]. The problem with this limit is that finding an action built out of classical degrees of freedom to which the large N limit of gauge theories are

[33] And since N is small, the CFT coupling at hand is not λ, but rather g_{YM}.

[34] Although these limits can be considered even for weakly coupled CFTs

[35] What is meant here by "classical" is that since large N implies small l_P, the metric does not fluctuate.

classical solutions is a challenge that had been outstanding since ‘t Hooft’s original paper.

Large λ Limit

After taking the ‘t Hooft limit, there is still the freedom to make λ large as well. In staring again at (2.40) and (2.41), taking $\lambda >> 1$ with $N >> 1$ implies a vanishing string length scale $l >> l_s$ (although still with $l_s >> l_P$). This is a natural limit to take on the gravity side since one can expand the Lagrangian around small α', which can be written in terms of λ

$$\alpha' = \frac{r_3^2}{2\lambda^{1/2}}, \tag{2.44}$$

and therefore large λ corresponds to small α' (and indeed, a vanishing string length. That is, the role of α' is replaced by $\lambda^{-1/2}$. Although large λ prevents the use of perturbation theory on the CFT side, the consequence of vanishing string length causes the gravity side to simplify down to classical type IIB supergravity. The AdS_5/CFT_4 correspondence implies that computations in supergravity will produce results which correspond to CFT observables (with the given $N \to \infty$ and large $\lambda >> 1$ limits) accordingly. That is,

$$\text{Classical IIB Supergravity on } AdS_5 \times S^5 \quad = \quad (\mathcal{N} = 4)\,\text{SYM}\,\Big|_{N\to\infty,\lambda>>1} \tag{2.45}$$

where now a semiclassical theory of gravity can be used to study strongly coupled SYM theory[36] [12].

Mapping Type IIB Fields and CFT Operators

The details of this is beyond the scope of the thesis, but one final piece of the puzzle to consider is how to set-up the “dictionary” part of the problem. That is, how do do the CFT operators get mapped to the Type II B fields (and vice-versa)? Well, another requirement for the AdS/CFT correspondence to hold is that the global symmetries of the two theories be identical. As mentioned section 2.2.2, the continuous global symmetry of $\mathcal{N} = 4$ SYM theory is the superconformal group $SU(2,2|4)$. It turns out that this global symmetry appears on the gravity side as well [12]. Furthermore, the SYM theory also has the previously mentioned S-duality symmetry, where the

[36] Since perturbation theory is no longer possible on the CFT side.

complex coupling constant τ as defined in (2.37) transforms in $PSL(2;\mathbb{Z})$. This, it turns out, shows up as a global discrete symmetry of Type IIB string theory. Therefore, S-duality is also a symmetry of both sides [12]. With that being said, it is only a symmetry in the "strong" form of the conjecture. Either the ‘t Hooft limit or the large λ limit eliminates this symmetry.

For all of these symmetries, it can be shown that representations of the subgroup $SU(2,2|4)$ also map to both sides. One example is that single trace operators on the SYM side correspond to single particle states on the gravity side, but there more scenarios one can consider [12]. Ultimately, this implies that one can map many different types of operators on the SYM side to the fields on the gravity side. Hence, computations on one side of the correspondence can allows for the study of physical observables on the other.

2.2.5 The $\text{AdS}_5/\text{CFT}_4$ Conjecture Summarized

The simplest instance[37] of the $\text{AdS}_5/\text{CFT}_4$ conjecture states an equivalence between the following two theories

- Type IIB superstring theory on $\text{AdS}_5 \times S^5$ where both AdS_5 and S^5 have the same radius r_3 (the radius of a D3 brane given in (2.31)) and string coupling g_s.
- $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 4-dimensions with gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its super conformal phase[38].

Note that the two couplings are related by equation (2.42), which implies that the D3 brane radius r_3 and ‘t Hooft coupling λ can be related to either coupling constant g_s or g_{YM}. This is known as the "strong" form of the conjecture, which holds for all values of N[39]. The different versions of this conjecture are summarized in Table 2.1, along with the natural choice of small parameter to expand around when performing computations [12].

[37] For other examples, see [12].

[38] This just means that $\langle X^i \rangle = 0$ for all $i = 1, ...6$. That is, the gauge algebra is unbroken; the opposite case is called the "spontaneously broken" or "Coulomb" phase [12].

[39] As mentioned, N here corresponds to both the gauge group $SU(N)$ and the number of stacked D3 branes making up the black brane.

Gravity on $\mathrm{AdS}_5 \times S^5$		$\mathcal{N} = 4$ SYM
• (Full Quantum) Type IIB String Theory • $r_3^4 = 4\pi g_s(\alpha')^2$	$\Longleftrightarrow$	• $\mathcal{N} = 4$ SYM • All N, g_{YM} • $g_s = g_{YM}^2$
• Classical Type IIB String Theory • g_s string loop expansion	$\Longleftrightarrow$	• 't Hooft limit $\lambda = g_{YM}^2 N$ fixed, $N \to \infty$ • $1/N$ expansion
• Classical Type IIB Supergravity • α' expansion	$\Longleftrightarrow$	• Large λ limit (for $N \to \infty$) • $\lambda^{-1/2}$ expansion.

Table 2.1: The three common instances of the AdS_5/CFT_4 correspondence. The listing is in-order of the "strength" of the conjecture, and includes natural choices for small parameter expansions. For example, if one wanted to study large CTF coupling (assuming large N gauge theories $SU(N)$ on the SYM side), perform computations on the gravity side by expanding in small parameter α' (equivalently, the string length is small and suppress higher order corrections of the quantum theory). Note that the S-duality is only present in the strongest version of the correspondence.

Chapter 3

Black Hole Entropy from the $\mathrm{AdS}_3/\mathrm{CFT}_2$ Correspondence

Another success of the AdS/CFT correspondence is that it correctly produces the Benkenstein-Hawking entropy of certain black holes. To understand how this works, the AdS/CFT correspondence relates the two partition functions

$$Z_{Gravity} = Z_{CFT_2}, \tag{3.1}$$

for some theory of gravity on AdS_3. For Einstein gravity, at high temperatures the left-hand-side is dominated by the black hole solution, while at low temperatures it is dominated by what is called thermal AdS_3 (the latter of which will be discussed in section 3.3). This black hole solution is known as the BTZ black hole, which is a solution to $(2+1)$-dimensional Einstein gravity with negative cosmological constant. The entropy of a BTZ black hole is given by the Bekenstein-Hawking formula[1]

$$\mathcal{S}_{BTZ} = \frac{A}{4G}, \tag{3.2}$$

through a computation performed in gravity. In AdS_3 gravity the Bekenstein-Hawking entropy is in fact equivalent to the Cardy formula [15]

$$\mathcal{S} = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{6}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\widetilde{L}_0 - \frac{c}{24}\right)}. \tag{3.3}$$

[1]The fact that the entropy of the black hole is entirely encoded on the surface is another example of the holographic prinicple.

This formula gives the entropy of any two-dimensional CFT in the high temperature regime, and the AdS_3/CFT_2 correspondence implies that entropy on the CFT will agree with entropy on the gravity side (and vice-versa) [13]. In this way, the entropy of a BTZ black hole can be accessed by computing the entropy of the two dimensional CFT, and vice-versa. The purpose of this chapter will be to derive this formula, compute the BTZ black hole entropy with it, and discuss the Cardy formula's components along the way: Virasoro algebra element L_0 and the Brown-Henneaux central charge c. The approach here will largely stick to the gravity side following [15]. For simplicity, only the non-rotating BTZ black hole will be considered.

3.1 AdS_3 Gravity Metrics

The Euclidean action for (2+1) pure gravity in the presence of a negative cosmological constant is:

$$S_E = -\frac{1}{16\pi G}\int d^3x\sqrt{g}(R + \frac{2}{l^2}) + S_{bdy},$$

where once again the negative cosmological constant $\Lambda \to -\frac{1}{l^2}$ in which l is the AdS_3 radius; the explicit form of the boundary action S_{bdy} will become clear when relevant. In stepping back to Lorentzian signature for a moment, the equations of motion for this action are the vacuum Einstein field equations with negative cosmological constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{g_{\mu\nu}}{2l^2} = 0. \tag{3.4}$$

The most general metric for this action which is invariant under time and angular translations, and time reversal is given by the line element

$$ds^2 = A(r)dy^2 + B(r)dr^2 + C(r)d\theta^2, \tag{3.5}$$

and in solving for the coefficients $A(r), B(r)$, and $C(r)$ there are two basic metric solutions which satisfy the field equations. The first is global AdS_3

$$ds^2 = (1 + r^2/l^2)dy^2 + \frac{dr^2}{1 + r^2/l^2} + r^2 d\theta^2, \tag{3.6}$$

where sending $y = it$ recovers Lorentzian signature. This chapter will additionally consider the following identifications[2]: $y \sim y + 2\pi Im[\tau]$ such that time y forms a

[2] In doing so, this metric is what is known as "thermal" AdS_3.

circle, and $\theta \sim \theta + 2\pi Re[\tau]$. The geometry of global AdS_3 in Euclidean signature forms a solid torus where the boundary is a torus of complex structure τ. Time y runs the length of the torus, and the θ coordinate spirals around the body (like a cylinder).

Another solution to the field equations with the same boundary conditions is the non-rotating BTZ black hole [22]

$$ds^2 = \frac{(r^2 - r_+^2)}{l^2} dy^2 + \frac{l^2}{(r^2 - r_+^2)} dr^2 + r^2 d\theta^2, \tag{3.7}$$

were r_+ is the event horizon with similar identifications as global AdS_3, only now let $y \sim y + \beta$ as in (2.4) with the usual definition $\beta = 1/T$. Here, the temperature T is the temperature of the Lorentzian black hole. The smoothness of the event horizon region will be discussed later in section 3.3, and is determined by r_+ and it's relation to β and τ. For now, note that (3.6) can be retrieved from this by substituting $r_+^2 \leftrightarrow -l^2$ in (3.7). And to be clear, the geometry of the chosen metrics must be asymptotically AdS_3 (by satisfying boundary conditions). Indeed, metrics (3.6) and (3.7) fulfill this requirement; both metrics have the topology of a solid torus with boundary tori of complex structure τ.

To get a preview of the non-rotating BTZ black hole entropy, consider the identification $y \sim y + \beta$. From here, note that the unitary operator for time evolution is of course $U(t) = e^{-itH}$ with H the hermitian Hamiltonian operator. Substituting the compactification of $t \to y$ whith $y \sim y + \beta$ into the unitary operator gives the following

$$U(t) = e^{-itH} \to e^{-\beta H}. \tag{3.8}$$

From this, summing over all energy eigenstates for a canonical ensemble of a quantum mechanical system is the definition of the partition function

$$Z \equiv tr(e^{-\beta H}), \tag{3.9}$$

as mentioned in section 2.1.1. Evidently, the presence of periodic time y implies that gravity can be used to compute statistical mechanical properties. The statistical mechanics property of interest corresponding to this black hole is its Bekenstein-Hawking entropy:

$$\mathcal{S} = \frac{A}{4G} = \frac{\pi r_+}{2G}, \tag{3.10}$$

where $A = 2\pi r_+$ is the circumference of the circle described by θ.

In order to verify that this entropy can be extracted from a two-dimensional CFT using the AdS_3/CFT_2 correspondence, the gravity side of the correspondence first needs to be explored in greater detail. In particular, boundary stress tensors on the gravity side will be shown to be rewritten in terms of Virasoro algebra; the Virsaoro algebra is what links the two sides of the AdS_3/CFT_2 correspondence.

3.1.1 AdS_3 Boundary Stress Tensor

To analyze this problem with greater specificity, either metric (3.6) or (3.7) can be written in terms of Gaussian normal coordinates:

$$ds^2 = d\eta^2 + h_{ij}dx^i dx^j, \tag{3.11}$$

where the metric h_{ij} contains the time y and angular θ part of the metric(s). From there, the bulk part of the action can be re-written as:

$$S_E = -\frac{1}{16\pi G}\int d^2x d\eta\sqrt{h}\left(R^{(2)} + (tr(K))^2 - tr(K^2) + \frac{2}{l^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{h}\, tr(K) \tag{3.12}$$

which is trivially shown by verifying the equality

$$\sqrt{h}\left(R^{(2)} + (tr(K))^2 - tr(K^2)\right) - 2\frac{\partial\left((\sqrt{h})\, tr(K)\right)}{\partial\eta} = \sqrt{g}R.$$

Here, $R^{(2)}$ is the two-dimensional scalar curvature associated with the metric h_{ij}, and the extrinsic curvature is given by

$$K_{ij} = \frac{1}{2}\partial_\eta h_{ij}. \tag{3.13}$$

Next, as discussed at the beginning of this chapter, a boundary action S_{bdy} is required. Without it, the variation of S_E with respect to the metric h_{ij} will have a contribution of $\delta\partial_\eta h_{ij}$. Dirichlet boundary conditions are chosen, and the metric is held fixed $\delta h_{ij} = 0$, and $\delta\partial_\eta h_{ij} \neq 0$. Yet after variation, both terms appear. Therefore, in order for the boundary terms of δS_E to vanish, one must add a counter-term to the initial action which cancels this out. This boundary action S_{bdy} is the Gibbons-Hawking-

York term

$$S_{bdy} \to S_{E,GH} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{h}\, tr(K),$$

which was previously defined in (2.15). This is exactly the equation to the boundary term $\delta S_{E,EH}$ after variation but with the opposite sign. Furthermore, note that the Gaussian normal metric h_{ij} is the induced metric on a constant η slice as discussed in section 2.1.2 when (2.15) was first introduced. Thus, the final action $S_E = S_{E,EH} + S_{E,GH}$ can be written as

$$S_E = \frac{1}{16\pi G}\int d\eta \int_{\partial\mathcal{M}} d^2x\sqrt{h}\left(R^{(2)} + (tr(K))^2 - tr(K^2) + \frac{2}{l^2}\right). \qquad (3.14)$$

In varying (3.14) with respect to the induced metric h_{ij}, $\delta S_E = 0$ is given when satisfying the equations of motion. However, recall (2.18). Applying this to the relevant AdS_3 spacetime and allowing the equations of motion to be satisfied, the variation can be written in terms of a boundary stress-energy tensor

$$\delta S_E = \frac{1}{2}\int_{\partial\mathcal{M}} \sqrt{h} T^{ij}\delta h_{ij},$$

where this is zero precisely because the metric is fixed $\delta h_{ij} = 0$ on the boundary[3]. In actually computing this term from straight-forwardly varying δS_E, this is

$$\delta S_E = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{h}(K^{ij} - tr(K)h^{ij})\delta h_{ij}, \qquad (3.15)$$

and therefore the boundary stress tensor is identified to be

$$T^{ij} \equiv -\frac{1}{8\pi G}(K^{ij} - tr(K)h^{ij}). \qquad (3.16)$$

This is the Brown-York stress tensor, and gets an extra contribution from the boundary term S_{bdy} required by holographic renormalization. Indeed, this result agrees with what was derived in their original paper [14]. Note that this result is valid in any coordinate system, and not specific to that given in the Gaussian normal coordinates. Another aspect to this stress-tensor is that it exists purely due to the extrinsic curvature K^{ij} of the boundary metric h_{ij}, and does not correspond with any matter in the theory.

[3] δh_{ij} is generally not zero in the bulk, but this is fine since the bulk term vanishes according to the equations of motion.

As a reminder, the AdS/CFT correspondence specifically relates a CFT to a bulk theory on AdS. In this simple (two derivative) example, the bulk theory is Einstein gravity with two different solutions: global AdS_3 and the non-rotating BTZ black hole. Therefore, to consider asymptotic AdS_3 behavior, send the coordinate $\eta \to \infty$ and examine how the metric(s) behave[4]. The general solution to Einsteins equations which includes both global AdS_3 and the BTZ black hole can be written as an expansion by powers of $e^{-2\eta/l}$, which is called the "Fefferman-Graham expansion" [23]. Explicitly, the metric is written as

$$h_{ij} = e^{2\eta/l} h_{ij}^{(0)} + h_{ij}^{(2)} + ..., \tag{3.17}$$

and since asymptotic AdS_3 lies in the large $\eta >> 1$ regime all higher order terms in the "..." may be dropped since they go by powers of $e^{-\eta/l}$. The leading term $h_{ij}^{(0)}$ is known as the "conformal boundary metric," and its this metric which is identified to be the boundary metric up to conformal rescalings in which a CFT "lives." That is, metrics $g_{\mu\nu}$ written as conformal rescalings of another are ones which can be written as

$$g_{\mu\nu} = e^{2\Omega(x)} g_{\mu\nu}^{(0)}. \tag{3.18}$$

This will be discussed in more detail shortly, but for now note that what is being done to the metric $h_{ij} \to h_{ij}^{(0)}$ is analogous to fixing the fields ϕ_0^i near the boundary as discussed when introducing the AdS/CFT dictionary (2.23) in section 2.1.4.

Since $h_{ij}^{(0)}$ now plays the role of the boundary metric, a new variationional principle which fixes $h_{ij}^{(0)}$ should be developed (the subleading terms can still vary). However, in applying this requirement straight-forwardly, $\delta S_E \neq 0$ will immediately occur for two reasons. First, the variation $\delta h_{ij}^{(2)}$ will appear, and secondly the action itself diverges at large $\eta \to \infty$. To fix both of these problems, a "counter-term" can be introduced into the overall S_E action [24, 25]

$$S_{counter} = -\frac{1}{8\pi G l} \int_{\partial \mathcal{M}} d^2x \sqrt{h}, \tag{3.19}$$

which causes the variation given by (2.18) to now become

$$\delta S_E = \frac{1}{2} \int_{\partial \mathcal{M}} d^2x \sqrt{h^{(0)}}\, T^{ij} \delta h_{ij}^{(0)}. \tag{3.20}$$

[4]In the original coordinates, instead send $r \to \lambda$ for some large cut-off λ and examine a constant slice of the spacetime at λ.

This is what was found before, but now $h_{ij} \to h^{(0)}_{ij}$ where no other variations of subleading terms from the metric expansion appear. Since $\delta h^{(0)}_{ij} = 0$ and the large η cut-off will also no longer cause a divergence, the variation therefore vanishes accordingly. In computing this variation straight-forwardly, it is the case that

$$\delta S_E = \frac{1}{16\pi Gl}\int_{\partial\mathcal{M}} d^2x\sqrt{h^{(0)}}\left(h^{(2)}_{ij} - tr(h^{(2)})h^{(0)}_{ij}\right)\delta h^{(0)}_{ij}, \tag{3.21}$$

and thus the asymptotic AdS_3 boundary stress-tensor is identified similarly to the stress tensor before aside from the difference in sign and extrinsic curvature being replaced with the second term in the metric expansion $K^{ij} \to h^{(2)}_{ij}$ (and including the AdS_3 radius as provided by the counter-term)[5]

$$T^{ij} \equiv \frac{1}{8\pi Gl}\left(h^{(2)}_{ij} - tr(h^{(2)})h^{(0)}_{ij}\right). \tag{3.22}$$

Recall that a CFT is a theory whose partition function is invariant by rescalings of the metric. In this way, the action of the CFT may be allowed to break invariance by varying with respect to the metric[6]. In fact, it is varying with respect to the conformal rescaling itself which preserves the metric in the quantum theory (which will certainly leave the partition function invariant). It further turns out that varying the gravity action with respect to conformal rescalings is proportional to the trace of the boundary stress tensor, which would therefore need to vanish in order to preserve conformal invariance since this is still a classical theory of gravity [15]. The trace of the boundary stress-tensor is written as

$$tr(T) = -\frac{1}{8\pi Gl}tr(h^{(2)}), \tag{3.23}$$

where the trace is performed over the boundary metric

$$tr(h^{(2)}) = (h^{(0)})^{ij}(h^{(2)})_{ij}. \tag{3.24}$$

In order to compute this, one requires $h^{(2)}_{ij}$. All subleading terms in this metric expansion can be found by solving Einstein's equations up to the given order. In

[5]Stress tensors in higher dimensional gravity can be found in [25, 26], and additional related work in [27, 28, 29, 30, 31].

