Broadcast Independence in Graphs

by

Linda Neilson
B.A.Sc., University of British Columbia, 1989
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ABSTRACT

The usual graph parameters related to independent and dominating sets can be adapted to broadcasts on graphs. We examine some possible definitions for an independent broadcast. We determine the minimum maximal and the maximum broadcast weight for all our independence parameters on both paths and grids. For graphs in general, we examine the relationships between these broadcast independence parameters and the existing minimum and maximum minimal broadcast domination weight (or cost). We also determine upper and lower bounds for maximum boundary independent broadcasts and a new upper bound for hearing independent broadcasts.

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My family.

Let G be a graph...

DEDICATION

To my Dorothys.

Chapter 1

Introduction, Background and New Definitions

1.1 Introduction

Imagine that a graph represents an empty framework. Each vertex carries the potential for activation. For example, the vertices could be locations where bacteria could grow or where a cell phone tower could be built. The *distance* between any two vertices is the length of a shortest path which connects them or, informally, the minimum number of edges which must be traversed to move from one vertex to the other. In our first analogy, all growth occurs at the same rate, moving outwards from the vertex to cover all other vertices at distance $k \geq 0$ after k days. Although growth occurs at the same rate, it does not necessarily start at the same time. We want to select some of the vertices to be active. Each location or vertex v which is selected as active is assigned a natural number $f(v)$ representing the number of days it has been growing. If $f(v) \geq d(u,v)$, the distance between u and v, then the growth originating at v has reached u. We say that u is within range of v or, equivalently, u is dominated by v. The time that v has been growing is $f(v)$. If every vertex is either active or is within range of at least one active vertex, then we say that the network or graph is *dominated*.

The historic analogy which is responsible for the title "broadcast independence" involves envisioning the graph as a network of possible transmission tower locations. If $f(v) > 0$, then we say that the vertex v is *broadcasting* and, in this case, if $d(u,v) \leq f(v)$, then u *hears* v. If every vertex is either transmitting or is within range of at least one active vertex, then we say that the network or graph is *dominated*. The sum of all of the individual weights is sometimes called the *broadcast cost*. However, it is unlikely that the cost of a transmission has a linear relationship to its range. Further, we will be looking to discuss the greatest cost broadcasts which maintain some form of independence, but maximizing cost is counter-intuitive. Hence, we have chosen to call the sum of the individual weights the *broadcast weight*.

If we restrict our broadcasting vertices so that they only broadcast to themselves and their immediate neighbours, i.e. $f(v) = 1$, then dominating the network is related to the already defined and well studied problem of finding a *dominating set*. The idea of a larger but constant range for each of the broadcasting vertices is similarly related to *k-distance domination* or *k-domination* and was introduced separately by Henning [13] and (in the context of packing and covering numbers of a tree) by A. Meir and J.W. Moon [16]. Erwin [9] was the first to look in depth at the idea of allowing a different range for each of the broadcasting vertices; this approach is referred to simply as *broadcasts*. The study of the minimal weight required to dominate the network (graph) with a *broadcast* is found mostly under the title of *dominating broadcasts*.

A subset of the vertices of a graph is an *independent set* if no two vertices of the subset are adjacent in the graph. When the restrictions of an *independent set* are applied to a *dominating set* we get an *independent dominating set*. For example, the graph below in Figure 1.1 can be dominated by two vertices but if we want an independent dominating set, then the minimum number required is three, see Figure 1.2.

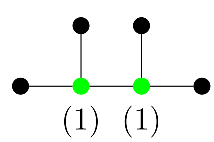


Figure 1.1: The green vertices form a dominating set which is not independent. If the characteristic function of the dominating set is taken as a broadcast, then the two broadcasting vertices hear each other.

The characteristic function of an independent set can be seen as a broadcast in which no broadcasting vertex is dominated by any another broadcasting vertex. Erwin [9] introduced the idea of restricting the interaction between the individual transmitting vertices of a broadcast. He defined a *hearing independent* broadcast in which no vertex transmits to any other transmitting vertex. He also investigated

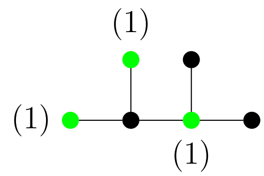


Figure 1.2: The green vertices form a dominating set which is independent. If the characteristic function of the dominating set is taken as a broadcast, then the broadcasting vertices do not hear each other.