[6]That is, as long as the partition function remains invariant.

solving for $h_{ij}^{(2)}$, this trace is evaluated to

$$tr(h^{(2)}) = -\frac{l}{16\pi G}R^{(0)}, \tag{3.25}$$

which is decidedly non-zero assuming $h_{ij}^{(2)}$ is not flat[7]. This non-zero trace of the boundary metric is what is known as a Weyl (conformal) anomaly. Even though this is still classical gravity, this has a quantum mechanical consequence on the CFT side, and is the result of the two-dimensional CFT being placed on a curved background (the boundary of the asymptocically AdS_3 spacetime). That is, this non-zero trace shows up on the CFT side of this problem, and the standard form of the Weyl anomaly is given by[15]

$$tr(T) = -\frac{c}{24\pi}R. \tag{3.26}$$

In comparing these two equivalent results, one can compute the central charge

$$c = \frac{3l}{2G}, \tag{3.27}$$

which is in perfect agreement with the original derivation by Brown and Henneaux in [32]. What this central charge is precisely will be discussed in at the end of section 3.2, but for now just note that the presence of the Weyl anomaly requires choosing a specific representative class of metrics $h_{ij}^{(0)}$. If there was no Weyl anomaly, then the action would not depend on the particular representative class of metric $h_{ij}^{(0)}$ up to conformal rescalings. That is, the CFT is by definition Weyl invariant; the CFT partition function Z_{CFT} does not change. Therefore, by (3.18), any metric h_{ij} as a conformal rescaling of $h_{ij}^{(0)}$ will leave the CFT action invariant. Furthermore, it can be shown that in two dimensions *any* metric can be written in the following form

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\Omega(x)}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{3.28}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. What this implies is that if the theory had no Weyl anomaly, any choice of (two-dimensional) metric would leave the CFT theory invariant, since any metric is related to flat space by a conformal rescaling. However, in the presence of a Weyl anomaly, Z_{CFT} depends on the choice of $h_{ij}^{(0)}$ representation.

[7]Similarly, the stress tensor which was previously computed without expanding the metric h_{iJ} would also give a non-zero trace unless h_{ij} itself described flat space.

3.1.2 Choosing a Boundary Metric Representation

Consider the choice of metric $h_{ij}^{(0)}$ as the flat metric placed on the cylinder (a choice of topology), and work in complex coordinates. The corresponding line element is given by

$$ds^2 = h_{ij}^{(0)} dx^i dx^j = dw d\bar{w}, \tag{3.29}$$

where the complex coordinate $w(z)$ is defined as the holomorphic coordinate

$$w \equiv \sigma_1 + i\sigma_2, \tag{3.30}$$

with the identification $w \sim w+2\pi$; the corresponding anti-holomorphic $\bar{w}(\bar{z})$ is defined similarly. The σ_1 coordinate stands in for the spatial direction and the σ_2 coordinate for the (imaginary) time direction. Solving Einstein's equations in this coordinate system results in the AdS_3 boundary stress tensor now having homomorphic and anti-holomorphic components

$$T_{ww} = \frac{1}{8\pi G l} h_{ww}^{(2)}, \quad T_{\bar{w}\bar{w}} = \frac{1}{8\pi G l} h_{\bar{w}\bar{w}}^{(2)}. \tag{3.31}$$

The details of quantizing the full theory of AdS_3 will be left for chapter 4, but for now simply consider how the stress tensors are related to symmetries in a two-dimensional CFT after quantization on the gravity side. To set this up properly, now consider the CFT side of the AdS_3/CFT_2 correspondence.

3.2 Getting to Virasoro from the CFT_2

The conformal group in $d+1$ spacetime dimensions is in general $SO(d+2, 2)$. However, an important exception to this in (1+1)-dimensions. In this case, the conformal group is infinite dimensional, and is generated by two copies of the infinite dimensional Virasoro algebra. Virasoro algebra is a Lie algebra which generates the the centrally extended diffeomorphism group of the circle; the group is often denoted as $\widehat{Diff(\mathbb{S}^1)}$[8]. As discussed in the previous subsection, it is conformal transformations which must preserve the gravity partition function, and therefore the Virasoro symmetry must be present in both quantum theories. The remainder of this section will discuss this symmetry in a general two-dimensional CFT.

[8]The group $Diff(\mathbb{S}^1)$ with no "hat" refers to the diffeomorphism group of the circle with no central extension.

3.2.1 Witt Algebra

To start consider a two-dimensional classical field theory on flat space which is invariant under conformal transformations, or a "classical CFT"[9]. It turns out that, in two dimensions, conformal transformations are equivalent to holomorphic coordinate changes. Therefore, change to the holomorphic and anti-holomorphic (complex) coordinates $w(z)$ and $\bar{w}(\bar{z})$ respectively, and work with the flat metric given by (3.29)[10]. An infinitesimal, local coordinate transformation of the original complex coordinates

$$z \to z + \epsilon(z) \tag{3.32}$$

and derivatives transform like

$$\partial_\mu \to -\epsilon(z)\partial_z. \tag{3.33}$$

Combined, this acts on a scalar field $\Phi(z, \bar{z})$ as

$$\Phi \to \Phi - \epsilon(z)\partial_z\Phi - \epsilon(\bar{z})\partial_{\bar{z}}\Phi. \tag{3.34}$$

The algebra elements which generate these transformations are

$$L_n = -z^{n+1}\partial_z, \quad \widetilde{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{3.35}$$

These generators satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [\widetilde{L}_m, \widetilde{L}_n] = (m-n)\widetilde{L}_{m+n}, \tag{3.36}$$

where the two different copies commute with each other. This algebra generates the complexification of $Diff(\mathbb{S}^1)$, and is also known as "Witt algebra." This is an infinite-dimensional algebra, and includes the important subalgebra $\mathfrak{sl}(2,\mathbb{C})$. Specifically, the elements of the subalgebra are $L_0, L_{\pm 1}, \bar{L}_0, \bar{L}_{\pm 1} \in \mathfrak{sl}(2,\mathbb{C})$. This will be important in chapter 4, since including this subalgebra over the reals puts them in the product group[11] [2]

$$SL(2,\mathbb{R}) \times SL(2,R) \sim SO(2,2), \tag{3.37}$$

[9] This is technically not a CFT, since the theory is not yet a quantum theory.

[10] At this moment, the metric at hand has no relation to the $h_{ij}^{(0)}$ found on the gravity side, but the fact that they are the same is not a coincidence due to the AdS/CFT correspondence.

[11] The emergence of SL(2,$\mathbb{C}$) or $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ depends on choice of metric signature. The former corresponds with these complex coordinates (Euclidean signature), while the latter corresponds with the usual Lorentzian signature.

which will be important in chapter 4. For now, the classical stress tensor is defined by performing the Noether procedure for this symmetry.

Since the action of the "classical CFT" is invariant under conformal transformations, corresponding conserved currents exist. In complex coordinates, these conserved currents are the holomorphic and anti-holomorphic stress tensors, which obey

$$\partial_{\bar{z}} \left(\epsilon(z)T(z)\right) = 0, \quad \partial_z \left(\epsilon(\bar{z})\bar{T}(\bar{z})\right) = 0, \tag{3.38}$$

with definitions

$$T_{zz} \equiv T(z), \quad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}), \tag{3.39}$$

and where the currents are defined as

$$J^{\bar{z}} \equiv \epsilon(z)T(z), \quad J^{z} \equiv \epsilon(\bar{z})\bar{T}(\bar{z}). \tag{3.40}$$

Furthermore, it can be shown that the trace of the "overall" stress-tensor $T_{z\bar{z}} = 0$ vanishes.[12,13] As stated in section 3.1, the non-zero trace of the stress tensor appears when the (quantum) CFT is placed on a curved spacetime. On the gravity side, $h_{ij}^{(0)}$ was set to be flat space on the cylinder. Of course, performing a conformal rescaling of $h_{ij}^{(0)}$ could be done on the CFT side, since any conformal transformation will leave the CFT invariant by construction. However, it is also the case that conformally rescaling $h_{ij}^{(0)}$ would result in the new metric space being curved, and hence reproducing the Weyl anomaly. In this way, the central charge c can be obtained on the CFT side by seeing how the CFT stress tensor transforms under a conformal symmetry.

3.2.2 Central Charge

The quantum version of the Noether procedure in field theory is the derivation of Ward identities. It is therefore desirable to understand the Ward identities of a two-dimensional CFT. In classical field theory, a classical theory with a $U(1)$ symmetry has an associated current J^{mu}. From here, the Noether procedure is conserved under the equations of motion, $\partial_{mu} J^{mu} = 0$. Now, for the relevant quantum theory at hand, the action and path integral measure $D[\theta]$ are invariant under some conformal trans-

[12] Consider a rescaling symmetry, and vary the gravity action. It is immediately apparent that the only way the action can vanish is if the trace of the stress tensor is zero [15].

[13] What is meant by "overall" is simply that the traces of these individual holomorphic and anti-holomorphic stress tensors are not zero.

formation. Therefore, the quantum mechanical version of the conservation equation is a Ward identity given by the operator equation

$$\langle \partial_\mu J^\mu(y) O_1(x_1)...O_n(x_n)\rangle = 0, \quad \text{if} \quad x_i \neq y. \tag{3.41}$$

If a specific $x_i = y$, then the corresponding operator also transforms $O_i \to O_i + \delta O_i$ and the integration measure needs to be more carefully dealt with. Stated without derivation, this updates the conservation equation to [2]

$$\partial_\mu \langle J^\mu(y) O_1(x_1)...O_n(x_n)\rangle = \sum_i \delta(y - x_i)\langle O_1...\delta O_i...O_n\rangle, \tag{3.42}$$

for any number of operators O_i relating to the current through $y = x_i$; this formula is called the Ward identity.

This is a general result. However, for the relevant two-dimensional CFT in complex coordinates, the Ward identity corresponding to conformal symmetries can be rewritten as an operator equation

$$\delta_{\epsilon,\bar{\epsilon}} O(x) = -res_{z\sim x}[\epsilon(z)T(z)O(x)]. \tag{3.43}$$

where $J(z)$ is the conserved current corresponding to the conformal symmetry. The operators in the brackets are evaluated by performing the operator product expansion (OPE)

$$O_1(x)O_2(y) = \sum_j f_{12j}(x-y)O_j(y), \tag{3.44}$$

with OPE coefficients $f_{12j}(x-y)$; the sum is over all local operators in the theory. Note that this holds true for any correlation function

$$\langle O_1(x_1)O_2(x_2)...O_n(x_n)\rangle = \sum_j f_{12j}(x-y)\langle O_j(y)O_3(x_3)...O_n(x_n)\rangle, \tag{3.45}$$

since the OPE is a general operator relationship [33]. Therefore, in choosing the operator $O(x)$ to be the stress tensor itself, the OPE computed is the TT OPE, and the behavior of the stress tensor $T(z)$ under a conformal transformation can be computed from the now known TT OPE along with the Ward identity. The TT OPE is [2]

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{3.46}$$

where the symbol '∼' indicates that only the singular terms have been included[14]. Combining (3.46) with (3.43) thus computes the conformal transformation of the stress-energy tensor T [2]

$$\delta T(z) = -res_{z\sim w}[T(z)T(w)] = -\epsilon(z)T'(z') - 2\epsilon'(z')T(z) - \frac{c}{12}\epsilon'''(z'''), \qquad (3.47)$$

where the $'$ indicates that a term has undergone a conformal transformation $T(z) \rightarrow T'(z')$ with respect to complex coordinate z; c is said to be the central charge of the CFT.

It turns out that this is an anomalous term corresponding to the quantum theory. That is, this central charge c is the same central charge corresponding to the Weyl anomaly in AdS_3 gravity, and therefore the same c which appears in the stress tensor trace in curved space from (3.27). The way this is shown to be the case is to transform the AdS_3 boundary stress tensor by the same conformal transformations as discussed in the CFT. The result is [2]

$$\delta_\epsilon T(z) = -\epsilon(z)T(z') - 2T(z)\epsilon'(z') - \frac{l}{8}\epsilon'''(z'''). \qquad (3.48)$$

Notice that the AdS_3 boundary stress tensor transforms in the same way as the CFT stress tensor corresponding to the conserved current of the conformal symmetries. Therefore, just as before (restoring G)

$$c = \frac{3l}{2G},$$

which agrees exactly with (3.27). Another useful result is that exponentiating the infinitesimal transformation (3.48) provides the finite transformation law [2]

$$T'(z') = \left(\frac{dz'}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{z', z\}\right], \qquad (3.49)$$

where the term is curly brackets is the Schwarzian derivative

$$\{f(z), z\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \qquad (3.50)$$

[14] There is also an infinite series of non-singular contributions at $z = w$ [2].

3.2.3 Virasoro Algebra

This number c is known as the central charge because it is the centrally extended element of the Witt algebra; it commutes with every other element of the Witt algebra. The algebra which encompasses the full central extension of the Witt algebra is known as the Virasoro algebra, and is defined by the following commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{3.51}$$

The Virasoro algebra contains the same information as the TT OPE. The way this works is that the stress tensors are broken down into modes

$$\begin{aligned} L_n - \frac{c}{24}\delta_{n,0} &= \oint dw\, e^{-inw}\, T_{ww} \\ \widetilde{L}_n - \frac{c}{24}\delta_{n,0} &= \oint d\bar{w}\, e^{in\bar{w}}\, T_{\bar{w}\bar{w}}, \end{aligned} \tag{3.52}$$

noting that the $\widetilde{L}_n$ satisfies an independent set of Virasoro algebra with a corresponding central charge c[15]. Evidently, the symmetry algebra of a two dimensional CFT[16] is given by the Virasoro algebra. The two copies correspond with the left-moving holomorphic and right-moving anti-holomorphic algebras, respectively.

The Virasoro algebra will also be shown to be a symmetry of a quantum theory of AdS_3 in section 4.4. For now, there are some physical properties of the AdS_3 side which can be related to the Virasoro charges. In particular, the mass and angular momentum are related to the charges by the L_0 element

$$L_0 - \frac{c}{24} = \frac{1}{2}(Ml - J), \quad \widetilde{L}_0 - \frac{c}{24} = \frac{1}{2}(Ml + J). \tag{3.53}$$

As an example, consider the non-rotating BTZ metric (3.7) with black hole mass M and angular momentum $J = 0$. Taking the metric expansion (3.17) and solving Einsteins equations finds $h^{(2)}_{ww} = h^{(2)}_{\bar{w}\bar{w}} = r_+^2/4$, and hence the stress tensors in (3.31). Using this combined with (3.52), the L_0 is computed directly

$$L_0 = \widetilde{L}_0 = \frac{l}{16G}\left(1 + \frac{r_+^2}{l^2}\right), \tag{3.54}$$

[15] In general, this may be a different number $\widetilde{c}$, but for the purpose of this thesis $\widetilde{c} = c$ will always be true.

[16] The reason why tress tensor transformed at all is because the conformal transformation changed the flat spacetime to a curved one.

and therefore

$$M = \frac{r_+^2}{8Gl^2}, \quad J = 0, \tag{3.55}$$

where M and J are the mass and angular momentum of the BTZ black hole. It turns out that the well known fact that the mass is proportional to the square of the event horizon radius can be directly computed from the Virasoro algebra.

3.3 Getting to The Cardy Formula

3.3.1 Higher Derivative Generalizations

Thus far, the discussion on the gravity side has remained in the classical regime with a two derivative action. For any more complicated quantum theory (such as string theory), this action would correspond to the leading portion of a more general effective action. Since higher order terms may contain an arbitrary number of derivatives, these results must be generalized. That is, the entropy-area relation $\mathcal{S} = A/4G$ will not necessarily hold in a general theory of quantum gravity; a more general expression for entropy $\mathcal{S}$ is desired. The details of this generalization process is unnecessary to cover for the purposes of this thesis, so the result will be stated without proof [15]

$$c = \frac{l}{2G} g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}, \tag{3.56}$$

where $\mathcal{L} = \mathcal{L}(g^{\mu\nu}, \nabla_\mu, R_{\mu\nu})$ is the Lagrangian density of the higher-derivative gravity action [15][17]. The metric $g_{\mu\nu}$ is still one which satisfies Einstein's equations in pure AdS_3 gravity (whether that be global AdS_3 or the BTZ black hole) in it's asymptotic region. A quick check verifies that this is consistent with the two derivative (leading) action. In the leading two derivative action, $\frac{\partial \mathcal{L}}{\partial R_{\mu\nu}} = g^{\mu\nu}$, which retrieves the Brown-Henneaux central charge $c = 3l/2G$ as given in (3.27).

3.3.2 Canonical Partition Functions

As stated more generally in 2.1.1, the AdS_3/CFT_2 correspondence relates the partition functions

$$Z_{Gravity}(h^{(0)}) = Z_{CFT_2}(h^{(0)}), \tag{3.57}$$

[17] This higher derivative gravity action would be in the previously mentioned derivative expansion involving the full quantum theory.

for consistent quantum theories of gravity on AdS_3. Note the explicit dependence on the boundary metric is highlighted here. In particular, consider $h^{(0)}_{ij}$ to still be the flat metric, but now place it on a torus of complex structure τ instead of on a cylinder. This adds a second identification to the complex coordinate w

$$w \sim w + 2\pi\tau. \tag{3.58}$$

From here, the Cardy formula may be derived starting from Z_{CFT_2} as a path integral on the torus as done in [34]. However, one may also consider the canonical formulation (2.2) by substituting in the relationship between the Virasoro $L_0, \widetilde{L}_0$, and central charges and the energy and angular momentum using (3.53). The resulting partition function is

$$Z_{CFT_2}(\tau,\bar{\tau}) = tr\left[e^{2\pi i\tau(L_0-\frac{c}{24})}e^{-2\pi i\bar{\tau}(\widetilde{L}_0-\frac{c}{24})}\right]. \tag{3.59}$$

To examine the above equation from a more physically intuitive perspective, this paritition function can be re-written. In particular, the Virasoro terms can be traded for mass Ml and angular momentum J from (3.53). Then, the complex structure τ multiplies these separate terms in the following way: $Im[\tau]$ multiplies Ml and $Re[\tau]$ multiples J. The partition function takes the form

$$Z_{CFT_2}(\tau,\bar{\tau}) \propto tr\left[e^{\pi(iRe[\tau]J-Im[\tau]Ml)}\right], \tag{3.60}$$

by noting that

$$H = Ml = L_0 - \frac{c}{24} + \widetilde{L}_0 - \frac{c}{24}, \quad J = L_0 - \widetilde{L}_0, \tag{3.61}$$

and therefore the imaginary part τ (and $\bar{\tau}$) can be seen to be playing the role of inverse temperature β, and the real part acts as the chemical potential μ for angular momentum. That is,

$$\tau = \frac{(\mu + i\beta)}{2\pi}. \tag{3.62}$$

Attempting to write down $Z_{Gravity}$ in canonical form requires reckoning with the fact that the Hilbert space is difficult to define depending on the energy regime. At low energies, Hilbert space is understood to be comprised of a gas of particles moving on AdS, while at high energies black hole solutions emerge. Since black holes carry entropy, they cannot be interpreted as individual states of a theory; how does one take the trace over these kind of states? The way to deal with this is to write $Z_{Gravity}$

in path integral formulation, since black holes can be included as additional saddle points of the functional integral weighted by their corresponding action. That is,

$$Z_{Gravity}(\tau,\bar{\tau}) = \sum e^{-S_E}, \tag{3.63}$$

where the sum runs over all saddle points of the full gravity action.[18,19]. To start, consider the (simplest) saddle point: pure AdS_3 gravity given by the global AdS_3 metric.

3.3.3 Low Temperature Gravity Behaves as Thermal AdS_3

In proceeding, define $w = \theta + iy/l$, with the identification provided in (3.58). Without knowing S_E explicitly, the way to obtain this portion of the full action is to integrate the pure AdS_3 variation (3.20).[20] To do this, first use coordinates with fixed periodicity by defining

$$z = \frac{i-\bar{\tau}}{\tau-\bar{\tau}}w - \frac{i-\tau}{\tau-\bar{\tau}}\bar{w}, \tag{3.64}$$

such that z acts under the identification $z \sim z + 2\pi i$. Doing this causes complex structure τ to now appear in the boundary AdS_3 metric $h^{(0)}$

$$ds^2 = \left|\frac{1-i\tau}{2}dz + \frac{1+i\tau}{2}d\bar{z}\right|^2. \tag{3.65}$$

Then, writing (3.20) in these new z coordinates, and then converting back to the w coordinates gives the variation

$$\delta S_{E,Thermal} = 4\pi^2 i\left(T_{\bar{w}\bar{w}}\delta\bar{\tau} - T_{ww}\delta\tau\right), \tag{3.66}$$

where the stress tensor has naturally been decomposed into the holomorphic and anti-holomorphic terms previously discussed.[21] Something else to note: the pure AdS_3 metric is invariant under $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ group of isometries generated

[18]The "full" action refers to all corrections from string and loop corrections.

[19]These saddle points include metrics which are equivalent under a coordinate transformation (which will be shown in later in the section).

[20]Although the metric dependence of this integral is actually on $h^{(0)}_{ij}$, recall that it originally came from either the global AdS_3 metric (3.6) or the non-rotating BTZ black hole metric (3.7).

[21]The subscript "thermal" indicates that the pure AdS_3 action behaves as a system with statistical mechanics due to τ playing the role of inverse temperature.

by $L_{0,\pm 1}$ and $\widetilde{L}_{0,\pm 1}$. This will be discussed in greater detail in sections 4.1 and 4.4, but for now this implies that $L_0 = \widetilde{L}_0 = 0$ for pure AdS_3 [15]. This allows one to compute the stress tensors from (3.52)

$$T_{ww} = -\frac{c}{48\pi}, \quad T_{\bar{w}\bar{w}} = -\frac{c}{48\pi}, \tag{3.67}$$

and hence the action corresponding to thermal AdS_3 is

$$S_{E,Thermal} = \frac{i\pi}{12}(c\tau - c\bar{\tau}). \tag{3.68}$$

In just including this portion of the action in the partition function, the exact low temperature behavior of $Z_{Gravity}$ is found

$$Log\, Z_{Gravity}(\tau, \bar{\tau}) = -\frac{i\pi}{12}(c\tau - c\bar{\tau}) + ..., \tag{3.69}$$

where the "..." are exponentially suppressed loop corrections at low temperatures $Im(\tau) \to \infty$.

3.3.4 High Temperature Gravity Behaves as a BTZ Black Hole

As previously discussed, periodic imaginary time plays the role of inverse temperature $y \sim y + \beta$ for non-rotating BTZ black hole states. And to avoid a conical singularity at the event horizon $r = r_+$, make the further identification $y \sim y + 2\pi l^2/r_+$, and for pedagogical reasons consider only purely imaginary τ going forward. Taking these two identifications together, the Hawking temperature is therefore given by

$$T = r_+/(2\pi l^2). \tag{3.70}$$

Yet also recall that time had the identification $y \sim y + 2\pi\tau$ (τ is now purely imaginary), where τ is the boundary torus structure. Therefore, the non-rotating BTZ black hole metric only contributes to the partition function when $r_+ = il/\tau$.

To compute the action of the non-rotating BTZ black hole, define the coordinates

$$w' = -\frac{w}{\tau}, \quad r' = \frac{l}{r_+}\sqrt{r^2 - r_+^2}, \tag{3.71}$$

with $w' = \theta' + iy'/l$. From here, one could again integrate

$$\delta S_E = \frac{1}{2}\int_{\partial\mathcal{M}} d^2x\sqrt{h^{(0)}}\,T^{ij}\delta h_{ij}^{(0)},$$

only this time using the non-rotating BTZ metric (3.7) instead of global AdS_3 (3.6)[22]. First, (3.7) can be rewritten by using these new primed coordinates

$$ds^2 = (1 + r'^2/l^2)dy'^2 + \frac{l^2 dr'^2}{1 + r'^2/l^2} + r'^2 d\theta'^2. \tag{3.72}$$

Previously the global AdS_3 (3.6) and non-rotating BTZ black hole (3.7) metrics were related by a simple coordinate transformation, but now (3.72) is in the exact form of global AdS_3. The main difference between this BTZ metric and the original global AdS_3 metric is that now the identification is $w' \sim w' + 2\pi\tau'$, in which $\tau' = -1/\tau$. In this way, thermal AdS_3 is equivalent to the BTZ black hole up to the coordinate transformation $\tau \to -1/\tau'$ (and vice-versa)[23]

$$\text{Thermal AdS}_3 \text{ with } \tau \quad \Leftrightarrow \quad \text{BTZ Black Hole with } \tau' = -1/\tau. \tag{3.73}$$

As a result of this equivalence between a coordinate transformation, the action is therefore immediately of the equivalent form to thermal AdS_3

$$S_{E,BTZ} = \frac{i\pi}{12}(c\tau' - c\bar{\tau}') = -\frac{i\pi}{12}\left(\frac{c}{\tau} - \frac{c}{\bar{\tau}}\right), \tag{3.74}$$

where now at high temperature $Im(\tau) \to 0$ and the non-rotating BTZ black hole action will dominate the partition function

$$Log\, Z_{Gravity} = \frac{i\pi}{12}\left(\frac{c}{\tau} - \frac{c}{\bar{\tau}}\right) + \text{"....."} \tag{3.75}$$

3.3.5 The Cardy Formula

Equations (3.69) and (3.75) describe how the gravity partition function behaves in the low and high temperature regimes respectively[24]. The leading term in the former

[22]Specifically, one would compute $h_{ij}^{(2)}$ first, which then leads to the differing stress tensors.

[23]This has only been established for purely imaginary τ, the generalization of which is performed in [15].

[24]At some critical temperature, the system undergoes a phase transition between thermal AdS_3 and the BTZ black hole, which then causes the latter action to dominate the partition function (or

is given by thermal AdS_3 and the latter by the non-rotating BTZ black hole. Thus, for a purely imaginary τ, the high temperature regime in gravity is well approximated by a non-rotating BTZ black hole.

Using the high temperature solution (3.75), one can compute the entropy of the non-rotating BTZ black hole $\mathcal{S}_{BTZ}$; for the non-rotating BTZ black hole, $L_0 = \widetilde{L}_0$. For further simplicity, consider only two-derivative gravity. In order proceed, return to the CFT side and use the saddle point approximation on (3.59)

$$Log\, Z_{CFT_2} \approx \mathcal{S} + 2\pi i\tau\left(L_0 - \frac{c}{24}\right) - 2\pi i\bar{\tau}\left(L_0 - \frac{c}{24}\right), \tag{3.76}$$

where the central charges c is given by (3.56). Furthermore, notice that the above expression is in the same form as the definition of Helmholtz free energy

$$-\frac{F}{T} = \mathcal{S} + \beta E,$$

where E in this case contains both holomorphic and anti-holomorphic Virasoro components, and β contains the τ and $\bar{\tau}$ up to a factor of $2\pi i$. Specifically, $\beta \simeq 2\pi i\tau$ (or $\bar{\tau}$), and therefore in solving for entropy in both of the above equations

$$\mathcal{S} = 2\pi i\bar{\tau}\left(L_0 + \frac{c}{24}\right) - 2\pi i\tau\left(L_0 - \frac{c}{24}\right) + Log\, Z_{CFT_2}, \tag{3.77}$$

and comparing this with[25]

$$\mathcal{S} = \beta(E - F) \Rightarrow 2\pi i\tau(E + \widetilde{E} - F), \tag{3.78}$$

its the case that there are two energies associated with the holomorphic and anti-holomorphic Virasoro terms

$$E = L_0 - \frac{c}{24}, \quad \widetilde{E} = L_0 - \frac{c}{24}. \tag{3.79}$$

To compute these, go back to (3.59) and take derivatives with respect to τ and $\bar{\tau}$

$$\begin{aligned} L_0 - \frac{c}{24} &= \frac{1}{2\pi i}\frac{\partial(Log\, Z_{CFT_2})}{\partial\tau} \approx -\frac{c}{24\tau^2} \\ L_0 - \frac{c}{24} &= -\frac{1}{2\pi i}\frac{(\partial Log\, Z_{CFT_2})}{\partial\bar{\tau}} \approx -\frac{c}{24\bar{\tau}^2}, \end{aligned} \tag{3.80}$$

vice-versa). This is called the Hawking-Page transition [35].

[25] Noting that $\bar{\tau} = -\tau$ for a purely imaginary τ.

where the right-hand-sides were computed by invoking the AdS/CFT correspondence and substituting in the leading order term from (3.75) into the $Log\, Z_{CFT_2}$ derivatives here. Therefore, substituting in the far right-hand-sides of (3.80) and the leading order action from (3.75) into (3.77) gives a new expression for the entropy

$$\mathcal{S} = \frac{i\pi c}{6\tau} - \frac{i\pi c}{6} \Rightarrow \frac{i\pi c}{6\tau}(c + c), \tag{3.81}$$

where the right-hand-side used the fact $\bar{\tau} = -\tau$ for purely imaginary τ.

For a moment, consider only the "non-tilded" part of the entropy. This can be re-written in terms of the Virasoro generator L_0 and central charge c. Showing this is quite straight-forward:

$$\frac{i\pi c}{6\tau} = 2\pi\sqrt{-\frac{c}{6}\frac{c}{24\tau^2}},$$

and using (3.80) and (3.79) allows one to write

$$\Rightarrow 2\pi\sqrt{\frac{c}{6}E} = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)},$$

$$\therefore \; \frac{i\pi c}{6\tau} = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}. \tag{3.82}$$

This can identically be done for the other portion of the entropy (again making use of(3.80) and (3.79) and the fact that $\bar{\tau} = -\tau$)

$$\frac{i\pi c}{6\tau} = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}. \tag{3.83}$$

Finally, substituting (3.82) and (3.83) into (3.81) results in

$$\mathcal{S} = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}. \tag{3.84}$$

This is the Cardy formula (3.3), and agrees with what was derived in [34] when evaluating Z_{CFT_2} as a path integral on the torus. The Cardy formula gives the entropy at high temperature for any two dimensional CFT, and by the AdS/CFT correspondence also accurately computes the entropy of the non-rotating BTZ black hole. In fact, the Cardy formula above holds true for rotating BTZ black holes ($L_0 \neq \widetilde{L}_0$), generic central charges $c \neq \widetilde{c}$, and higher derivative theories of gravity.

Consider the simple case of two-derivative gravity. The non-rotating BTZ black hole entropy can be computed by three independent computations. The first method is to use the Bekenstein-Hawking entropy equation

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G} = \frac{(\pi l)^2 T}{G},$$

in which the right-hand side came from substituting Hawking temperature T in for the event horizon r_+ from (3.70). The second method is to compute the entropy from classical statistical mechanics and general relativity

$$\mathcal{S} = -\frac{dF}{dT}, \quad F \equiv -T\, Log\, Z_{Gravity}, \tag{3.85}$$

in which the classical limit has

$$Log\, Z_{Gravity} \approx -S_{E,BTZ}, \tag{3.86}$$

and therefore

$$\mathcal{S} = -\left(S_{E,BTZ} + T\frac{dS_{E,BTZ}}{dT} \right). \tag{3.87}$$

The the Euclidean action (3.14) in which the metric is given by (3.7) (in Gaussian normal coordinates) was computed to be

$$S_{E,BTZ} = -\frac{(\pi l)^2 T}{2G}, \tag{3.88}$$

and therefore the entropy is

$$\mathcal{S} = \frac{(\pi l)^2 T}{G}. \tag{3.89}$$

Finally, the Cardy formula (3.3) can be used directly. For two derivative gravity, $c = 3l/2G$ from (3.27) and the Virasoro generator L_0 for the non-rotating BTZ black hole are given by (3.54). Substituting (3.27) in for c and (3.54) in for L_0 into (3.3) immediately gives

$$\mathcal{S} = \frac{(\pi l)^2 T}{G}. \tag{3.90}$$

It has been shown that the Bekenstein-Hawking entropy of the non-rotating (electrically neutral) BTZ black hole from (3.85) agrees exactly with a direct computation of the entropy by treating the BTZ black hole as a semiclassical thermodynamic system as in (3.89). Furthermore, this black hole entropy exactly agrees with the entropy as computed by the Cardy formula (3.3).

This agreement between gravity and CFT is noteworthy, as it gives a state-counting interpretation of the BTZ black hole microstates in terms of high-energy states in a dual CFT. However, it turns out that both the Bekenstein-Hawking entropy and Cardy formula receive corrections. The former due to higher-derivative corrections to the gravity action as discussed at the beginning of section 3.3, plus from quantum effects in the bulk. The latter gains corrections to the density of states at large (but not infinite) energy. With that said, the nature of those corrections, both in gravity and in CFT, is beyond the scope of this thesis."

Since equation (3.3) was derived using the AdS/CFT correspondence (and can be derived solely from the CFT side as in [34]), it has thus been shown that the leading contribution to the BTZ black hole entropy can be computed either on the gravity side or the CFT side of the AdS/CFT correspondence. This historical result provides another successful application of the AdS/CFT correspondence.

Chapter 4

A Quantum Theory of Pure AdS_3 Gravity

In the previous chapter, it was left vague as to the exact nature of either theory pertaining to the AdS_3/CFT_2 correspondence. What is a theory of quantum gravity on AdS_3, and what gauge theory is it dual to? One example is the configuration of D-branes known as the D1-D5 system, which give rise to five-dimensional black holes with near-horizon geometry to $AdS_3 \times S^3$ [13, 15, 36, 37]. Its dual description is described by a sigma model whose target space is the moduli space of instantons on X (X is T^4 or $K3$). But one question that can be asked following these more complicated theories is: can a quantum theory of pure AdS_3 exist? And if so, what CFT is it dual to?

Simple models of quantum gravity with lower spacetime dimensions may be desired for study[1], and one such successful model is that of the $(1+1)$-dimensional dilaton[2] gravity with negative cosmological constant. This toy model is known as Jackiw-Teitelboim (JT) gravity, and has relevance in the $NAdS_2/NCFT_1$ correspondence[3]; the "N" refers to the name "nearly" AdS_2 and "nearly" CFT_1 [39]. A pure AdS_2 gravity theory turns out to not be consistent with any configuration with non-zero energy, since the variation of the metric imposes that the stress tensor of matter vanishes. In two dimensions the Einstein-Hilbert action is topological, which implies

[1]The hope is that they can provide insight into higher dimensional theories.

[2]A dilaton is a type of scalar particle which appears in theories with extra dimensions when the volume of the compactified dimensions varies, and first appeared as a radion (excitations of the $g_{\mu\nu}$) in the Kaluza-Klein theory (the unification of general relativity with classical electrodynamics).

[3]JT gravity is not actually dual to a single one-dimensional CFT, but rather an ensemble of them [38].

the Einstein-Hilbert action only depends on the genus[4] of the spacetime. As such, the scalar curvature does not contribute to the equation of motion for the metric. Therefore, instead one considers a "nearly" AdS2 theory of gravity with a dilaton coupled to gravity, the model that correctly captures a large number of situations where AdS_2 arises from a higher dimensional system [39]. In particular, the partition function of JT gravity has been shown to correspond to a genus expansion of a certain matrix integral [38]. More work done here and the $NAdS_2/NCFT_1$ correspondence can be found in [40, 39, 41].

If JT gravity provides a UV complete consistent quantum theory of gravity in $(1+1)$-dimensions while also being applicable to the $NAdS_2/NCFT_1$ correspondence, one may wonder if the number of dimensions can be raised to $(2+1)$-dimensions? To state again, does pure AdS_3 gravity exist as a consistent theory of quantum gravity in and of itself? The short answer is probably not[5], but it could still work as an effective field theory in which physical observables could be computed. In particular, AdS_3 gravity was found not to be dual to a two-dimensional CFT, but if it is consistent, it is dual to an ensemble which generalizes random matrix theory [11]. And on the gravity side of the correspondence, it has been shown in [9] that a quantum AdS_3 theory can be used to compute identity Virasoro conformal blocks[6] for large central charge c via a diagramatic expansion to one loop order; identity Virasoro conformal blocks serve as building blocks for correlation functions (and hence physical observables).

To explain the concept of identity Virasoro conformal blocks in slightly more detail, recall that−as a feature of the AdS/CFT Correspondence dictionary−coupling matter to the gravity side allows one to turn on sources for operators dual to the bulk matter fields. In this case, this amounts to inserting multi-local operators into the Alekseev-Shatashvili theory, and computing some expectation value[7] which encodes the gravitational contributions to bulk scattering processes. These gravitational contributions to bulk scattering have an independent mathematical definition, and are known as identity Virasoro conformal blocks.

The random matrix theory side of this holographic correspondence is beyond the scope of this thesis, so the remaining chapter will focus strictly on the AdS_3 gravity side. In order for this to be possible, one needs to solve quantum AdS_3 gravity exactly.

[4]The genus g can be thought of as the "number of holes" in a topological space, and is related to the topological invariant by the Euler characteristic.

[5]For the full rundown as to why it may or may not be, see [9].

[6]Previous work computing Virasoro blocks are found in [42] and [43].

[7]For example, $\langle O_{12}O_{34}\rangle$ where the subscripts indicate multiple positions $z_{1,2}$ and z_{34}.

One starts by first solving the classical theory, which was first done in [5, 44]. It turns out that classical AdS_3 gravity can be understood equivalently as a classical Chern-Simons gauge theory, and therefore one can then quantize the theory as a Chern-Simons theory[8]. A Chern-Simons theory is a $(2+1)$-dimensional topological gauge theory (developed in [46]) which has equations of motion of the form $F_{\mu\nu} = 0$[9]. This quantum theory of gravity turns out to be equivalent to the quantization of certain coadjoint orbits[10] of the Virasoro group; this quantization was first done by Anton Alekseev and Samson Shatashvili in [10].

As a result, the final quantum pure AdS_3 gravity action can be written equivalently as two copies of the phase space path integral quantization of the coadjoint orbit $Diff(S^1)/PSL(2;\mathbb{R})$[11]. The purpose of this chapter is to perform this derivation in full, with the end goal of obtaining the quantum theory of AdS_3 written as the "Alekseev-Shatashvili" action. The approach taken will primarily follow that given in [9].

4.1 Preliminaries

As discussed in chapter 3, the action[12] for global AdS_3 is given by the Einstein Hilbert action

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{2}{l^2}\right) + S_{bdy},$$

where l is the radius of the spacetime and S_{bdy} is some suitable boundary action. AdS_3 gravity has no propagating degrees of freedom, but does have edge modes[13]; the edge modes are called "boundary gravitons." These gravitons are generated by diffeomorphisms and Lorentz transformations which do not vanish at the boundary (while still preserving the boundary conditions).

In principle, to have a complete quantum theory one needs to be able to compute the full path integral (such as in (2.3)) which would also include summing over all metrics for all possible topologies[14]. However, in abandoning the ambition to writing

[8] For additional explorations of quantizing a Chern-Simons theory, see [45].

[9] A Yang-Mills gauge theory has equations of motion of the form $\partial_\mu F^{\mu\nu} = 0$.

[10] For a complete classification of the Virasoro orbits, see [47].

[11] Other coadjoint orbits correspond to adding matter content to the AdS_3 gravity theory [9].

[12] Stated in Lorentzian signature instead.

[13] This idea is discussed in greater detail in [5], but it's simple to show that there are no gravitational waves in lower dimensional theories of gravity.

[14] This is the case for JT gravity, but this idea was also briefly touched upon implicitly when using

down a fully UV complete theory, one can restrict the focus to specific topologies. In this case, consider summing over metrics on a solid cylinder. Indeed, one such example metric is global AdS_3 in (3.6)[15].

This choice turns out to be a good one because it has previously been shown that AdS_3 gravity on a cylinder is classically equivalent to a certain winding sector of $SO(2,2)$ Chern-Simons theory. This winding sector can be quantized[16], leading to an equivalent description in terms of two copies of Alekseev-Shatashvili theory[17] , which one can think of as the quantum field theory of the boundary gravitons.

This theory, which originated by quantizing the coadjoint orbit of the Virasoro group $Diff(\mathbb{S}^1)/PSL(2,\mathbb{R})$, directly corresponds with a boundary action in AdS_3 gravity[18]. From here, Wick-rotate and perform relevant identifications so that the bulk geometry is that of a solid torus (a disk time a time circle), the same as that of Euclidean BTZ, leading to a boundary effective action for the Euclidean black hole which can be easily quantized. Computations of the Euclidean path integral with this new boundary action have been shown to be 1-loop exact in [9].

4.2 Global AdS_3 Gravity as a Classical Chern-Simons Theory

As was just mentioned, Alekseev and Shatashvili performed their Virasoro quantization using a phase space path integral formulation [10]. Since phase space path integrals are dependent on the Hamiltonian, one requires the action to be written in “first-order” formulation[19]. There is additional motivation to utilize phase space path integrals over the usual Feynman approach, and this will be briefly discussed in chapter 5 when Euclidean wormholes will be the focus of discussion.

(3.63).

[15]There are of course infinitely many metrics one could write down which still can be placed on spacetimes with the topology of a cylinder, however.

[16]The natural choice of gauge theory corresponding to these diffeomorphisms is the double cover of $SO(2,2)$ given by $SO(2,1)_L \otimes SO(2,1)_R$ corresponding with the left and right asymptotic boundaries, respectively. However, this choice causes the left- and right-moving boundary gravitons to be completely decoupled [9].

[17]The two copies correspond with chirality, which will be discussed in 4.4.

[18]Because this quantum Chern-Simons theory's phase space is the same as the phase space for AdS_3 gravity, the two quantum theories are equivalent.

[19]That is, the action should only have a single time derivative such that the theory is in Hamiltonian form.