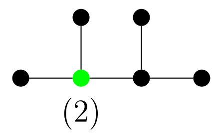


Figure 1.3: A dominating broadcast of weight 2.

minimal dominating hearing independent broadcasts. The graph in Figure 1.3 is dominated by a minimal dominating hearing independent broadcast with weight two. Placing different restrictions on the interactions between the individual broadcasting vertices and studying how this affects the weight of a dominating broadcast is the main goal of this dissertation.

Returning to a broadcast generated from the characteristic function of an independent set, we extend the restrictions on this broadcast to broadcasts in general. This leads to nine distinct possible definitions for an independent broadcast; seven new definitions and Erwin's already defined hearing independence definitions. Each definition has different consequences on the choice and strength of the broadcast vertices and the overall broadcast weight. We obtain general results as well as specific results for paths, trees, cycles and grids. Note that, when independence is introduced, we are no longer looking for the smallest weight broadcast to dominate the graph but rather for the largest weight broadcast which can occur without violating the broadcast's independence. Since many systems can be divided into subsystems which interact at, between and sometimes beyond their boundaries, there is potential for practical application of this work.

1.2 Background and Definitions

Any definition or terminology not defined in this thesis can be found in West [18]. The weight of a broadcast on a graph with more than one component is the sum of the weight of the broadcast on each individual component, hence we assume that all our graphs are connected. Further, multiple edges and loops do not change the ability of a vertex to dominate itself or another vertex, so only simple graphs are considered. Imagine that each vertex of a graph G is a possible location for a transmitting or broadcasting tower with the distance between two vertices representing the strength of the transmission needed to transmit from one vertex to the other. To choose the location and strength of our towers we use a *broadcast* which is a function $f : V(G) \to \{0, 1..., \text{diam}(G)\}$ where no vertex v is assigned a value larger than its *eccentricity*, $e(v) = \max\{d(v, u) : u \in V(G)\}$. Notice that if a vertex broadcasts with a strength equal to its eccentricity, then it will dominate the entire graph. If $f(v) > 0$, then there is a broadcast tower located at v which has the strength to transmit to all vertices within a distance of $f(v)$ from v; we call this set of vertices the *f-neighourhood* of v which is defined as $N_f(v) = \{u : u \in V(G) \text{ and } d(u, v) \leq f(v)\}$. If $u \in N_f(v)$, then we say that u *hears* or is $f-$*dominated* by v or equivalently that v $f-$*dominates* u. If X is a set of vertices, then $N_f(X) = \bigcup_{v \in X} N_f(v)$. If $f(v) > 0$, then we say that v is a *dominating* or *broadcasting vertex* and we let $V_f^+(G) = \{v : v \in V(G) \text{ and } f(v) > 0\}$ be the set of all f-dominating vertices. The subset of $V_f^+(G)$ consisting of *broadcasting* vertices with strength one is $V_f^1(G) = \{v : v \in V(G) \text{ and } f(v) = 1\}$ and $V_f^{++}(G) = V_f^+(G) - V_f^1(G)$ is the set of all broadcasting vertices with $f(v) > 1$. For brevity, if the context is clear we may suppress the reference to the graph, the broadcast, or both in our notation. For example, V_f^+, $V^+(G)$ or V^+ may be used in place of $V_f^+(G)$.

If every vertex of G is f-dominated, then we say that f is a *dominating broadcast*. For example, if a broadcast f assigns a value of $\text{diam}(G)$ to a vertex of G of maximum eccentricity and zero to all other vertices, then f is a *dominating broadcast*. Or, if a broadcast g assigns a value of $\text{rad}(G)$ to a central vertex and zero to all others, then g is also a dominating broadcast, and since $\text{rad}(G) \leq \text{diam}(G)$ the *weight* of g, $\sigma(g) = \sum_{v \in V_g^+} g(v)$, is less than or equal to $\sigma(f)$ (the weight of f). An interesting problem which has been studied [10] is to find $\gamma_b(G)$, the *minimum (weight) dominating broadcast* for G, where $\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast of } G\}$. Note that $\gamma_b(G) \leq \text{rad}(G)$ for all graphs G. If $\gamma_b(G) = \text{rad}(G)$ we say that G is a *radial* graph.