For now, simply choose to begin with the first-order formulation of AdS_3 gravity. To start, consider the line element for global AdS_3

$$ds^2 = -(r^2+1)dt^2 + r^2 d\theta^2 + \frac{dr^2}{r^2+1}, \tag{4.1}$$

where θ has the angular periodicity of 2π, and the boundary is located at $r \to \infty$. The time describes the length along the cylinder, and therefore constant time slices are disks. Next, decompose the metric into a dreibein by

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu, \tag{4.2}$$

where A, B are flat indices raised and lowered by the Minkowski metric η_{AB} and μ, ν are the bulk spacetime indices. Then, in introducing the spin-connection[20] $\omega^A_{B\mu}$ which satisfy

$$\omega_{ABC} = -\omega_{BAC},$$

the Einstein-Hilbert action (4.1) can be fully written in terms of these first-order variables

$$S = -\frac{1}{16\pi G}\int \epsilon_{ABC}\, e^A \wedge \left(d\omega^{BC} + \omega^B_{\ D} \wedge \omega^{DC} + \frac{1}{3} e^B \wedge e^C \right) + S_{bdy}, \tag{4.3}$$

where the "wedge" operator is a way of compactly writing an anti-symmetric tensor product

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu, \tag{4.4}$$

and the one-forms given by

$$e^A = e^A_\mu dx^\mu, \quad \omega^A_{\ B} = \omega^A_{\ B\mu} dx^\mu. \tag{4.5}$$

Varying this action with respect to the spin-connection give the torsion-free condition, and varying with respect to the dreibein gives Einstein's field equations; both together verify that (4.3) is equivalent to (4.1). Note also that this action only has a single time derivative, which implies it's in Hamiltonian form[21].

[20] A spin connection can be regarded as a gauge field for local Lorentz transformations.
[21] To recover the Lagrangian, simply perform a Legendre transformation.

The metric in (4.1) for global AdS_3 can be expressed in terms of the dreibein

$$e^0 = \sqrt{r^2+1}dt, \quad e^1 = rd\theta, \quad e^2 = \frac{dr}{\sqrt{r^2+1}}. \tag{4.6}$$

From here, one can solve the torsion-free condition for the spin-connection ω_{ABC}. Next, consider acting with diffeomorphisms and local Lorentz transformations[22] on the dreibein and spin connection describing global AdS_3. Under these transformations the drebein and spin connection transform as

$$e_\mu^A \to e_\mu^A + \left(\mathcal{L}_s e_\mu^A - V_B^A e_\mu^B\right), \tag{4.7}$$

and

$$\omega_{B\mu}^A \to \omega_{B\mu}^A + \left(\mathcal{L}_s \omega_{B\mu}^A + D_\mu V_B^A\right), \tag{4.8}$$

where $\mathcal{L}_s$ is the Lie derivative[23],

$$\mathcal{L}_s e_\mu^A = \xi^\nu \partial_\nu e_\mu^A + e_\mu^A \partial_\nu \xi^\nu. \tag{4.9}$$

Here, ξ^μ is a coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, V_B^A is some infinitesimal Lorentz transformation in $SO(2,1)$

$$\Lambda_B^A \to \delta_B^A + \epsilon V_B^A + ..., \tag{4.10}$$

and the covariant derivative is defined as

$$D_M V_B^A = \partial_\mu V_B^A + [\omega_\mu, V]_B^A = \partial_\mu V_B^A + \omega_{C\mu}^A V_B^C - V_C^A \omega_{B\mu}^C. \tag{4.11}$$

Equations (4.7) and (4.8) collect the diffeomorphisms and local Lorentz transformations which are symmetries of (4.3). In fact, notice that the covariant derivative given above in (4.11) behaves as if ω and V are non-Abelian gauge fields.

In the classical equivalence between AdS_3 gravity and Chern-Simons theory, the dreibein and spin connection terms are collected into the vector one-forms

$$A^A = \frac{1}{2}\epsilon^{ABC}\omega_{BC} + e^A, \quad \bar{A}_\mu^A = \frac{1}{2}\epsilon^{ABC}\omega_{BC} - e^A. \tag{4.12}$$

[22] Local Lorentz transformations $\Lambda(x^\mu)$ act on the flat indices, while the diffeomorphisms act on the spacetime indices.

[23] In ordinary three-dimensional space, the Lie derivative of a vector field $\vec{A}$ along a constant vector field $\vec{v}$ is given by $\mathcal{L}_s\vec{A} = (\vec{v}\cdot\vec{\nabla})\vec{A}$.

From here, notice that the infinitesimal element of $SO(2,1)$ comes from the Lie algebra $\mathfrak{so}(2,1)$,

$$\Lambda^A_B \to \delta^A_B + \epsilon\theta^C (J_C)^A_B, \tag{4.13}$$

where I identify $V^A_B = \theta^C (J_C)^A_B$ by comparing with (4.10). The generators $J_A \in \mathfrak{so}(2,1)$ satisfy

$$[J_A, J_B] = \epsilon_{ABC} J_C, \quad tr\,(J_A J_B) = \frac{1}{2}\eta_{AB}. \tag{4.14}$$

However, it is more common to write these in the fundamental representation of $\mathfrak{sl}(2;\mathbb{R})$

$$J_0 = -\frac{i}{2}\sigma_2, \quad J_1 = \frac{1}{2}\sigma_1, \quad J_2 = \frac{1}{2}\sigma_3 \tag{4.15}$$

where σ_A are the Pauli matrices[24]. Using these, the full gauge fields are defined by

$$A \equiv A^A J_A, \quad \bar{A} \equiv \bar{A}^A \bar{J}_A, \tag{4.16}$$

where $\bar{J}_A$ forms another copy of $\mathfrak{sl}(2,\mathbb{R})$ with the difference being that ϵ_{ABC} is defined to have the opposite sign in the commutator relation. These gauge fields are now 2×2 matrices, and allow the action (4.3) to be written as a classical Chern-Simons theory

$$S = -\frac{1}{64\pi G}\int \left(I[A] - I[\bar{A}]\right) + S_{bdy}, \quad I[A] = tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{4.17}$$

Since this is a Chern-Simons action, the equation of motion after varying the "gauge field" A sets the corresponding field strength tensor $F = 0$. In this case, noting that F is non-Abelian, this looks like

$$F = dA + A \wedge A = 0, \quad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = 0, \tag{4.18}$$

which implies that A and $\bar{A}$ are "flat-connections." These equations are equivalent to those of first-order gravity in (4.3), which was the torsion free condition along with the field equations. Classically, the diffeomorphisms and local Lorentz transformations associated with AdS_3 action (4.3) act in the same way as infinitsimal $SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$ gauge transformations on gauge fields A, and $\bar{A}$ in the Chern-Simons action (4.17). Phrased another way, A and $\bar{A}$ transform as independent $SO(2,1)$ connections under infinitesimal local Lorentz transformations and diffeomorphisms. In either case, the same equations of motion emerge in (4.17) and (4.3), and are therefore classically

[24] The two groups are related by the isomorphism $SL(2,\mathbb{R})/Z_2 = SO(2,1)$.

equivalent on-shell, as the latter can be mapped to the former.

4.3 Constraints and Boundary Conditions

Before quantizing the theory, write the gauge fields explicitly in terms of the spatial and temporal parts[25]

$$A = A_M dx^M = A_0 dt + \widetilde{A}_i dx^i, \quad \bar{A} = \bar{A}_0 dt + \widetilde{\bar{A}}_i dx^i, \tag{4.19}$$

where the $\widetilde{A}$ labels the spatial part in the absence of indices. Writing the gauge fields in this way, the full action given by

$$S = S[A] - S[\bar{A}] + S_{bdy}, \tag{4.20}$$

where now

$$S[A] = -\frac{k}{2\pi}\int_{\mathcal{M}} dt \wedge tr'\left(-\frac{1}{2}\widetilde{A} \wedge \dot{\widetilde{A}} + A_0 \widetilde{F}\right), \tag{4.21}$$

noting that $k = \frac{1}{4G}$, $\widetilde{F}$ is the spatial field strength

$$\widetilde{F} = \widetilde{d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A}, \tag{4.22}$$

$\widetilde{d}$ is the spatial derivative, and tr' is the trace taken in the fundamental representation of $SL(2;\mathbb{R})$. Furthermore, a good choice for the boundary action such that a good variational principle is established is

$$S_{bdy} = -\frac{k}{4\pi}\int_{\partial\mathcal{M}} d^2x \left(tr'(A_\theta^2) + tr'(\bar{A}_\theta^2)\right). \tag{4.23}$$

One then must ask: what are the boundary conditions? To discuss them, let us find the A and $\bar{A}$ corresponding to global AdS_3. These are computed from (4.16)

$$A = \frac{r^2+1}{d}x^+ J_0 + r dx^+ J_1 + \frac{dr}{\sqrt{r^2+1}} J_2,$$

$$\bar{A} = \frac{r^2+1}{d}x^- \bar{J}_0 + r dx^- \bar{J}_1 + \frac{dr}{\sqrt{r^2+1}} \bar{J}_2. \tag{4.24}$$

---

[25] Performing this step is well understood in Chern-Simons gauge theory, but this was only done for the first time in [9] within the context of AdS_3 gravity.

Then, in using the fact that $J_A \in \mathfrak{sl}(2;\mathbb{R})$, these can be written as 2×2 matrices

$$A = \begin{pmatrix} \frac{dr}{2\sqrt{r^2+1}} & -\frac{(\sqrt{r^2+1}-r)dx^+}{2} \\ \frac{(\sqrt{r^2+1}+r)dx^+}{2} & -\frac{dr}{2\sqrt{r^2+1}} \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -\frac{dr}{2\sqrt{r^2+1}} & -\frac{(\sqrt{r^2+1}+r)dx^-}{2} \\ \frac{(\sqrt{r^2+1}-r)dx^-}{2} & \frac{dr}{2\sqrt{r^2+1}} \end{pmatrix}. \tag{4.25}$$

Next, sending $r \to \infty$ updates these to [26]

$$A = \begin{pmatrix} \frac{dr}{2r} + \mathcal{O}\left(\frac{1}{r}\right) & \mathcal{O}\left(\frac{1}{r^2}\right) \\ rdx^+ + \mathcal{O}\left(\frac{1}{r^2}\right) & -\frac{dr}{2r} + \mathcal{O}\left(\frac{1}{r}\right) \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -\frac{dr}{2r} + \mathcal{O}\left(\frac{1}{r}\right) & -rdx^- + \mathcal{O}\left(\frac{1}{r^2}\right) \\ \mathcal{O}\left(\frac{1}{r^2}\right) & \frac{dr}{2r} + \mathcal{O}\left(\frac{1}{r}\right) \end{pmatrix}. \tag{4.26}$$

By construction, the way these gauge fields asymptote at the boundary successfully cause the variation of the full action to vanish. That is, the boundary conditions are applied to A and $\bar{A}$, which are then integrated over such that the action vanishes

$$\delta S = -\frac{k}{\pi} \int_{\partial \mathcal{M}} d^2x \left(tr'(A_- \delta A_\theta) + tr'(\bar{A}_+ \delta \bar{A}_{theta}) \right) = 0, \tag{4.27}$$

where

$$A_\pm = \frac{1}{2}\left(A_0 \pm A_\theta\right). \tag{4.28}$$

This is evident by the fact that at the boundary

$$\delta A_\theta \propto \mathcal{O}\left(\frac{1}{r}\right), \quad A_\pm \propto \mathcal{O}\left(\frac{1}{r}\right), \tag{4.29}$$

implying that

$$tr'(A_- \delta A_\theta) \propto \mathcal{O}\left(\frac{1}{r^2}\right) \to 0, \tag{4.30}$$

and similarly for $\bar{A}$.

Since the equation of motion has it be that the field strength tensor vanishes,

$$F = dA + A \wedge A = 0, \tag{4.31}$$

a choice for A such that this equation is at least locally satisfied is[27]

$$A = g^{-1}dg, \quad \bar{A} = \bar{g}^{-1}d\bar{g}, \tag{4.32}$$

[26] Higher-order terms $\mathcal{O}(r^{-n})$ can be dropped for the purposes of this analysis.

[27] It's possible that $F = 0$ isn't true everywhere, implying the existence of a delta function (due to some source) at some location.

for $g(\vec{x},t) \in SL(2;\mathbb{R})$. For the profiles of A and $\bar{A}$ in (4.25) describing global AdS$_3$, a representation for g and $\bar{g}$ is

$$g = \begin{pmatrix} \rho\, cos\left(\frac{x^+}{2}\right) & -\rho^{-1}\, sin\left(\frac{x^+}{2}\right) \\ \rho\, sin\left(\frac{x^+}{2}\right) & \rho^{-1}\, cos\left(\frac{x^+}{2}\right) \end{pmatrix}, \tag{4.33}$$

and

$$\bar{g} = \begin{pmatrix} \rho\, cos\left(\frac{x^-}{2}\right) & -\rho^{-1}\, sin\left(\frac{x^-}{2}\right) \\ \rho\, sin\left(\frac{x^-}{2}\right) & \rho^{-1}\, cos\left(\frac{x^-}{2}\right) \end{pmatrix}, \tag{4.34}$$

where $x^\pm = \pm t + \theta$ and

$$\rho = \sqrt{\sqrt{r^2+1} + r}. \tag{4.35}$$

However, g is double valued[28]. That is, in going around the circle from $\theta \to \theta + 2\pi$, $g \to -g$. To see this, fix time $t = 0$ and compare $\theta = 0$ and $\theta = 2\pi$

$$g\Big|_{t=0,\theta=0} = \mathbb{I}, \quad g\Big|_{t=0,\theta=2\pi} = -\mathbb{I}.$$

Because g and $\bar{g}$ are double valued, the Wilson loop of A and $\bar{A}$ in the fundamental representation of $SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$ is non-trivial

$$\mathcal{P}e^{\int_0^{2\pi} d\theta A_\theta} = -\mathbb{I}, \quad \mathcal{P}e^{\int_0^{2\pi} d\theta \bar{A}_\theta} = -\mathbb{I}, \tag{4.36}$$

and hence $F = 0$ is only satisfied locally; A and $\bar{A}$ are singular as $SL(2;\mathbb{R})$ connections [9, 5].

Because the starting point was global AdS$_3$, it should be required that the gauge configuration is non-singular everywhere. This can be achieved by a suitable choice of the gauge group; there are two ways of doing this. One choice is to take the global gauge group to be

$$SL(2;\mathbb{R})/\mathbb{Z}_2 \times SL(2;\mathbb{R})/\mathbb{Z}_2 = PSL(2;\mathbb{R}) \times PSL(2;\mathbb{R}) = SO(2,1) \times SO(2,1),$$

which identifies $g \sim -g$ and $\bar{g} \sim -\bar{g}$. In this way, demanding that AdS3 be described by a non-singular gauge configuration forces the gauge group to be $SO(2,1) \times SO(2,1)$

[28] A basic example of a double valued function is $f(x) = \pm\sqrt{x}$, because any value of x gives two values of $f(x)$. Indeed, since 0 and 2π are at the same location on the circle, the two values of g imply that g is double valued.

[9]. However, it was clarified in [11] that

$$(SL(2;\mathbb{R}) \times SL(2;\mathbb{R}))\,/\mathbb{Z}_2 = SO(2,2),$$

is a better choice since in this case the boundary symmetries are directly related. However, for pedagogical purposes, the former choice will be utilized for the remainder of this chapter[29].

4.4 Quantizing AdS$_3$ Gravity and Alekseev-Shatashvili Theory

It's actually the case that the full action (4.20) is really a constrained system, and in fact reduces to a boundary action. This is evident by the fact that the fields A_0 and $\bar{A}_0$ appear as Lagrange multipliers. That is, one can integrate out A_0 and $\bar{A}_0$, which enforces the constraints $\widetilde{F} = 0$ (and similarly for the barred equation).

In the previous section it was discussed A and $\bar{A}$ can be written as (4.32) since they are flat connections. This is similarly true for the spatial connections $\widetilde{A}$ and $\widetilde{\bar{A}}$, which are the flat connections on the disk. The spatial connections can be parameterized as

$$\widetilde{A} = g^{-1}\widetilde{d}g, \quad \widetilde{\bar{A}} = \bar{g}^{-1}\widetilde{d}\bar{g}, \tag{4.37}$$

for $g \in PSL(2;\mathbb{R})$. However, it turns out this decomposition is redundant. The same spatial connection for A can be obtained from either $g(\vec{x},t)$ or $h(t)g(\vec{x},t)$ with $h(t) \in PSL(2;\mathbb{R})$. Yet for now, just consider $g \in PSL(2,\mathbb{R})$.

The now constrained action (4.20) can be written in terms of g, which is the difference of chiral Weiss-Zumino-Witten (WZW) actions [45, 9]

$$S = S_-[g] + S_+[\bar{g}], \tag{4.38}$$

where

$$S_\pm[g] = \frac{k}{2\pi}\left(\int_{\partial\mathcal{M}} d^2x tr'\left((g^{-1})'\partial_\pm g\right) \mp \frac{1}{6}\int_{\mathcal{M}} tr'\left(g^{-1}\wedge dg \wedge g^{-1}dg \wedge g^{-1}dg\right)\right), \tag{4.39}$$

[29]The difference between the two contains an extra global subtlety which is invisible in perturbation theory.

noting that $\partial_\pm = \frac{1}{2}(\partial_\theta \pm \partial_t)$ and $'$ is the angular derivative ∂_θ. A chiral WZW action describes a two-dimensional (almost) conformal field theory invariant under the affine Lie algebra. An affine Lie algebra is an infinite dimensional algebra constructed out of the conserved currents associated with the finite Lie algebra[30].

In this case, the Lie group is $PSL(2;\mathbb{R})$, and therefore associated Noether currents can be computed $j = j^A J_A$. The currents are expanded in modes

$$j^A(z) = \sum_{n\in\mathbb{Z}} j^A_n z^{-n-1}, \tag{4.40}$$

where z is a holomorphic coordinate on infinite plane. With these currents, the affine Lie algebra can be constructed. The commutation relations of the affine Lie algebra are therefore

$$[j^A_m, j^B_n] = f_{ABC} j^C_{n+m} + cn\delta^A_B \delta_{m+n,0}. \tag{4.41}$$

where "c" is the centrally extended charge that commutes with the other elements of the affine algebra. This commutation relation is the chiral symmetry algebra associated with the left-moving currents; the second copy of the affine algebra produced by the $\bar{j}_A$'s give the chiral symmetry algebra associated with the right-moving currents.

Going back to (4.38), let's write the g's in terms of Gauss parameterizations as in [45, 9]

$$g = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}, \tag{4.42}$$

and

$$\bar{g} = \begin{pmatrix} 1 & -F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\lambda}^{-1} & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Psi & 1 \end{pmatrix}, \tag{4.43}$$

where λ, F, and Ψ are all real scalar fields[31]. These are then substituted into (4.37),

$$\widetilde{A} = \begin{pmatrix} \widetilde{d}ln\,\lambda - \Psi(\lambda^2 \widetilde{d}F) & 2\Psi\widetilde{d}ln\,\lambda + \widetilde{d}\Psi - \Psi^2(\lambda^2\widetilde{d}F) \\ \lambda^2\widetilde{d}F & -\widetilde{d}ln\,\lambda + \Psi(\lambda^2\widetilde{d}F) \end{pmatrix}, \tag{4.44}$$

and similarly for $\bar{A}$. It now remains to impose the boundary conditions, which will fix λ and Ψ in terms of r and F. This is done by sending (4.44) to the boundary

[30] The Lie algebra must be "semi-simple."

[31] The F here is not the field strength tensor.