The weight of a dominating broadcast depends on which vertices are chosen to dominate so another important problem is finding the minimum weight dominating broadcast for a given set of dominating vertices. For two broadcasts g and f, we say that $g \leq f$ if $g(v) \leq f(v)$ for all $v \in V(G)$ and we say that $g < f$ if $g \leq f$ and there is at least one vertex u such that $g(u) < f(u)$. Similarly, $g > f$ if $f < g$. A dominating broadcast f is *minimal dominating* if there is no dominating broadcast g with $g < f$. The largest weight for a minimal dominating broadcast is $\Gamma_b(G) = \max\{\sigma(f) : f \text{ is a minimal dominating broadcast of } G\}$.

Dominating broadcasts can also be seen as a natural extension of the much studied problem of dominating sets. A *dominating set* S is a subset of $V(G)$ such that every vertex $v \in V(G) - S$ is adjacent to a vertex in S. The characteristic function f of a dominating set is a dominating broadcast with $f : V(G) \to \{0, 1\}$.

If all the towers broadcast with the same strength (or all growth started at the same time), then we can represent this situation by putting restrictions on the codomain of f. If $f : V(G) \to \{0, k\}$ where k is any positive integer, then we call f a *k-broadcast* which is related to the previously mentioned k-domination. Similar to more general broadcasts, let $\gamma_k(G)$ be the weight of a minimum *k-dominating* broadcast. Since a k-broadcast is a broadcast, $\gamma_b(G) \leq \gamma_k(G)$ for all $k \geq 1$. If $k = 1$ and f is a dominating broadcast, then $V_f^+(G) = V_f^1(G)$ is a dominating set. Hence, $\gamma_b(G) \leq \gamma_1(G) = \gamma(G)$ where $\gamma(G)$ is the size of a *minimum dominating set.*

An *independent set* S is a subset of $V(G)$ such that no two vertices of S are adjacent. Since broadcast signals (or growth) may interfere and interact with each other, we are interested in using and extending the definition of independence to broadcasts. To further our discussion, we define the following concepts for a connected graph G and a broadcast, f (see Figure 1.4):

Definition 1.2.1. *The* boundary *or* boundary set *of a broadcasting vertex v is*

$$B_f(v) = \{u : d(u, v) = f(v)\}$$

and if $u \in B_f(v)$, then we refer to u as a boundary vertex *of v.*

Definition 1.2.2. *The* set of broadcasting vertices heard by a vertex u *is*

$$H_f(u) = \{v : d(u, v) \leq f(v) \text{ and } f(v) > 0\}.$$

Definition 1.2.3. *The* private boundary of a broadcasting vertex *v is $PB_f(v) = N_f(V(G)) - N_{f'}(V(G))$ where f' is the broadcast with $f'(x) = f(x) - 1$ for $x = v$ and $f'(x) = f(x)$ otherwise, or, informally, the set of vertices dominated by v which are no longer dominated when the strength of the broadcast at v is reduced by 1.*

Since $f(v) \leq e(v)$ every broadcasting vertex has a non-empty boundary set, or equivalently $B_f(v) \neq \emptyset$ for all $v \in V_f^+$. Notice that if $f(v) \geq 2$, then $PB_f(v) \subseteq B_f(v)$ but if $f(v) = 1$, then it is possible that $v \in PB_f(v)$.

Definition 1.2.4. *If $uv \in E(G)$ and $u, v \in N_f(x)$ for some $x \in V_f^+(G)$ such that at least one of u and v does not belong to $B_f(x)$, then we say that the edge uv is* covered *in f by x. The* set of all covered edges *is denoted CE_f.*

Definition 1.2.5. *If $uv \in E(G)$ and uv is not covered by any $x \in V_f^+(G)$, then we say that the edge uv is* uncovered *(by f). The* set of all uncovered edges *is denoted UE_f.*

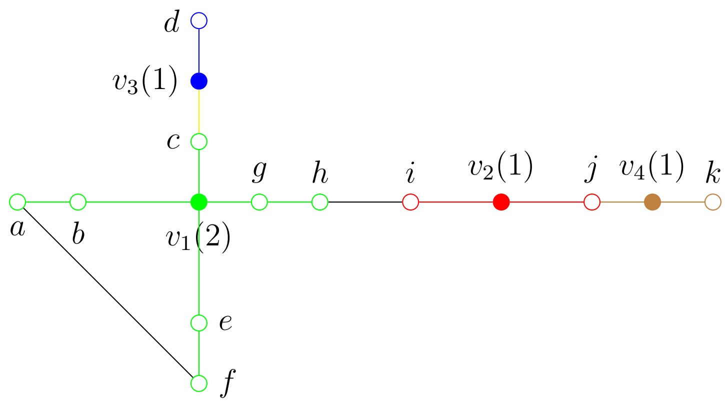


Figure 1.4: A dominating broadcast f with broadcast strengths shown in brackets: $V_f^+ = \{v_1, v_2, v_3, v_4\}$, $V_f^{++} = \{v_1\}$ and $V_f^1 = \{v_2, v_3, v_4\}$. The only edges uncovered by f are in black, or, equivalently $UE_f = \{af, hi\}$. The green edges are covered by v_1, the red by v_2, the blue by v_3, the brown by v_4 and the yellow edge is covered by v_3 and v_1. The neighbourhoods of V_f^+ are $N_f(v_1) = \{a, b, v_1, c, v_3, e, f, g, h\}$, $N_f(v_2) = \{i, v_2, j\}$, $N_f(v_3) = \{c, v_3, d\}$ and $N_f(v_4) = \{j, v_4, k\}$. The boundaries are $B_f(v_1) = \{a, f, h, v_3\}$, $B_f(v_2) = \{i, j\}$, $B_f(v_3) = \{c, d\}$ and $B_f(v_4) = \{j, k\}$. And the private boundaries are $PB_f(v_1) = \{a, f, h\}$, $PB_f(v_2) = \{v_2, i\}$, $PB_f(v_3) = \{d\}$ and $PB_f(v_4) = \{v_4, k\}$. The only vertices with $|H_f(v)| > 1$ are c, v_3 and j; $H(v_3) = H(c) = \{v_3, v_1\}$ and $H(j) = \{v_2, v_4\}$.

Remark 1.2.6. *If f is a broadcast on a graph G such that $UE_f = \emptyset$ and no edge is covered by more than one broadcasting vertex, then $\cup_{v \in V_f^+} B_f(v)$ forms an independent set on G. If f is a broadcast on a bipartite graph G, then since G is does not have any odd cycles, for any $v \in V_f^+$, $B_f(v)$ is an independent set.*

Erwin [10] proved the following result which we restate using the notion of private boundaries.

Proposition 1.2.7. [10] *A dominating broadcast f is minimal dominating if and only if $PB_f(v) \neq \emptyset$ for each $v \in V_f^+$.*

A broadcast is *irredundant* if $PB_f(v) \neq \emptyset$ for every $v \in V_f^+$ or, equivalently, if no broadcasting vertex can have its broadcast neighbourhood reduced without increasing the number of non-dominated vertices. Hence, Proposition 1.2.7 says that any dominating broadcast f is minimal dominating if and only if it is irredundant. In an *irredundant broadcast* every broadcasting vertex is either broadcasting with a strength of 1 and hears no other broadcasts, or it has a *boundary vertex* which does not hear any other broadcasting vertex or both. An irredundant broadcast f is *maximal irredundant* if no broadcast g with $f < g$ is irredundant. A maximally irredundant broadcast is not necessarily dominating, see Figure 1.5.