$(r \to \infty)$ and comparing with (4.26). For λ, equate the bottom left matrix entries

$$rdx^+ = rd\theta = \lambda^2 \widetilde{d}F,$$

$$\Rightarrow rd\theta = \lambda^2 F' d\theta,$$

$$\therefore\ \lambda = \sqrt{\frac{r}{F'}}. \tag{4.45}$$

And to determine Ψ, matching the diagonal components finds

$$\Psi = -\frac{F''}{2rF'}. \tag{4.46}$$

It can be similarly worked out for the barred fields that

$$\bar{\lambda} = \sqrt{\frac{r}{\widetilde{\bar{F}}}}, \quad \bar{\Psi} = -\frac{\bar{F}''}{2r\bar{F}'}. \tag{4.47}$$

At the boundary, the field F can be parameterized as

$$F\Big|_{\partial} = tan\left(\frac{\phi}{2}\right),$$

where ϕ is an angular variable $\phi(\theta, t)$. In section 4.3, it was enforced that group elements $g \in SL(2;\mathbb{R})$ were single valued by taking the quotient with respect to $\mathbb{Z}_2$ such that now $g \in PSL(2;\mathbb{R})$. Additionally, include the winding boundary condition $\phi(\theta + 2\pi, t) \sim \phi(\theta, t) + 2\pi$. This boundary condition is imposed to account for the same property in the angular coordinate for global AdS_3. Therefore, at fixed time, $\phi \in Diff(\mathbb{S}^1)$. However, as mentioned, there is a redundancy in the description; either $g(\vec{x}, t)$ or $h(t)g(\vec{x}, t)$ for $h(t) \in PSL(2, \mathbb{R})$ describes the flat connection for $\widetilde{A}$ and $\widetilde{\bar{A}}$. In using $g(\vec{x}, t) \to h(t)g(\vec{x}, t)$, the Gauss parameters transform like

$$\lambda \to (cF + d)\lambda, \quad \Psi \to \frac{c\lambda^{-2}}{cF + d}, \quad F \to \frac{aF + b}{cF + d}, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.48}$$

Substituting these into the spatial connection (4.44) will produce consistent boundary terms for λ, Ψ, and F as worked out above. Under this transformation the boundary

F is mapped as

$$F\big|_\partial = tan\left(\frac{\phi(\theta,t)}{2}\right) \to \frac{a(t)tan\left(\frac{\phi(\theta,t)}{2}\right)+b(t)}{c(t)tan\left(\frac{\phi(\theta,t)}{2}\right)+d(t)}. \tag{4.49}$$

That is, at fixed time ϕ is more precisely an element of $Diff(\mathbb{S}^1)/PSL(2;\mathbb{R})$, which is a coadjoint orbit of the Virasoro group[32]. Plugging in (4.42) and (4.43) into (4.38), the action is written in terms of the Gauss parameters

$$S_\pm[g] = -\frac{k}{\pi}\int_{\partial\mathcal{M}} d^2x \left(\frac{\lambda'(\partial_\pm\lambda)}{\lambda^2} + \lambda^2 F'(\partial_\pm\Psi)\right). \tag{4.50}$$

Yet, since this is just a boundary action, the Gauss parameters at the boundary can be used instead, including the redefinition of $F\big|_\partial$. This results in an action only in terms of ϕ

$$S_\pm[\phi] = -\frac{c}{24\pi}\int d^2x \left(\frac{\phi''\partial_\pm\phi'}{\phi'^2} - \phi'\partial_\pm\phi\right), \quad c = 6k = \frac{3}{2G}. \tag{4.51}$$

The left-moving S_+ is exactly the Alekseev-Shatashvili quantization of $Diff(\mathbb{S}^1)/PSL(2;\mathbb{R})$ [10], and the coupling constant c is the Brown-Henneaux central charge corresponding with two-derivative gravity as discussed in chapter 3 (with $l = 1$). Thus, the Chern-Simons description of AdS_3 gravity on a cylinder with gauge group symmetry $SO(2,2)$ is quantum mechanically equivalent to two copies of the phase space path integral quantization of $Diff(\mathbb{S}^1)/PSL(2;\mathbb{R})$[33].

4.5 Euclidean AdS_3 and Black Holes

Before concluding this chapter, there is some physics which can be extracted from this model as a quick check that it does correctly describe asymptotic AdS_3 spacetimes. In particular, a description was found for black hole thermodynamics in chapter 3 by starting with some quantum theory of gravity in which global AdS_3 gravity and the non-rotating BTZ black hole were saddle points of the gravity path integral. Since the present chapter is examining a quantum theory of global AdS_3, one can wonder if the Cardy-like behavior remains.

[32] The "first exceptional orbit" to be more specific [9].

[33] They both share the same phase space as JT gravity on the hyperbolic disk as well [9].

Upon Wick-rotating the global AdS_3 metric goes to (3.6) (with the identification $\theta \sim \theta + 2\pi$), and the left-moving Euclidean action pertaining to the quantum theory of global AdS_3 given in (4.51) is now written as

$$S_+[\phi] = \frac{c}{24\pi} \int d^2x \left(\frac{\phi'' \bar{\partial} \phi'}{\phi'^2} - \phi' \bar{\partial} \phi \right), \tag{4.52}$$

where

$$\bar{\partial} = \frac{1}{2} \left(\partial_\theta + i \partial_y \right). \tag{4.53}$$

The right-moving portion is the same, but with the change $\bar{\partial} \to \partial$ (the complex conjugate). Then, similarly to what was done in chapter 3, make the further identification $y \sim y + 2\pi Im[\tau]$ and $\theta \sim \theta + 2\pi Re[\tau]$ such that the spacetime is now on a torus of complex structure τ. This modifies the boundary conditions to[34]

$$\begin{aligned} \phi(\theta + 2\pi, y) &= \phi(\theta, y) + 2\pi \\ \phi(\theta + 2\pi Re[\tau], y + 2\pi Im[\tau]) &= \phi(\theta, y). \end{aligned} \tag{4.54}$$

Then, in just considering the left-moving (holomorphic) part of the action, varying with respect to the field ϕ finds the equation of motion

$$\bar{\partial} T = 0, \tag{4.55}$$

where T is the boundary stress tensor

$$T = -\frac{c}{12} \{ tan \left(\frac{\phi}{2} \right), \theta \}, \tag{4.56}$$

and is computed from (3.49)[35]. After imposing the boundary conditions in (4.54), the unique solution for the boundary ϕ_0 is found to be [9]

$$\phi_0 = \theta - \frac{Re[\tau]}{Im[\tau]} y. \tag{4.57}$$

From here, following section 3.3, the action(s) evaluated at the boundary $\phi = \phi_0$ (and

[34]The first line in this equation was previously true as well; the second line is the new boundary condition.

[35]The leading order term is the Casimir energy of the circle [2, 9].

$y \to 2\pi i\tau$ (or $\bar{\tau}$) give the saddle point approximation of the torus partition function[36]

$$-Log\, Z = S_+[\phi_0] + S_-[\phi_0] + ..., \tag{4.58}$$

or rather

$$-Log\, Z = \frac{\pi i c}{12}(\tau - \bar{\tau}) + ..., \tag{4.59}$$

in which the higher-order terms are loop corrections. It turns out that this partition function for AdS_3 on a torus can be computed exactly[37], and the above equation is its large $c >> 1$ limit. And indeed, like thermal AdS_3 in (3.69), this term in the partition function dominates at low temperatures $Im[\tau] \to \infty$.

Recall that the non-rotating BTZ black hole saddle point contribution to the gravity partition function is equivalent to the thermal AdS_3 contribution under the transformation $\tau \to -1/\tau$. Therefore, applied to this versions of the AdS_3 theory, this is equivalent to imposing the boundary conditions

$$\begin{aligned} \phi(\theta + 2\pi, y) &= \phi(\theta, y) \\ \phi(\theta + 2\pi Re[\tau], y + 2\pi Im[\tau]) &= \phi(\theta, y) + 2\pi, \end{aligned} \tag{4.60}$$

and this time the unique solution for the boundary ϕ_0 is

$$\phi_0 = \frac{y}{Im[\tau]}. \tag{4.61}$$

This gives the saddle point approximation of the BTZ black hole[38]

$$Log\, Z = \frac{\pi i c}{12}\left(\frac{\tau - \bar{\tau}}{|\tau|^2}\right).... \tag{4.62}$$

If τ was purely imaginary, note that this is of the same form as (3.75). Indeed, in the high temperature limit $Im[\tau] \to 0$, this saddle point dominates the partition function, and will therefore correspond with the black hole entropy computed in the Cardy formula (3.3). Evidently, even if this quantum theory of AdS_3 is not a fully consistent one, computations can still be made with it that do correspond with real

[36]It is required to have $c >> 1$, otherwise this action would not dominate the partition function.
[37]See [9] for the computation.
[38]Recall that it was never necessary to change metrics during this process. In chapter 3, it was shown that the BTZ black hole metric was completely equivalent to global AdS_3 in (3.72) by coordinate transformations. Modifying the boundary conditions is enough to make this change from thermal AdS_3 to the BTZ black hole.

physics.

Chapter 5

The Eternal Traversable AdS_3 Wormhole

This chapter can be divided into two categories. The first will introduce the concept of AdS_3 wormholes: what they are, why they appear, and what kind of physics do they provide for a quantum theory of AdS_3 gravity. These sections will follow primarily from [11] and [7], with the end goal of constructing a boundary action with an eternal traversable wormwhole solution (this action will be examined in the second category of this chapter). Specifically, the Euclidean boundary action will be quickly derived using arguments similar to that in the previous chapter[1], and the Lorentizan action follows by analytically continuing back to real time. The Lorentzian action will also contain a double trace deformation which connects the two boundaries by a series of light matter fields[2] which "hold open" the throat. This action has a saddle point, an eternal traversable wormhole, which will turn out to not violate causality[3]. This action is the AdS_3 version of similar work done in [4] which studied a deformation on JT gravity and the SYK model.

The second category of this chapter will present the new work done in [8]. Because this wormhole is traversable, the two boundaries are causally connected, and one can therefore send particles through the bulk from one end to the other. This process can be studied purely from the boundary action for the wormhole by looking at the spectrum of fluctuations around the wormhole along with the associated propagator.

[1] For specifics, see [11].

[2] "Light" here means that the mass of the fields are of order $\mathcal{O}(1)$ in units of the AdS radius.

[3] The wormhole solution actually is supported by NEC-violating matter. The way this works is that each matter field only violates a small $\mathcal{O}(\hbar)$ amount of energy, but is supported by are many matter fields.

5.1 Context for AdS_3 Wormholes

AdS_3 gravity and JT gravity share some common features[4]. For one, both theories do not have propagating degrees of freedom; the theories only have edge modes. Another is that the Schwarzian path integral of JT gravity on a disk can be written in terms of a field $\phi \in Diff(S1)/PSL(2;\mathbb{R})$ [11]. The phase space of AdS_3 gravity where the constant time slice is a disk (the AdS_3 theory in chapter 4) is twice the size of the configuration space of the JT theory. And yet an additional interesting feature of JT gravity is that a genus expansion of the gravity path integral includes in its leading-order term a sum of geometries smoothly connecting asymptotic regions (filling out the bulk geometry which satisfy the spacetime boundary conditions). In particular, the leading-order term of these off-shell contributions has a boundary of two disks, with the geometry being smoothly connected by a throat; this describes a Euclidean wormhole. Therefore, given these similarities between JT gravity and AdS_3 gravity, are off-shell wormholes also present in the AdS_3 theory? If so, what do they imply?

The short answer to the first question is yes, but that is not interesting in and of itself. For one, in the context of the AdS/CFT correspondence, almost all known wormholes are known to be unstable[5] [49, 11]. However, it was stated in the previous chapter that the quantum AdS_3 gravity theory may not be a consistent quantum theory. Specifically, Maloney and Witten computed the leading contribution to the torus partition function[6] of this quantum AdS_3 gravity theory in [6]. They did so by summing over saddle points of the kind discussed in the previous chapter, plus the metrics continuously connected to them[7]. Their results implied that the density of states is not strictly non-negative [50][8]. However, it was proposed in [51] that including non-saddle point contributions could cause the density of states to become non-negative. These non-saddle points would asymptote to a torus boundary, although they may not have solid torus topologies as discussed in the previous chapter.

The work in [11] suggests that AdS_3 is not in fact dual to a single two-dimensional CFT, but instead a statistical ensemble analogous to the holographic duality between JT gravity and a matrix model. This was achieved by first computing the partition function of AdS_3 gravity placed on spactetimes which are topologically a torus times

[4] At least, when it comes to their classical solutions.

[5] For a discussion on the taxonomy of wormhole stability, see [48].

[6] A torus partition function is the CFT dual to the gravity path integral. Recall that the CFT ''lives" on the AdS boundary, which in this case is the torus of complex structure.

[7] Although all metrics had the topology of a solid torus.

[8] Among other issues discussed in [6, 11].

a time interval $\mathbb{T}^2 \times I$ (or equivalently, an annulus times a time circle). That is, these are Euclidean wormholes which smoothly connect two asymptotic regions where each has a torus boundary, and hence is expected to be the leading contribution of the non-saddle points in the AdS_3 gravity path integral.

The holographic dictionary in this case relates summing over geometries with multiple boundaries, and from there computing the associated correlation functions. Then, for a geometry with two torus boundaries, one sums over all possible bulk geometries with these boundaries, and therefore computes the connected two point-function

$$\langle Z(\tau_1)Z(\tau_2)\rangle_{con} = \langle Z(\tau_1)Z(\tau_2)\rangle - \langle Z(\tau_1)\rangle\langle Z(\tau_2)\rangle, \tag{5.1}$$

which goes like [11]

$$\langle Z(\tau_1)Z(\tau_2)\rangle_{con} = Z_{\mathbb{T}^2\times I} + ..., \tag{5.2}$$

where the leading-order contribution is a single Euclidean wormhole geometry (a throat connecting the two torus boundaries[9]). After the torus partition function, this is the next simplest partition function of AdS_3 gravity which one can compute (saddle points). And in showing evidence for this theory of quantum AdS_3 gravity having a holographic dual to a statistical ensemble, it implies that the former theory of AdS_3 gravity on a solid torus could be made consistent if the non-saddle points were accounted for. Note that the reason why the AdS_3 theory was initially placed on a solid torus rather than the annulus times a time circle is simply that global AdS_3 naturally takes the topology of a cylinder (and therefore thermal AdS_3 is a torus).

5.2 Double Trace Deformation

Regardless as to whether or not pure AdS_3 gravity is a fully consistent quantum theory, the fact there exists a Euclidean wormhole should have some physical consequence. To be more specific, are the associated Lorentzian (traversable) wormholes physically allowed? Traversable wormholes have long been an interest as a source of long distance transportation. However, traversable wormhole configurations require matter which violates the average null energy condition (ANEC). An outline of the various types of energy conditions are not particularly relevant to the discussion at hand, but generically the idea is that the energy density of any region of space should be classically non-negative (aside from Casimir effects). Therefore, the question of

[9] These boundaries are the BTZ black hole solutions of AdS_3 gravity [9, 11].

"how does one hold the wormhole open" arises, since classically there does not seem to be any mass of negative null energy density. Furthermore, imagine the scenario in which one applies a Lorentz boost to one end of a wormhole. One could attempt to create a configuration with closed time-like curves (CTCs). That is, a traversable wormhole implies the existence of time machines, and hence violations of causality. If the two boundaries of the spacetime could be causally connected, this must not allowed to be the case.

Starting from a two-sided black hole in AdS_{d+1}, it was found in [7] that, through the modification of boundary conditions on a large number of bulk matter fields, certain interactions which couple the two boundaries could be added to the theory. This leads to a bulk stress tensor with negative energy, and therefore a controlled violation of ANEC which was also enough to support a wormhole without introducing any CTCs or violating causality[10]. Hence, they rendered a wormhole traversable after gravitational backreaction[11].

The way boundary conditions of the bulk matter fields are modified (which changes the metric at the classical level) was by "turning on" a coupling. This coupling is what is known as a double trace deformation. Generically, a double trace goes as follows [7]

$$S_{DT} = \sum_i \int dt d^{d-1} x h(t,x) O_R^i(t,x) O_L^i(t,x), \tag{5.3}$$

with deformation $h(t,x)$, the scalar operator O has dimension less than $d/2$, and is dual to a scalar field on the gravity side. Forming a traversable wormhole from a black hole with two asymptotically AdS regions is a simple example considered by [7], while an eternal traversable wormhole was constructed in JT gravity placed on a strip in [4]. Time runs the length of the strip and the two boundaries of the strip are each asymptotic AdS boundaries; this configuration immediately does not contain any CTCs. This wormhole was constructed by introducing a new boundary interaction term to the action with a sum of N operators with conformal dimension Δ. As per the AdS/CFT correspondence, the bulk theory has at least N matter fields, with an interaction across the two boundaries, and these matter fields directly correspond with operators on the boundary. This interaction naturally produces a finite amount of negative energy in the bulk via the Casimir effect, which forms an

[10]More accurately, no one has yet found CTCs in this setting.

[11]Actually, a signal sent down the wormhole was found to be scrambled by gravitational backreaction before making it to the other side in [52].

eternal traversable wormhole.

For our case (inspired by [4][12]), there is a map between the coordinate boundary time t and boundary position x with the global spacetime scalar fields $\Phi_1(x,t), \Phi_2(x,t), \bar{\Phi}_1(x,t)$, and $\bar{\Phi}_2(x,t)$[13]. Note that in addition to the right- and left-moving boundary fields (Φ and $\bar{\Phi}$) as discussed in the previous chapter, there are also two different boundaries to account for (denoted by the subscripts). The double trace deformation leads to an interaction between the reparameterization degrees of freedom on the two boundaries given by

$$S_{DT} = \frac{C}{24\pi}\int dt\, dx\, \eta \left(\left[\frac{\Phi_1'\Phi_2'}{cosh^2\left(\frac{\Phi_1-\Phi_2}{2}\right)} \right]^h \times \left[\frac{\bar{\Phi}_1'\bar{\Phi}_2'}{cosh^2\left(\frac{\bar{\Phi}_1-\bar{\Phi}_2}{2}\right)} \right]^{\bar{h}} \right), \tag{5.4}$$

where η describes the interaction strength[14]; the fields Φ_1 and Φ_2 will be introduced in the next section. The h and $\bar{h}$ are right- and left- moving scaling weights of the operators dual to the bulk matter field, and are related to its conformal dimension and spin by

$$h = \frac{\Delta + s}{2}, \quad \bar{h} = \frac{\Delta - s}{2}. \tag{5.5}$$

And finally, the overall $\frac{C}{24\pi}$ factor is a choice which follows the derivation of the Euclidean boundary action for the wormhole found in [11]. The following section will present and motivate its construction.

5.3 Constructing the Boundary Action for the Wormhole

The goal at hand is to construct an AdS_3 traversable wormhole, which will be divided into two subsections. In the first, I will reconstruct the boundary action analogous to the Alekseev-Shatashvili theory discussed in the previous chapter. This boundary action was used to study Euclidean wormholes in [11], which will be a good starting

[12]They start with the following double trace deformation: $S_{DT} = g\sum_i \int du O_L^i(u) O_R^i(u)$, where u is the boundary time and g is a coupling constant.

[13]This is analogous to the relationship between proper boundary time and global times at the two boundaries in [4].

[14]It is related to a coupling constant g, conformal dimension Δ, and number of matter fields N; see [8] and [4].

point for the work done in [8]. Although the full derivation will not be presented[15], this section will walk through many of the steps and arguments presented in [11] to obtain this AdS_3 boundary action. In the second subsection, I will analytically continue to real time and add the double trace deformation (5.4).

5.3.1 The Euclidean Boundary Action

Euclidean thermal AdS_3 has the line element[16]

$$ds^2 = cosh^2(\rho)dy^2 + sinh^2(\rho)dx^2 + d\rho^2, \tag{5.6}$$

in which the following identifications are made: $z \sim z + 2\pi$[17] and $z \sim z + 2\pi\tau$ where[18] $z = x + iy$. Although this metric space has the topology of a solid torus with a boundary torus of complex structure τ, the topology of the overall theory will be taken to be that of a torus times a time interval $\mathbb{T}^2 \times I$ instead. To account for both boundary i of complex structure τ_1 and τ_2, it is useful to redefine the coordinates

$$x' \equiv x - \frac{Re[\tau_{1,2}]}{Im[\tau_{1,2}]}\, y, \quad y' \equiv \frac{y}{Im[\tau_{1,2}]}, \tag{5.7}$$

where $\tau_{1,2}$ applies to the specified boundary torus. In doing this, the identification $z \sim z + 2\pi$ (still) implies that $(x', y') \sim (x' + 2\pi, y')$, while the identification $z \sim z + 2\pi\tau_{1,2}$ implies $(x', y') \sim (x', y' + 2\pi)$. In this way, dealing with the two separate boundary tori of respective complex structures $\tau_{1,2}$ can now be dealt with in a symmetric way. That is, their boundary conditions are of an identical form (this will be made apparent later in the section). For the remainder of the chapter, coordinates x and y are defined in (5.7)[19].

The way this new topology $\mathbb{T}^2 \times I$ will be implemented into the boundary action will become apparent shortly, but for now the reason for this choice of topology is that the Eucliean wormhole of interest will have the equivalent topology: an annulus times S^1. Here, S^1 plays the role of time, which is important in that similar manipulations as in chapter 4 can be performed on the boundary theory. Indeed, the procedure to quantize gravity on this space largely follows that discussed in chapter 4, but now

[15] For the sake of brevity; see [11] for the full discussion.

[16] This really is just (3.6) with slightly different identifications.

[17] Equivalently, $x \sim x + 2\pi$

[18] Equivalently, $x \sim x + 2\pi Re[\tau]$ and $y \sim y + 2\pi Im[\tau]$.

[19] That is, they should be understood to be x' and y'.

with two key differences.

To set-up the first, recall that the quantized AdS_3 theory only has edge modes, and therefore reduces to a boundary action. Yet, with the annulus times time topology of the spacetime, the theory now contains two torus boundaries which are connected via a shared bulk. Therefore, action now has two boundary components[20]. As a result, two boundary terms S_{bdy} need to be added to the action so as to have a good variational principle. That is, the Euclideanized gravity action (as a Chern-Simons theory) is [11]

$$S_E = -\frac{ik}{4\pi}\int d^3x\epsilon^{ij}tr\left(-A_i\partial_y A_j + A_y F_{ij}\right) - (A \to \bar{A}) + S_{E,bdy}, \tag{5.8}$$

where the boundary action now includes both boundaries[21]

$$S_{E,bdy} = \frac{ik}{4\pi}\left(\int_{\rho\to\infty} d^2xtr\left(\bar{\tau}_1 A_x^2 - \tau_1\bar{A}_x^2\right) + \int_{\rho\to-\infty} d^2xtr\left(\bar{\tau}_2 A_x^2 - \tau_2\bar{A}_x^2\right)\right). \tag{5.9}$$

The quantization of this theory then largely follows the same kind of procedure done in the previous chapter, but with one remaining difference. Recall the spatial gauge fields can be written in (4.37) as

$$\widetilde{A} = g^{-1}\widetilde{d}g, \quad \widetilde{\bar{A}} = \bar{g}^{-1}\widetilde{d}\bar{g},$$

since one can integrate out A_y and $\bar{A}_y$ by enforcing the fact that the spatial connections are flat. However, now that a constant time slice is an annulus, there are more general flat connections parameterized by holonomies[22] which can be partially gauge-fixed with single-valued $\widetilde{g}$

$$g = e^{b(y)xJ_1}\widetilde{g}, \quad \bar{g} = e^{\bar{b}(y)x\bar{J}_1}\widetilde{\bar{g}}, \tag{5.10}$$

and the redundancy $g(x,y,\rho) \to h(y)g(x,y,\rho) \in SL(2;\mathbb{R})$

$$h = e^{a(y)J_1}, \quad \bar{h} = e^{\bar{a}(y)\bar{J}_1}, \tag{5.11}$$

[20]As a whole, the two boundaries being connected by a shared bulk makes up the wormhole geometry.

[21]The dreibein is still created from thermal AdS_3; the new two-boundary topology is accounted for with these new two boundary terms.

[22]A flat connection with a holonomy implies a gauge configuration which is singular if the holonomy is around a contractible circle. In the case of the wormhole, there is no contractible coordinates, and so no singularity arises.

in which $J_1 \in \mathbf{sl}(2;\mathbb{R})$ from (4.15). These $a(y)$ and $b(y)$ parameters can be collected in the fields $\phi, \bar{\phi}$ in the following combinations

$$\Phi(x,y,\rho) = b(y)x + \phi(x,y,\rho), \quad \bar{\Phi}(x,y,\rho) = \bar{b}(y)x + \bar{\phi}(x,y,\rho), \tag{5.12}$$

in which the old fields ϕ and $\bar{\phi}$ follow the identification

$$\phi(x,y,\rho) \sim \phi(x,y,\rho) + a(y), \quad \bar{\phi}(x,y,\rho) \sim \bar{\phi}(x,y,\rho) + \bar{a}(y). \tag{5.13}$$

These $b(y)$ and $\bar{b}(y)$ parameters in particular have an important physical interpretation, which will be discussed in section 5.4.2.

In going toward either boundary, the Chern-Simons terms are

$$\rho \to \infty: \; A \approx \frac{1}{2}\begin{pmatrix} d\rho & 0 \\ e^{\rho}(dx + \bar{\tau}_1 dy) & -d\rho \end{pmatrix}, \quad \bar{A} \approx \frac{1}{2}\begin{pmatrix} -d\rho & -e^{\rho}(dx + \tau_1 dy) \\ 0 & d\rho \end{pmatrix}, \tag{5.14}$$

and

$$\rho \to -\infty: \; A \approx \frac{1}{2}\begin{pmatrix} d\rho & e^{-\rho}(dx + \bar{\tau}_2 dy) \\ 0 & -d\rho \end{pmatrix}, \quad \bar{A} \approx \frac{1}{2}\begin{pmatrix} -d\rho & 0 \\ -e^{-\rho}(dx + \tau_2 dy) & d\rho \end{pmatrix}, \tag{5.15}$$

which will be imposed as boundary conditions such that the action can be reduced to a boundary one similar to what was done in 4.3. The geometry of the two boundaries are tori with respective complex structures τ_1 and τ_2, and the identifications $(x,y) \sim (x+2\pi, y)$ and $(x,y) \sim (x, y+2\pi)$. Note that near the boundaries $\rho \to \pm\infty$, the line element approximates to that of the torus

$$ds^2 \approx \frac{e^{2\rho}}{4}|dx + \tau_{1,2}dy|^2 + d\rho^2, \tag{5.16}$$

and this does indeed describe the conformal boundaries with tori structures $\tau_{1,2}$. Then, similar to the previous chapter, the Gauss parameters are solved at the boundary. They are found to be [11]

$$\lambda \approx Log\left(\frac{e^{\rho}}{\Phi'}\right), \quad \Psi \approx -\frac{e^{-\rho}\Phi''}{\Phi'}, \tag{5.17}$$

and the $\bar{\lambda}$ and $\bar{\Psi}$ fields are the same for $\Phi \to \bar{\Phi}$. Furthermore, since there are two boundaries, these Gauss parameters will appear for each limit. This is done through the Φ fields

$$\Phi_1 \equiv lim_{\rho\to\infty}\Phi, \quad \Phi_2 \equiv lim_{\rho\to-\infty}\Phi, \tag{5.18}$$

and similarly for the $\bar{\Phi}$ fields.

Before putting all of this together, it's important to note that the path integral with these boundary conditions is not the complete gravity path integral on the annulus times time. In the path integral, one sums over all metrics of topology $\mathbb{T}^2 \times I$, and the bulk geometry of this topology can take many different forms. Specifically, the bulk geometry is allowed to twist[23] along one boundary with respect to the other; there are infinitely many twists which still allow for the boundaries to be smoothly connected in the bulk. These twists are called Dehn twists, and are in the group $PSL(2;\mathbb{Z})$. The full path integral is related to each individual choice of twist by[24]

$$Z_{\mathbb{T}^2\times I}(\tau_1,\tau_2) = \sum_{\gamma\in PSL(2;\mathbb{Z})/\Gamma_\infty} Z(\tau_1,\gamma\tau_2), \tag{5.19}$$

where the left-hand-side is the full path integral on the annulus times time, and the right-hand side is the sum over the individual partition functions corresponding to a choice of twist. Specifically, the twist as it related to the torus structure $(\tau_1,\tau_2)\to(\tau_1,\tau_2+1)$ is generated by the axial T transformation, and the Γ_∞ is the subgroup generated by T.

In substituting in (4.37) with the choices made in this section (including the asymptotic Gauss terms) into the action given by (5.8) and (5.9), the Euclidean boundary action is therefore

$$\begin{aligned} S_E = \frac{C}{24\pi}\int d^2x\Bigg(&\frac{\Phi_1''\partial_1\Phi_1'}{\Phi_1'^2} + \frac{\bar{\Phi}_1''\bar{\partial}_1\bar{\Phi}_1'}{\bar{\Phi}_1'^2} + \frac{i}{2}\left(\bar{\tau}_1\Phi_1'^2 - \phi_1'\partial_y\phi_1 - \tau_1\bar{\Phi}_1'^2 + \bar{\phi}_1'\partial_y\bar{\phi}_1\right) \\ &+\frac{\Phi_2''\partial_2\Phi_2'}{\Phi_2'^2} + \frac{\bar{\Phi}_2''\bar{\partial}_2\bar{\Phi}_2'}{\bar{\Phi}_2'^2} + \frac{i}{2}\left(\bar{\tau}_2\Phi_2'^2 + \phi_2'\partial_y\phi_2 - \tau_2\bar{\Phi}_2'^2 - \bar{\phi}_2'\partial_y\bar{\phi}_2\right)\Bigg) \\ &-\frac{iC}{24}\int_0^{2\pi} dy\,\left(b^2\partial_y Y - \bar{b}^2\partial_y\bar{Y}\right), \end{aligned} \tag{5.20}$$

where the fields $\Phi_{1,2}$ and $\bar{\Phi}_{1,2}$ are given in (5.12). The derivatives are defined as

$$\partial_1 = \frac{i}{2}(\bar{\tau}_1\partial_x - \partial_y), \qquad \partial_2 = \frac{i}{2}(\bar{\tau}_2\partial_x + \partial_y). \tag{5.21}$$

[23]This should not to be confused with torsion;the theory is still torsion free.
[24]Refer to [11] for details.

And furthermore, to account for the relative twisting of the boundaries, the following terms are added to the above action (5.20)

$$Y(y) = \frac{1}{2\pi b(y)} \int_0^{2\pi} dx \, (\phi_1(x,y) - \phi_2(x,y)) \,, \tag{5.22}$$

$$\bar{Y}(y) = \frac{1}{2\pi \bar{b}(y)} \int_0^{2\pi} dx \, \left(\bar{\phi}_1(x,y) - \bar{\phi}_2(x,y)\right) . \tag{5.23}$$

This action is indeed invariant under (5.13). The action has a single time derivative and therefore is in Hamiltonian form.

5.3.2 The Eternal Traversable Wormhole

To finish the construction of the eternal traversable wormhole, one must add the double trace deformation (5.4) to the Lorentzian version of (5.20). Therefore, let's analytically continue (5.20) to real time term-by-term. In performing the (reverse) Wick rotation

$$y \equiv it \Rightarrow dy = idt \Rightarrow \partial_y = -i\partial_t,$$

all dynamical variables and twist functionals now depend on real time t. Recall the fact that Euclidean and Lorentizan actions are related by $S = iS_E$ in terms of path integral language. That is,

$$\int d[\Phi] e^{iS} \sim \int d[\Phi] e^{-S_E}. \tag{5.24}$$

Therefore, this changes the twist part of the boundary action to

$$S_{twist}[x,y] = -\frac{iC}{24} \int_0^{2\pi} dy \, \left(b^2 \partial_y Y - \bar{b}^2 \partial_y \bar{Y}\right) = \frac{C}{24} \int_0^{\infty} dt \, \left(b^2 \partial_t Y(t) - \bar{b}^2 \partial_t \bar{Y}\right) .$$

Let's then integrate the $Y(t)$ term by parts

$$\int_{-\infty}^{\infty} dt \, b^2 \partial_t Y \Rightarrow - \int_{-\infty}^{\infty} dt \, (\partial_t b^2) Y = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} dt \, dx \, (\partial_t b) \, (\phi_1 - \phi_2)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} dt \, dx \, b \, (\partial_t \phi_1 - \partial_t \phi_2) \, .$$

Doing the same for the $\bar{Y}(t)$ term yields similar results. Therefore, the twist part of the boundary action taken into real time is

$$S_{twist}[x,t] = \frac{C}{24\pi}\int_{-\infty}^{\infty}\int_{0}^{2\pi} dt\, dx\, \left(b\left(\partial_t\phi_1 - \partial_t\phi_2\right) - \bar{b}\left(\partial_t\bar{\phi}_1 - \partial_t\bar{\phi}_2\right)\right). \tag{5.25}$$

Yet, in going to real time, the boundary tori are now boundary cylinders. In order to get these, one needs to undo the identification made on time y, and therefore $y \in \mathbb{R}^{25}$. Now that the boundaries are cylinders and no longer have complex structures, the τ_1 and τ_2 must be removed by redefinitions of boundary coordinates. This will be done by setting $\tau_{1,2} = i$ and $\bar{\tau} = -i$. Applying this to the remainding terms in (5.20), the derivative terms are now written as

$$\partial_1 = \frac{1}{2}(\partial_x - i\partial_y) = \bar{\partial}_2, \quad \partial_2 = \frac{1}{2}(\partial_x + i\partial_y) = \bar{\partial}_1,$$

and therefore the Alekseev-Shatashvili contribution of the action is rewritten as

$$S_{AS} = \frac{C}{24\pi}\int d^2x \Bigg(\frac{\Phi_1''\partial_1\Phi_1'}{\Phi_1'^2} + \frac{\bar{\Phi}_1''\bar{\partial}_1\bar{\Phi}_1'}{\bar{\Phi}_1'^2} + \frac{1}{2}\left(\Phi_1'^2 - i\phi_1'\partial_y\phi_1 + \bar{\Phi}_1'^2 + i\bar{\phi}_1'\partial_y\bar{\phi}_1\right)$$
$$+\frac{\Phi_2''\partial_2\Phi_2'}{\Phi_2'^2} + \frac{\bar{\Phi}_2''\bar{\partial}_2\bar{\Phi}_2'}{\bar{\Phi}_2'^2} + \frac{1}{2}\left(\Phi_2'^2 + i\phi_2'\partial_y\phi_2 + \bar{\Phi}_2'^2 - i\bar{\phi}_2'\partial_y\bar{\phi}_2\right)\Bigg). \tag{5.26}$$

To proceed fully into Lorenzian signature, the derivatives are updated to

$$\partial_1 = \bar{\partial}_2 = \partial_- = \frac{1}{2}(\partial_x - \partial_t), \quad \partial_2 = \bar{\partial}_1 = \partial_+ \frac{1}{2}(\partial_x + \partial_t), \tag{5.27}$$

and therefore

$$S_{AS}[x,t] = -\frac{C}{24\pi}\int dtdx \Bigg(\frac{\Phi_1''\partial_-\Phi_1'}{\Phi_1'^2} + \frac{\bar{\Phi}_1''\partial_+\bar{\Phi}_1'}{\bar{\Phi}_1'^2} + \frac{1}{2}\left(\Phi_1'^2 - \phi_1'\partial_t\phi_1 + \bar{\Phi}_1'^2 + \bar{\phi}_1'\partial_t\bar{\phi}_1\right)$$
$$+\frac{\Phi_2''\partial_+\Phi_2'}{\Phi_2'^2} + \frac{\bar{\Phi}_2''\partial_-\bar{\Phi}_2'}{\bar{\Phi}_2'^2} + \frac{1}{2}\left(\Phi_2'^2 + \phi_2'\partial_t\phi_2 + \bar{\Phi}_2'^2 - \bar{\phi}_2'\partial_t\bar{\phi}_2\right)\Bigg). \tag{5.28}$$

Before bringing everything together, return to the double trace deformation in (5.4). To understand how this is constructed, recall the general double trace deformation equation (5.3). One needs to therefore compute the cross-wormhole two-point

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[25]Formally, to get to a cylinder from a torus, take in the limit $Im(\tau) \to \infty$. This causes the time circle is become infinitely long, and hence acts as the length of the cylinder.

function at the same time[26] $\langle O_L O_R \rangle$. As will be discussed in the next section, the $b(t)$ and $\bar{b}(t)$ parameters will be taken as constants. It turns out the two-point function is proportional to [8]

$$\langle O_L O_R \rangle \propto \left[\frac{1}{cosh\left(\frac{b\Delta x}{2}\right)} \right]^h , \tag{5.29}$$

which can be reparameterized in terms of the boundary degrees of freedom Φ_1 and Φ_2

$$\langle O_L O_R \rangle \propto \left[\frac{1}{cosh\left(\frac{\Phi_1 - \Phi_2}{2}\right)} \right]^h , \tag{5.30}$$

and similarly for the barred fields and $h \to \bar{h}$

$$\langle \bar{O}_L \bar{O}_R \rangle \propto \left[\frac{1}{cosh\left(\frac{\bar{\Phi}_1 - \bar{\Phi}_2}{2}\right)} \right]^{\bar{h}} . \tag{5.31}$$

Then, to take into account these two types of fields (Φ and $\bar{\Phi}$), take the product of the two above equations. Therefore, to add the double trace deformation to the action, one needs to simply add this term to the Lagrangian with a coupling constant

$$S_{DT} = \frac{C}{24\pi} \int dt\, dx\, \eta \left(\left[\frac{\Phi_1' \Phi_2'}{cosh^2\left(\frac{\Phi_1 - \Phi_2}{2}\right)} \right]^h \times \left[\frac{\bar{\Phi}_1' \bar{\Phi}_2'}{cosh^2\left(\frac{\bar{\Phi}_1 - \bar{\Phi}_2}{2}\right)} \right]^{\bar{h}} \right),$$

where–as stated in the previous section–the coupling constant η describes the interaction strength.

To bring everything together, the action which allows for an eternal traversable wormhole solution is the combination of (5.28), (5.25), and (5.4)

$$S = S_{AS}[x,t] + S_T[x,t] + S_{DT}[x,t],$$

and thus[27]

$$S = -\frac{C}{24\pi} \int dx\, dt \left(\frac{\Phi_1'' \partial_- \Phi_1'}{\Phi_1'^2} + \frac{\bar{\Phi}_1'' \partial_+ \bar{\Phi}_1'}{\bar{\Phi}_1'^2} + \frac{1}{2}\left(\Phi_1'^2 - \phi_1' \partial_t \phi_1 + \bar{\Phi}_1'^2 + \bar{\phi}_1' \partial_t \bar{\phi}_1 \right) \right.$$

[26] This is because, for small coupling g, $\langle e^{ig \sum_i \int dx^3 O_L^i O_R^i} \rangle \sim e^{ig \sum_i \int dx^3 \langle O_L^i O_R^i \rangle}$, where the exponents here are the double trace deformation of the form in [4].

[27] All computations made with this action were done after the field redefinition $\Phi \leftrightarrow \bar{\Phi}$, which is allowed since this action is invariant under a change of chirality (and are therefore equivalent).

$$+\frac{\Phi_2''\partial_+\Phi_2'}{\Phi_2'^2}+\frac{\bar\Phi_2''\partial_-\bar\Phi_2'}{\bar\Phi_2'^2}+\frac{1}{2}\left(\Phi_2'^2+\phi_2'\partial_t\phi_2+\bar\Phi_2'^2-\bar\phi_2'\partial_t\bar\phi_2\right)\Bigg)$$

$$+\frac{C}{24\pi}\int dx\,dt\,\eta\left(\left[\frac{\Phi_1'(x,t)\Phi_2'(x,t)}{cosh^2\left(\frac{\Phi_1(x,t)-\Phi_2(x,t)}{2}\right)}\right]^h\times\left[\frac{\bar\Phi_1'(x,t)\bar\Phi_2'(x,t)}{cosh^2\left(\frac{\bar\Phi_1(x,t)-\bar\Phi_2(x,t)}{2}\right)}\right]^{\bar h}\right) \tag{5.32}$$

$$+\frac{C}{24\pi}\int dt\,dx\,\left(b(t)\left(\partial_t\phi_1-\partial_t\phi_2\right)-\bar b(t)\left(\partial_t\bar\phi_1-\partial_t\bar\phi_2\right)\right).$$

5.4 Wormhole Solutions

In section 5.1, it was stated that the wormhole is a saddle point of the gravity path integral. However, it turns out that it is only a saddle point for fixed $b(t)=b$ and $\bar b(t)=\bar b$. Furthermore, these constants are not arbitrary, and are determined by the equations of motion. The claim asserted at the begging of the chapter was that the action has wormhole solutions to the equations of motion, although it turns out that there are no wormhole solutions to the equation of motion for the action (5.20) [11]. Its only after one adds the double trace deformation that present equations of motion which are satisfied by wormhole solutions. These solutions will be found in this subsection; the remainder of the section will discuss the geometry of these wormholes.

5.4.1 Constructing an Ansatz

One considers solutions to the equations of motion which are translationally invariant[28]. However, before looking for these solutions, let's first examine the gauge redundancy.

Recall that the action is invariant under the variation (5.13) (now in real time)

$$\phi_{1,2}(x,t)\sim\phi_{1,2}(x,t)+a(t), \tag{5.33}$$

and similarly with the barred fields. Indeed, note that all spatial derivative terms as well as the double trace in (5.32) are immediately manifestly invariant under this transformation. The only terms which require extra steps to show that the action as

[28]There are, in general, infinitely many solutions to the field equations. For our purposes, we only consider this symmetric configuration.

a whole is invariant under the transformation are

$$\phi_1' \partial_t \phi_1 - \bar{\phi}_2' \partial_2 \bar{\phi}_i, \tag{5.34}$$

where the barred fields have an identical series of terms as well. Under the transformation, the above equation becomes

$$\phi_1' \partial_t \phi_1 - \bar{\phi}_2' \partial_2 \bar{\phi}_i \to \left(\phi_1' \partial_t \phi_1 - \bar{\phi}_2' \partial_2 \bar{\phi}_i\right) + (\phi_1' - \phi_2')\partial_t a(t), \tag{5.35}$$

and yet the last term can be rewritten as a total derivative

$$(\phi_1' - \phi_2')\partial_t a(t) = ((\phi_1 - \phi_2)\partial_t a(t))'. \tag{5.36}$$

Therefore, since (5.36) is a total derivative, it vanishes under integration over x. Thus, this action is gauge invariant under the variation (5.13).