Given a dominating broadcast f on a graph G, if $N_f(v) \cap N_f(u) = \emptyset$ for all distinct $u, v \in V_f^+(G)$, or equivalently if $|H(v)| = 1$ for all $v \in V(G)$, then f is an *efficient broadcast* and our signals will not interfere or *overlap*. If f is a broadcast such that every vertex x which hears more than one broadcasting vertex also satisfies $d(x, u) \geq f(u)$ for all $u \in V_f^+$, then we say that the broadcast *only overlaps in boundaries*. If $f(v) - d(u, v) > 0$, then we say that u is *overdominated* (by v). If $|H(u)| > 1$ and u is *overdominated*, then the broadcast *overlaps beyond its boundaries.* In Figure 1.4, the broadcast values assigned to v_2 and v_1 meet the criteria for an efficient broadcast, the values assigned to v_2 and v_4 meet the criteria for a broadcast which only overlaps in boundaries but not those for an efficient broadcast and the values assigned to v_3 and v_1 overlap beyond their boundaries and thus do not meet the criteria for either of these two broadcast types.

Recall that the characteristic function of any independent set can be considered to be a broadcast f with $f : V(G) \to \{0, 1\}$; it has the following features which we generalize to define three different types of broadcast independence:

1 *bn*-**independent type:** Broadcasts only overlap in boundaries.

2 *h*-**independent type:** Broadcast vertices only hear themselves.

3 *s*-**independent type:** Broadcast vertices form an independent set.

We consider applying each of these conditions to broadcasts in general and note that $1 \implies 2 \implies 3$.

For dominating sets and independent sets we have the following established inequalities. For any graph G,

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$$

where $\gamma(G)$ and $\Gamma(G)$ are, respectively, the size of a minimum and maximum minimal dominating set and $i(G)$ and $\alpha(G)$ are, respectively, the size of a minimum maximal and maximum independent set. This inequality chain is the direct result of the fact that any maximal independent set is also a minimal dominating set. We would like a definition of broadcast independence which provides a similar chain for broadcasts.

In Figures 1.5, 1.9, and 1.12 we see, respectively, that while maximal $bn-, h-, s-$ independent broadcasts are dominating they are not necessarily minimal dominating broadcasts. A broadcast is *dominating* and *irredundant* if and only if it is minimal dominating and maximal irredundant [1]. Hence we consider our parameters with the additional condition of irredundance; as in Definitions 1.3.2, 1.3.6 and 1.3.9.

Recall that even a maximal irredundant broadcast can have non-dominated vertices as in Figure 1.5. This is why we will also consider our parameters with the additional condition of minimal domination; as in Definitions 1.3.3, 1.3.7 and 1.3.10.

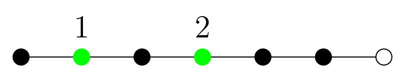


Figure 1.5: A maximal irredundant broadcast which is not dominating.

The *broadcast irredundance number*, $ir_b(G)$, and the *upper broadcast irredundance number*, $IR_b(G)$, of a graph G are defined as

$$ir_b(G) = \min\{\sigma(f) : f \text{ is a maximal irredundant broadcast of } G\},$$

and

$$IR_b(G) = \max\{\sigma(f) : f \text{ is an irredundant broadcast of } G\}.$$

As shown by Ahmadi et al [1]:

$$ir_b(G) \leq \gamma_b(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \Gamma_b(G) \leq IR_b.$$

1.3 Definitions for Independence

For broadcast independence, we introduce nine definitions consisting of three different categories which each contain three definitions. In general, if a broadcast f on a graph G meets one of our definitions of independence and there is no broadcast g such that $g > f$ and g still meets our definition of independence, then we say that f is a *maximal independent* broadcast for this type of independence. Otherwise f is not maximally independent and can be *extended* (for example, to g) to a larger weight broadcast which still meets the given definition of independence.

For all our independence definitions, the weight of a maximal independent broadcast depends on the choice of broadcasting vertices. Hence, for any graph G we will have a minimum maximal weight independent broadcast, denoted as an $i_{\text{independence type}}$ -broadcast, and a maximum independent broadcast, denoted as an $\alpha_{\text{independence type}}$ -broadcast. The weight of these minimum maximal and maximum broadcasts for a graph G will be denoted as $i_{\text{independence type}}(G)$ and $\alpha_{\text{independence type}}(G)$, respectively. These are the parameters that we are interested in and we illustrate that all nine definitions are distinct in this regard. Where expedient we may refer to a $i_{\text{independence type}}$ -broadcast on a graph G as an $i_{\text{independence type}}(G)$-broadcast, and similarly for an $\alpha_{\text{independence type}}(G)$-broadcast.