Next, note that translationally invariant solutions are necessarily characterized by translationally invariant configurations for the gauge fields. Therefore, the gauge invariant fields should all be made to be constants

$$\Phi_1 - \Phi_2 = \phi_1 - \phi_2 = const., \tag{5.37}$$

$$\Phi_1' = b(t) + \phi_1' = const., \tag{5.38}$$

and

$$\Phi_2' = b(t) + \phi_2' = const., \tag{5.39}$$

with the same requirement holding for the barred terms. The most general ansatz which satisfies these is

$$\Phi_1 = bx + c_1, \quad \Phi_2 = bx \tag{5.40}$$

and

$$\bar{\Phi}_1 = \bar{b}x + \bar{c}_1, \quad \bar{\Phi}_2 = \bar{b}x, \tag{5.41}$$

in which $b, \bar{b}, c_1$, and $\bar{c}_1$ are all constants[29]. In summary, the translation invariant ansatz is

$$b(t) = b, \quad \bar{b}(t) = \bar{b}, \tag{5.42}$$

[29] The "2" fields could also have had their own "c" constant, but a choice of gauge can immediately send them to zero.

with

$$\phi_1 = c_1, \quad \bar{\phi}_1 = \bar{c}_1, \quad \phi_2 = \bar{\phi}_2 = 0. \tag{5.43}$$

Extremizing the action under this translationally invariant ansatz gives the following two equations of motion for the c_1 and $\bar{c}_1$ constants

$$\frac{d}{dc_1}\left[\left(b^2 Sech^{2h}\left(\frac{c_1}{2}\right)\right)\left(\bar{b}^2 Sech^{2\bar{h}}\left(\frac{\bar{c}_1}{2}\right)\right)\right] = 0, \tag{5.44}$$

$$\frac{d}{d\bar{c}_1}\left[\left(b^2 Sech^{2h}\left(\frac{c_1}{2}\right)\right)\left(\bar{b}^2 Sech^{2\bar{h}}\left(\frac{\bar{c}_1}{2}\right)\right)\right] = 0, \tag{5.45}$$

which are satisfied under $c_1 = \bar{c}_1 = 0$. The full ansatz is therefore

$$\phi_1 = \phi_2 = \bar{\phi}_1 = \bar{\phi}_2 = 0, \tag{5.46}$$

along with (5.42) for specific b and $\bar{b}$. To solve for these constants, the action is extremized on this ansatz (5.46) and (5.42), and the b and $\bar{b}$ equations of motion are

$$\frac{d}{db}\left(\frac{(b^2+\bar{b}^2)c}{24\pi} - b^{2h}\bar{b}^{2\bar{h}}\eta\right) = 0, \tag{5.47}$$

and

$$\frac{d}{d\bar{b}}\left(\frac{(b^2+\bar{b}^2)c}{24\pi} - b^{2h}\bar{b}^{2\bar{h}}\eta\right) = 0. \tag{5.48}$$

Relating the scaling weights h and $\bar{h}$ to conformal dimension Δ and spin s by (5.5), these specific constants which solve these equations of motion and are therefore

$$b = \left[2\left(\Delta - s\right)^{\left(\frac{s-\Delta}{2}\right)}\left(\Delta + s\right)^{\left(\frac{\Delta-s}{2}-1\right)}\eta^{-1}\right]^{\left(\frac{1}{2(\Delta-1)}\right)}, \tag{5.49}$$

and

$$\bar{b} = \left[2\left(\Delta - s\right)^{\left(\frac{s+\Delta}{2}-1\right)}\left(\Delta + s\right)^{-\left(\frac{s+\Delta}{2}\right)}\eta^{-1}\right]^{\left(\frac{1}{2(\Delta-1)}\right)}. \tag{5.50}$$

And thus, using this same ansatz, a general fluctuation is characterized by

$$\Phi_1(x,t) \to b\,x + (db(t)\,x + \phi_1(x,t)), \quad \Phi_2(x,t) \to b\,x + (db(t)\,x + \phi_2(x,t)), \tag{5.51}$$

and

$$\bar{\Phi}_1(x,t) \to \bar{b}\,x + (d\bar{b}(t)\,x + \bar{\phi}_1(x,t)), \quad \bar{\Phi}_2(x,t) \to \bar{b}\,x + (d\bar{b}(t)\,x + \bar{\phi}_2(x,t)), \tag{5.52}$$

where $db(t), \phi(x,t)$, etc. are small and the constants b and $\bar{b}$ are given by (5.49) and (5.50) respectively.

Before concluding this section, it should be stated now that the procedure to compute the spectrum of fluctuations will involve transforming over to Fourier space. In doing so, return to the general fluctuations (5.51) and (5.52) and note the terms. Since $db(t)$ only depends on time, one should treat the $k = 0$ separately from the other $k \neq 0$ modes[30]. The way this modifies the starting fluctuations(5.51) and (5.52) will be stated in their relevant sections.

5.4.2 The Wormhole Geometry

Before finding the spectrum of fluctuations, how are the b and $\bar{b}$ of this ansatz best physically interpreted[31]? To answer this question, return to the Euclidean boundary action (5.20) (and before the double trace deformation was added). In employing the ansatz given by (5.46) and (5.42), the saddle point action evaluated from (5.20) is [11] [32]

$$S_E = -\frac{i\pi C}{12}(\bar{b}^2(\tau_1 + \tau_2) - b^2(\bar{\tau}_1 + \bar{\tau}_2)), \tag{5.53}$$

and therefore it is confirmed that the wormhole is only a saddle when b and $\bar{b}$ are fixed. From this saddle point, the classical approximation to the path integral is given by

$$Z \approx e^{-S_E} = \left(e^{2\pi(\tau_1+\tau_2)}\right)^{\frac{C\bar{b}^2}{24}} \left(e^{2\pi i(\bar{\tau}_1+\bar{\tau}_2)}\right)^{\frac{Cb^2}{24}}. \tag{5.54}$$

That is, the left- and right-moving Virasoro terms are embedded within this

$$L_0 - \frac{c}{24} \sim \frac{C\bar{b}^2}{24}, \quad \widetilde{L}_0 - \frac{\widetilde{c}}{24} \sim \frac{Cb^2}{24}. \tag{5.55}$$

And therefore the b and $\bar{b}$ are related to the left- and right-moving energies given in by the Virasoro generators. In fact, in chapter 4 it was the case that $b^2 = \bar{b}^2 = -1$, which corresponded with the coadjoint orbit $Diff(S^1)/PSL(2;\mathbb{R})$; this was the quantization of vacuum AdS_3. However, for "heavy enough" matter (heavy enough to make a black hole), one can again obtain a wormhole saddle point, only this time with two boundaries. However, one needs to first add the double trace deformation (otherwise

[30]In fact, the twist part (5.25) part of the action (5.32) is only non-zero after variation for $k = 0$.

[31]The interpretation of the field ansatz (5.46) is understood to just be the ground state of the (chiral) boundary gravitons.

[32]Note the similarities with (3.69).

the saddle point does not exist), where now there is a boundary graviton theory related to the coadjoint orbit parameterized by $b^2, \bar{b}^2 > 0$ [8]. The new coadjoint orbit is $Diff(S^1)/U(1)$, where $U(1)$ now acts as the group stabilizer generated only by L_0[33] [9]. Finally, it's the case that the wormhole can be thought of in terms of two separate "trumpets" in which each "trumpet" contains the holomorphic and anti-holomorphic Alekseev-Shatashvili theories (plus the twist terms). Said Alekseev-Shatashvili actions are further identified to include fields which are indeed elements of the Virasoro coadjoint orbit $Diff(S^1)/U(1)$.

Back in real time and with the double trace deformation, how does one understand the geometry? The two-point functions come from computing a Wilson line in AdS$_3$ gravity that goes from one boundary to the other. Therefore, the double trace deformation amounts to putting a uniform density of these Wilson lines into the bulk geometry, extended in the radial coordinate ρ, from one boundary to the other[34]. And for the particular simple case of $s = 0$[35] and the saddle, these Wilson lines are then extended in ρ, and therefore only modify the equations of motion that come from varying A_ρ and $\bar{A}_\rho$.

In particular, this implies that the constraint equations which come from varying with respect to A_0 and $\bar{A}_0$ are unmodified, and therefore one still has $\widetilde{F} = \widetilde{\bar{F}} = 0$. It turns out this is enough to determine the spatial gauge field[36]

$$A_i dx^i = \frac{1}{2}\begin{pmatrix} d\rho & e^{-\rho} b dx \\ e^{\rho} b dx & -d\rho \end{pmatrix}, \quad \bar{A}_i dx^i = \frac{1}{2}\begin{pmatrix} -d\rho & -e^{\rho}\bar{b} dx \\ -e^{-\rho}\bar{b} dx & d\rho \end{pmatrix}. \tag{5.56}$$

And this choice of radial gauge gives a spacetime metric which is block diagonal

$$ds^2 = d\rho^2 + g_{ij}(x,t) dx^i dx^j, \tag{5.57}$$

in which $g_{ij}(x,t)$ is the boundary metric. And for the scalar particle $s = 0$ case, their Wilson lines do not source torsion, and therefore the geometry in question is torsion free. One can then trade dreibein components e_t and e_x for the components of g_{ij} with t and x indices. Therefore, the wormhole solves the components of Einstein's

[33] This latter collection of coadjiont orbits is what is known as "ordinary" orbits. The former case in which $PSL(2;\mathbb{R})$ acts as the group stabilizer is called the first exceptional orbit [9].

[34] This amounts to adding terms to the boundary action which depend on A_ρ and $\bar{A}_\rho$.

[35] The $s = 0$ case is chosen for pedagogical reasons. In the case of $s = 1/2$, gravity couples to Fermions. This creates a torsion background, and is much more complicated.

[36] This is in agreement with the spatial gauge field for the Euclidean wormhole in [11].

equations that follow from variation with respect to g_{tt}, g_{tx}, and g_{xx}[37]. Then, the stress tensor components T_{tt}, T_{tx}, and T_{xx} are enough to determine the full spacetime geometry, which is

$$ds^2 = b^2 cosh^2(\rho)(-dt^2 + dx^2) + d\rho^2. \tag{5.58}$$

This descries a bottleneck geometry with the minimum length geodesic around the x-circle at $\rho = 0$. The length of the bottleneck is parameterized by b and $\bar{b}$[38]. Note also that for any given value of ρ, the metric is a conformally flat one, and is additionally Lorentz invariant. Upon performing a Lorentz transformation $(t, x) \to (t', x')$ the metric naturally goes back to itself in a boosted frame. This motivates one to accept that the boundary action is also Lorentz invariant[39].

5.5 Spectrum of Fluctuations

Having identified wormhole saddles in the previous section, one can now study the spectrum of fluctuations. As stated in section 5.4.1, a general fluctuation is characterized by equations (5.51) and (5.52)

$$\Phi_1(x,t) \to b\,x + (db(t)\,x + \phi_1(x,t)), \quad \Phi_2(x,t) \to b\,x + (db(t)\,x + \phi_2(x,t)),$$

and

$$\bar{\Phi}_1(x,t) \to \bar{b}\,x + (d\bar{b}(t)\,x + \bar{\phi}_1(x,t)), \quad \bar{\Phi}_2(x,t) \to \bar{b}\,x + (d\bar{b}(t)\,x + \bar{\phi}_2(x,t)),$$

where $db(t), \phi(x,t)$, etc. are once again small. Furthermore, recall that the since the fluctuations $db(t)$ and $d\bar{b}(t)$ only depend on time, the modes with zero spatial momentum and nonzero spatial momentum must be treated separately.

[37] That is, the tt, tx, and xx components of Einstein's equations.

[38] Additionally, the spin-connection can be shown to be parameterized by $b - \bar{b}$, which corresponds to the twisting of the wormhole throat, although in this case there is no twisting.

[39] The action (5.32) is not manifestly Lorentz invariant, but the fact that the boundary geometry is manifestly Lorentz invariant is promising.

5.5.1 Dispersion Relations for $k \neq 0$

Begin with fluctuations in Fourier space which carry nonzero spatial momentum. These are characterized by

$$\Phi_1(x,t) = b\,x + \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_0^\infty d\omega e^{i(kx-\omega t)}\phi_1(k,\omega), \tag{5.59}$$

$$\Phi_2(x,t) = b\,x + \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_0^\infty d\omega e^{i(kx-\omega t)}\phi_2(k,\omega), \tag{5.60}$$

$$\bar{\Phi}_1(x,t) = \bar{b}\,x + \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_0^\infty d\omega e^{i(kx-\omega t)}\bar{\phi}_1(k,\omega), \tag{5.61}$$

and

$$\bar{\Phi}_2(x,t) = \bar{b}\,x + \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_0^\infty d\omega e^{i(kx-\omega t)}\bar{\phi}_2(k,\omega). \tag{5.62}$$

The linearized equations of motion evaluated on this configuration become a set of four coupled algebraic equations for the dynamical variables $\phi_1, \phi_2, \bar{\phi}_1$, and $\bar{\phi}_2$. These equations are

$$\chi\phi_1 - (2hk^2 + b^2)\phi_2 - \left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\bar{\phi}_1 - \left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\bar{\phi}_2 = 0, \tag{5.63}$$

$$-(2hk^2 + b^2)\phi_1 + \chi\phi_2 - \left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\bar{\phi}_1 - \left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\bar{\phi}_2 = 0, \tag{5.64}$$

$$-\left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\phi_1 - \left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\phi_2 + \gamma\bar{\phi}_1 - \left(\frac{b^2\bar{h}(\bar{b}^2 + 2\bar{h}k^2}{\bar{b}^2 h}\right)\bar{\phi}_2 = 0, \tag{5.65}$$

and

$$-\left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\phi_1 - \left(\frac{2b\bar{h}k^2}{\bar{b}}\right)\phi_2 - \left(\frac{b^2\bar{h}(\bar{b}^2 + 2\bar{h}k^2}{\bar{b}^2 h}\right)\bar{\phi}_1 + \gamma\bar{\phi}_2 = 0, \tag{5.66}$$

where χ and γ are defined as

$$\gamma \equiv \frac{b^2\bar{h}(\bar{b}^2 - 2k^2(\bar{h}-1)) + 2hk(\bar{b}^2 + k^2)(k+\omega)}{\bar{b}^2 h}, \tag{5.67}$$

and

$$\chi \equiv b^2 + 2\frac{k^3(k-\omega)}{b^2} - 2k(\omega + k(h-2)). \tag{5.68}$$

These four equations can be collected into a 4×4 matrix equation with associated column vector $\mathbf{\Phi}$ in terms of the dynamical variables $(\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2)$. Written in this compact way, the equations of motion can be written as the matrix equation

$$\widetilde{G}^{-1}\mathbf{\Phi} = \mathbf{0}, \tag{5.69}$$

and the determinant of this matrix $\widetilde{G}^{-1}$ (it will become clear why this is written as an inverse matrix in the next section) is

$$\begin{aligned} det\,\widetilde{G}^{-1} = \alpha\Big(& 2b^6\bar{h}^2(2\bar{h}+h-1) + b^2h\big(\bar{b}^2(2h(h-1)-\bar{h}) + 2h(h-1) \\ & +\bar{h}(\bar{h}-1)k^2\big)(k-\omega)(k+\omega) - h^2(\bar{b}^2+k^2)(k^2-\omega^2)^2 \\ & +b^4\bar{h}\big(2\bar{b}^2h(h-1) + 4h^2k^2 + 4hk^2(\bar{h}-1) + \bar{h}(2\bar{h}-1)(k-\omega)(k+\omega)\big)\Big), \end{aligned} \tag{5.70}$$

where α is defined as

$$\alpha \equiv -\frac{16k^4(b^2+k^2)^2(\bar{b}^2+k^2)}{b^4\bar{b}^4h^2}. \tag{5.71}$$

From here, solutions to the equations of motion are therefore characterized by frequencies ω such that $det\,\widetilde{G}^{-1} = 0$.

Despite the fact that (5.70) is not Lorentz-covariant[40], the frequencies ω do in fact satisfy the Lorentz-invariant dispersion relation

$$m^2 = \omega^2 - k^2, \tag{5.72}$$

where m is the Lorentz invariant mass. In substituting in the expressions of the ω values found from the zeros of the determinant along with equations (5.49), (5.50), two unique expressions for wormhole mass emerge. Furthermore, it's also the case that neither mass is dependent on wave number k, and both are homogeneous in η with the dependence $\propto \eta^{-\left(\frac{1}{\bar{h}+h-1}\right)}$. After substituting in (5.5) to replace h and $\bar{h}$ in favor of Δ and s, the two expressions for the wormhole mass are thus

$$\begin{aligned} m^2_{\mp}(\Delta, s) = & \, 2^{\frac{1}{\Delta-1}}\,(\Delta-s)^{\left(1+\frac{2+s-3\Delta}{2(\Delta-1)}\right)}\,(\Delta+s)^{-\left(\frac{8+s+6\Delta}{4(\Delta-1)}\right)} \\ & \Big[(\Delta+s)^{\left(2+\frac{3}{\Delta-1}\right)} - \frac{1}{2}(\Delta+s)^{\frac{3\Delta}{\Delta-1}} \mp \frac{1}{2}(\Delta-s)(\Delta-s-2)(\Delta+s)^{\frac{2+\Delta}{\Delta-1}} \end{aligned} \tag{5.73}$$

[40]It does not transform in the Lorentz representation, nor can it be written in terms of other Lorentz-covariant quantities.

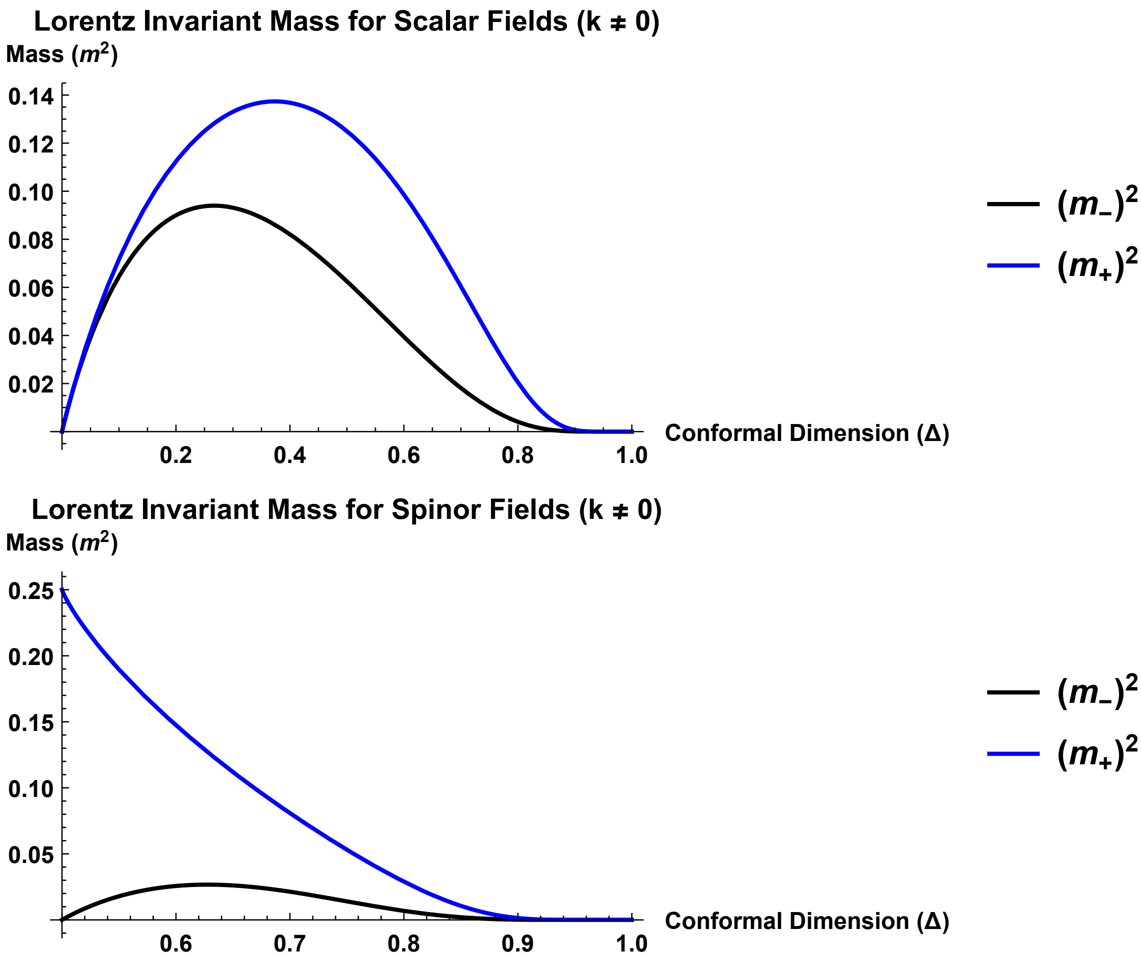

Figure 5.1: These two figures together contain the total range of Lorentz Invariant Masses in the $k \neq 0$ case for conformal dimension up to $0 \leq s < \Delta < 1$.

$$\mp(\Delta + s)^{\left(1+\frac{3}{\Delta-1}\right)} \sqrt{(\Delta^4 + s^4 + 2s^2(2 + (\Delta - 4)\Delta))}\Bigg].$$

The relevant solutions (those which let the mass remain positive) enforce the condition that $0 \leq s < \Delta < 1$. This condition is illustrated in figure 5.1 for both scalar and spinor fields. For the simple case $s = 0$, this dispersion relation is just

$$m_-^2 = 2b^2, \quad m_+^2 = 2(1 - 2h)b^2. \tag{5.74}$$

5.5.2 Dispersion Relations for $k = 0$

For the $k = 0$ case, the constants b and $\bar{b}$ must now be allowed to fluctuate [41]. Secondly, the gauge redundancy reduces the apparent six degrees of freedom back down to four[42]. Practically speaking, since the action is invariant under combinations $\phi \equiv \phi_1 - \phi_2$ and $\bar{\phi} \equiv \bar{\phi}_1 - \bar{\phi}_2$, this is equivalent to the choice of gauge where only the ϕ_1 and $\bar{\phi}_1$ fields are allowed to fluctuate. Therefore, in choosing the gauge where the following fluctuations vanish $\phi_2 = \bar{\phi}_2 = 0$,[43] the new starting fluctuations are

$$\Phi_1(x,t) \to b\,x + (db(t)\,x + \phi_1(t)), \quad \Phi_2(x,t) \to b\,x + db(t)\,x \tag{5.75}$$

and

$$\bar{\Phi}_1(x,t) \to \bar{b}\,x + (d\bar{b}(t)\,x + \bar{\phi}_1(t)), \quad \bar{\Phi}_2(x,t) \to \bar{b}\,x + d\bar{b}(t)\,x, \tag{5.76}$$

where now the field fluctuations $\phi(t)$ only depend on time[44]. The (Fourier) equations of motion are again collected in matrix form

$$\begin{pmatrix} -8(h-1) & \frac{8b\bar{h}}{\bar{b}} & 2i\omega & 0 \\ -\frac{8b\bar{h}}{\bar{b}} & 4 + \frac{4b^2(1-2\bar{h})\bar{h}}{\bar{b}^2 h} & 0 & -2i\omega \\ -2i\omega & 0 & b^2 & 0 \\ 0 & 2i\omega & 0 & \frac{b^2\bar{h}}{h} \end{pmatrix} \begin{pmatrix} db(\omega) \\ d\bar{b}(\omega) \\ \phi_1(\omega) \\ \bar{\phi}_1(\omega) \end{pmatrix} = \mathbf{0}, \tag{5.77}$$

where the ω values are again found form the zeros of the determinant

$$\begin{aligned} det\,\widetilde{G}^{-1} = -\frac{16}{\bar{b}^2 h^2}\Big(& 2b^6\bar{h}^2(h + 2\bar{h} - 1) + b^2\bar{b}^2 h(\bar{h} - 2h(h-1))\omega^2 \\ & -\bar{b}^2 h^2\omega^4 + b^4\bar{h}(2\bar{b}^2 h(h-1) - \bar{h}(2\bar{h}-1)\omega^2)\Big). \end{aligned} \tag{5.78}$$

It again turns out that the resulting mass terms are homogeneous in η (and are obviously not dependent on k). From here, the exact same procedure from section 5.5.1 is performed, and the Lorentz invariant masses computed from these equations of motion exactly agree with (5.73), and hence the constraints $0 \leq s < \Delta < 1$ shown in Figure 5.1. One might not have expected the two cases to exactly agree, since the

[41] And furthermore, the twist terms (5.25) will now be non-vanishing after variation.

[42] And furthermore, we only needed to take this into account for the $k = 0$ modes because the gauge parameter is only a function of time $a(t)$.

[43] That is, choose $a(t) = -\phi_2(t)$ and $\bar{a}(t) = \bar{\phi}_2(t)$.

[44] Their Fourier transformation is the same as before, only now one no longer needs to sum over non-zero k.

equations of motion and degrees of freedom for the $k = 0$ case are clearly different from the general $k \neq 0$ case. And yet–despite these differences–both are completely consistent with each other.

5.6 The Free Propagator of a Boundary Graviton

5.6.1 Free Propagator Preliminaries

A final point of curiosity for this project is to examine boundary graviton propagation through the bulk from one boundary to the other. That is, suppose that one excites a boundary graviton on one boundary. How does the excitation propagate, and does it excite the other boundary? The way one would start by approaching this problem is to compute the free-propagator for a boundary graviton. The propagator gives the probability amplitude for a particle to travel from one boundary to the other through a bulk supported by many scalar bulk matter fields, and the probability amplitude can be used to compute S-matrix elements. In the weak coupling expansion $1/c \sim G/l$, the propagator is approximately defined as the inverse of the differential operator appropriate to the particle[45] and are therefore known to be a ("retarded" or causal) Green's function of the equations of motion [53].

In QFT, S-matrix elements can be calculated using Feynman path integral formalism by[46]

$$\langle\Omega|\,\mathcal{T}\{\phi(x_1)...\phi(x_n)\}\,|\Omega\rangle = \frac{\int D[\phi]\phi(x_1)...\phi(x_n)e^{iS[\phi]}}{\int D[\phi]e^{iS[\phi]}}, \tag{5.79}$$

where $\mathcal{T}$ is the time-ordering operation, $S[\phi]$ is the action, and the integration measure $D[\phi]$ implies that one integrates over all possible field configurations. However, it is often more convenient to write this in terms of the generating functional

$$Z[J] = \int D[\phi]exp\{iS[\phi] + i\int d^4xJ(x)\phi(x)\}, \tag{5.80}$$

[45] That is, the operator which provides the equation of motion acting on ϕ.

[46] To be more specific, the coordinates x_i are understood to be $(\vec{x}', t')$ for some specific position and time $\vec{x}'$ and t'.

for some source $J(x)$. If no source is present, the vacuum is recovered

$$Z[0] = \int D[\phi] e^{iS[\phi]}, \tag{5.81}$$

which is just the denominator in (5.79). Using a generating functional is useful since it let's one compute (5.79) by simply taking derivatives of it[47]

$$(-i)^n \frac{1}{Z[0]} \frac{\partial^n Z[J]}{\partial J(x_1)...\partial J(x_n)}\bigg|_{J=0} = \langle\Omega|\, \mathcal{T}\{\phi(x_1)...\phi(x_n)\}\, |\Omega\rangle\,, \tag{5.82}$$

where the right-hand-side is exactly the left-hand-side of (5.79). In the case at hand, the boundary action given by (5.32) is taken to be a weakly coupled theory, and therefore cubic and other higher-order terms are neglected in the the generating functional. Therefore, it is useful to write the generating functional (5.80) as a Gaussian integral

$$Z[J] = \int_{-\infty}^{\infty} D[\phi] e^{-\frac{1}{2}\phi^{\dagger}(x)G^{-1}\phi(x)+J^{\dagger}(x)\phi(x)} = \sqrt{\frac{(2\pi)^d}{det\, G^{-1}}} e^{\frac{1}{2}J^{\dagger}GJ}, \tag{5.83}$$

where d is the number of spacetime dimensions. For a non-interacting (free) theory, the matrix G^{-1} is simply the differential operator corresponding to the quadratic action of a given theory[48]. And it is further the case that the matrix G is the corresponding free propagator of the theory. To see this, the matrix G and it's inverse G^{-1} satisfy[49]

$$G(x_1 - x_2)G^{-1}(x_1 - x_2) = -\delta^{(d-1)}(\vec{x}_1 - \vec{x}_2)\delta(t_1 - t_2). \tag{5.84}$$

Then, for some matrix $G^{-1}(x_1 - x_2)$ corresponding to the differential operator giving the equations of motion, the matrix G acts as a Green's function[50], and hence is defined to be the propagator.

---

[47]The generating functional can by understood as the QFT analog of the statistical mechanics partition function.

[48]For example, the free scalar field theory Lagrangian is $\mathcal{L} = \frac{1}{2}\phi(\Box + m^2)\phi$. Compare this with G^{-1} above; $G^{-1} = i(\Box + m^2)$.

[49]The negative sign on the right-hand-side comes from the imaginary i dependence on the left-hand-side.

[50]For an example, consider the Klein-Gordon propagator derivation in [54] or [53].

5.6.2 Computing the Boundary Graviton Free Propagator for the Wormhole

To compute the boundary graviton free propagator for the wormhole, the procedure is as follows. First, compute the quadratic action. That is, compute the corresponding differential operator which give the linearized equations of motion (upon "stripping-off" the field dependence); this was done in Fourier-space. This matrix which collects the linearized equations of motion (as discussed in section 5.5) is the matrix Fourier-space $\widetilde{G}^{-1}$, and therefore the free propagator is just $\widetilde{G}$. To go back to configuration space, one performs the following integral

$$G_{ij}(x,t) = \frac{1}{(2\pi)^2} \sum_{k\in\mathbb{Z}} \int_0^\infty d\omega\, \widetilde{G}_{ij}(k,\omega) e^{i(kx-\omega t)}, \tag{5.85}$$

where the matrix $\widetilde{G}_{ij}$ is just the inverse of the linearized equation of motion matrix $\widetilde{G}^{-1}$.

Unfortunately, the full configuration space boundary graviton free-propagator cannot be solved analytically. However, it turns out that free-propagators for specific k values can be, and then taken back to general time dependence t. Therefore, choose $k = 0$ and $s = 0$,[51] and the resulting equations of motion are the special case of (5.77) for $s = 0$

$$\begin{pmatrix} -8(h-1) & -8h & 2i\omega & 0 \\ -\frac{8b\bar{h}}{\bar{b}} & 4 + \frac{4b^2(1-2\bar{h})\bar{h}}{\bar{b}^2 h} & 0 & -2i\omega \\ -8h & -8(h-1) & 0 & -2i\omega \\ 0 & 2i\omega & 0 & b^2 \end{pmatrix} \begin{pmatrix} db \\ d\bar{b} \\ \phi_1 \\ \bar{\phi}_1 \end{pmatrix} = \mathbf{0}, \tag{5.86}$$

or in the more compact form[52]

$$\widetilde{G}^{-1}\mathbf{\Phi} = \mathbf{0} \tag{5.87}$$

in which the inverse of the matrix $(\widetilde{G}^{-1})^{-1} = \widetilde{G}$ is the free-propagator. The determinant is

$$\det \widetilde{G}^{-1} = 16(\omega^2 - 2b^2)(\omega^2 - 2b^2(1-2h)), \tag{5.88}$$

and the dispersion is given by (5.74).

[51] A similar process can be replicated for different, specific $k \neq 0$ values as well.

[52] The "vector" $\mathbf{\Phi}$ is just $(db, d\bar{b}, \phi_1, \bar{\phi}_1)$, where each entry is a function of ω.

For an example calculation, it is the case that

$$\widetilde{G}_{12} = \frac{8b^4 h}{16(\omega^2 - 2b^2)(\omega^2 - 2b^2(1-2h))}, \tag{5.89}$$

and since only a partial reverse Fourier transform is desired here, the (simplified) integral from (5.85)

$$G_{ij}(t) = \frac{1}{2\pi}\int_0^\infty d\omega\, \widetilde{G}_{ij}(k=0,\omega)e^{-i\omega t}, \tag{5.90}$$

is written as

$$G_{12}(t) = \frac{1}{2\pi}\int_0^\infty d\omega \frac{8b^4 h}{16(\omega^2 - 2b^2)(\omega^2 - 2b^2(1-2h))}\, e^{-i\omega t}. \tag{5.91}$$

This is evaluated as a contour integral with residues given by the determinant (5.88)

$$\omega_1 = \pm\sqrt{2}b, \quad \omega_2 = \pm\sqrt{2b^2(1-2h)},$$

which is indeed identical to the dispersion relation given in (5.74). That is,

$$G_{12} = \pm 2\pi i \left(res_{\omega=\omega_1}(f(\omega)) + res_{\omega=\omega_2}(f(\omega))\right), \tag{5.92}$$

where

$$f(\omega) = \frac{8b^4 h e^{-i\omega t}}{16(\omega^2 - 2b^2)(\omega^2 - 2b^2(1-2h))}. \tag{5.93}$$

The sign out-front of (5.92) is determined from the direction the contour is closed, which in turn is determined by the sign of time t. In considering $t > 0$, $Im[\omega] < 0$ must be the case[53]. Therefore, pick the positive roots, and close the contour from below (picking up a global minus sign). In stopping here, this would be the "retarded" propagator. However, considering $t < 0$ can instead provide the "advanced" propagator. For $t < 0$, then $Im[\omega] > 0$, and therefore close the contour above for the negative roots. It turns out that the only difference between $G_{12,R}$ and $G_{12,A}$ is an overall sign dictated by the the sign choice of t. Therefore, the final result accounting

[53] Recall that in the deformation, $\omega \in \mathcal{C}$. Then, the exponential $e^{-i\omega t} = e^{-iRe[\omega]t} + e^{Im[\omega]t}$, where $Im[\omega]$ is chosen such that the exponential decays to zero at infinity.

for both choices is computed to be

$$G_{12} = sgn(t)\,\frac{i\omega_1}{32}\left(\frac{\omega_1}{\omega_2}\,e^{-i\,|t|\,\omega_2} - e^{-i\,|t|\,\omega_1}\right).$$

Every other matrix element of the free propagator is computed in the exact same way. Thus, for the specific case of $k = 0$ and $s = 0$, the matrix elements of the (configuration space) Feynman free-propagator of a boundary graviton are computed to be

$$G_{11} = G_{22} = sgn(t)\,\frac{i\omega_1}{32}\left(e^{-i\,|t|\,\omega_1} + \frac{\omega_1}{\omega_2}\,e^{-i\,|t|\,\omega_2}\right), \tag{5.94}$$

$$G_{12} = G_{21} = sgn(t)\,\frac{i\omega_1}{32}\left(\frac{\omega_1}{\omega_2}\,e^{-i\,|t|\,\omega_2} - e^{-i\,|t|\,\omega_1}\right), \tag{5.95}$$

$$G_{33} = G_{44} = sgn(t)\,\frac{i}{2\omega_1^2}\left(\omega_1\,e^{-i\,|t|\,\omega_1} + \omega_2\,e^{-i\,|t|\,\omega_2}\right), \tag{5.96}$$

$$G_{34} = G_{43} = sgn(t)\,\frac{i}{2\omega_1^2}\left(\omega_1\,e^{-i\,|t|\,\omega_1} - \omega_2\,e^{-i\,|t|\,\omega_2}\right), \tag{5.97}$$

$$G_{31} = -G_{13} = G_{42} = -G_{24} = sgn(t)\,\frac{1}{8}\left(e^{-i\,|t|\,\omega_1} + e^{-i\,|t|\,\omega_2}\right), \tag{5.98}$$

and

$$G_{14} = -G_{41} = G_{23} = -G_{32} = sgn(t)\,\frac{1}{8}\left(e^{-i\,|t|\,\omega_1} - e^{-i\,|t|\,\omega_2}\right). \tag{5.99}$$

Thus, the propagator in terms of real time t corresponding to a scalar boson $s = 0$ with $k = 0$ is given by the matrix

$$G(t) = \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix}, \tag{5.100}$$

in which these matrix elements are given by the above results[54]. This propagator is written in terms of the full configuration space propagator as

$$G_{ij}(t) = \frac{1}{2\pi}\int_0^{2\pi} dx e^{-ikx} G_{ij}(x,t). \tag{5.101}$$

This same procedure could then be performed for any specific $k \neq 0$. In principle, if

[54]Notice that since $s = 0$, this propagator is completely determined by conformal dimension Δ and the interaction strength η from (5.49) and (5.50).

one could construct this propagator from all k, n-point correlation functions could be computed using (5.82) with the free generating functional $Z_0[J]$[55] given by

$$Z_0[J] = \sqrt{\frac{(2\pi)}{\det G^{-1}}} exp\{i \int d^2x' \int d^2x'' \frac{1}{2} J(x')G(x' - x'')J(x'')\}. \qquad (5.102)$$

As a final comment, the fact that this matrix has nonzero entries implies that an excitation of any one degree of freedom will excite the others. This is consistent with the idea that the two boundaries are able to send a signal to either side.

[55] $Z_0[J]$ is just the generating functional $Z[J]$ of a free (non-interacting) theory.

Chapter 6

Conclusions

This thesis discussed quantum AdS_3 gravity, both in its relation to the AdS/CFT correspondence and as a stand-alone theory. The AdS/CFT correspondence has allowed for great strides in the field of quantum gravity, because it provides a non-perterbative formulation of string theory by way of the duality between theories of quantum gravity which asymptote to AdS geometries and CFTs in one lower dimension. This was discussed in the context of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, where the gravity theory was given by type IIB string theory; the corresponding dual theory was $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory. The large $N \to \infty$ limit (in which N characterizes the number of stacked D-branes on the gravity side, or the $SU(N)$ gauge group on the CFT side) while keeping λ constant is known as the 't Hooft limit, which was the particular version of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence in which Juan Maldacena first proposed the AdS/CFT correspondence as a framework for holography.

The simpler application of the correspondence $\mathrm{AdS}_3/\mathrm{CFT}_2$ was then explored in the context of black hole entropy and the Cardy formula. In particular, it was shown how to derive the high-temperature entropy of a CFT from the gravity side. And due to the correspondence, the entropy of the BTZ black hole (the leading contribution of the gravity partition function at high temeprature) exactly agrees with the entropy of a corresponding CFT.

The quantization of global AdS_3 gravity theory was then shown in detail; the relevant topology was chosen to be that of the cylinder (the given topology of the usual global AdS_3 spacetime). This theory was shown to really be a boundary one. After Wick-rotating and identifying into periodic imaginary time, the boundary topology is then described as a torus with complex structure τ. The resulting AdS_3 theory is characterized only by edge modes, and has no bulk degrees of freedom. The theory can be decomposed into two decoupled chiral halves[1] (each written as a WZW theory), where each half is endowed with an identical Virasoro symmetry. At fixed time, the fields are elements of the coadjoint orbit of the Virasoro group $Diff(S^1)/PSL(2;\mathbb{R})$.

It was previously found that this theory of quantum AdS_3 gravity may not be a consistent theory of quantum gravity, since in computing the partition function (saddle points) the density of states was found to be negative.. A proposed solution to this problem is to account for off-shell (non-saddle points) of the partition function, which correspond to all of the ways the boundary topology can be connected; the subleading order contributions include Euclidean wormholes. However, this theory is still useful for computing Virasoro conformal blocks blocks (and ultimately correlation functions), and was previously done to one-loop order.

A new quantum theory of quantum AdS_3 was then considered with the topology now that of an annulus times time; the bulk geometry smoothly connects two boundary tori. This new geometry is a Euclidean wormhole, is parameterized by two constants b and $\bar{b}$, and these constants are related to a boundary description in terms of Virasoro coadjoint orbits labeled by said parameters.

Finally, this theory of gravity was coupled to a large number of light matter fields dual to relevant operators. The boundary conditions applied to this large number of matter fields is identified as a non-local[2] double trace deformation in the dual statistical ensemble to gravity. This causes eternal traversable wormholes to appear in the theory. Within this construction, the field theory of boundary gravitons were used to compute the spectrum of gravitational fluctuations and the gravitational contribution to scattering processes in the wormhole by way of the free propagator.

Future work could focus on continuing the work in [9] by calculating Virasoro blocks at the two-loop level, which could further provide consistency checks for this developing theory of quantum gravity. Another, more tangential line of inquiry could be developing this boundary theory of gravity by instead considering a $(2+1)$-dimensional gravity with a positive cosmological constant (de Sitter gravity)[3].

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[1]The theory as a whole is non-chiral.

[2]Between the dual operators, one exists on the left boundary, and the other on the right boundary.

[3]Prior work done in [55] and [56] considered JT gravity in nearly dS spacetime and pure $(1+1)$-dimensional dS spacetime; the Euclidean path integral was studied in [57]

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