The first type of definition is based on broadcast neighbourhoods only overlapping in boundaries or equivalently, on $N_f(v) \cap N_f(u) \subseteq B_f(v) \cap B_f(u)$ for all $u, v \in V_f^+$.

1.3.1 $bn-, bnr-, bnd$-Independence: based on broadcast neighbourhoods only overlapping in boundaries

Definition 1.3.1. *A broadcast is* bn-independent *if the broadcast neighbourhoods overlap only in their boundaries. The* minimum (maximum) weight of a maximal *bn-independent broadcast on a graphs G is* $i_{bn}(G)$ $(\alpha_{bn}(G))$*.*

Definition 1.3.2. *A broadcast is* bnr-independent *if it is bn-independent and irredundant. The* minimum (maximum) weight of a maximal bnr-independent broadcast *on a graph G is* $i_{bnr}(G)$ $(\alpha_{bnr}(G))$*.*

Definition 1.3.3. *A broadcast is* bnd-independent *if it is a minimal dominating bn-independent broadcast. The* minimum (maximum) weight of a maximal *bnd*-independent broadcast *on a graph G is* $i_{bnd}(G)$ $(\alpha_{bnd}(G))$*.*

In Figure 1.6, we see that a maximal *bn*-independent broadcast is not necessarily *bnd*- or *bnr*- independent. In Figure 1.7, we see that a maximal *bnr*- or *bnd*-independent broadcast need not be maximal *bn*-independent. However, if a broadcast is *bnr*-independent and dominating, then it is minimally dominating so it is a *bnd*-independent broadcast. Similarly any *bnd*-independent broadcast is *bnr*-independent or it would not be minimally dominating. Finally, in Figure 1.8, we see that a maximal *bnr*-independent broadcast need not be dominating.

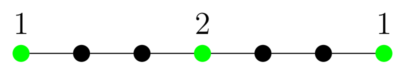


Figure 1.6: A maximal *bn*-independent broadcast which is not minimally dominating and hence not *bnr*− or *bnd*-independent.

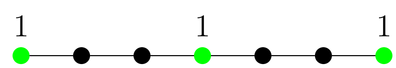


Figure 1.7: A maximal *bnr*−, *bnd*-independent broadcast which is not maximal *bn*-independent.

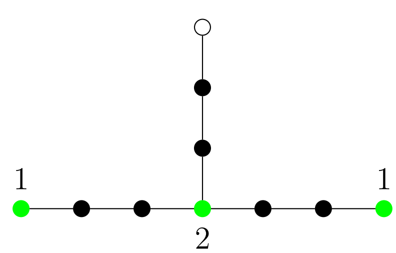


Figure 1.8: A maximal *bnr*-independent broadcast which is not maximal *bn*− or *bnd*-independent.

Remark 1.3.4. *Although the bnr-independent broadcast in Figure 1.8 can be extended to a maximal bn-independent broadcast it cannot be extended to a bnd-broadcast.*

1.3.2 h, hr, hd-Independence: based on broadcast vertices only hearing themselves

The second type of definition is based on broadcast vertices only hearing themselves. If a broadcast f is hearing independent, then, for all $u \in V_f^+$, $H(u) = \{u\}$. Erwin [9] introduced this definition and the definition with the additional restriction of minimal domination in his Ph.D. dissertation. It has also been considered further by Erwin and by others [10, 9, 5, 2, 3]. We mention some results due to Erwin [10], Bessy and Rautenbach [3], and Ahmane, Bouchemakh, and Sopena [2] in Chapter 2 and further results due to Bouchemakh and Zemir [5] for grids in Chapter 4.

Definition 1.3.5. *A broadcast is* h-independent *if every broadcast vertex only hears itself. The* minimum (maximum) weight of a maximal h-independent broadcast *on a graph G is $i_h(G)$ $(\alpha_h(G))$.*

Definition 1.3.6. *A broadcast is* hr-independent *if it is an irredundant h-independent broadcast. The* minimum (maximum) weight of a maximal hr-independent broadcast *on a graph G is $i_{hr}(G)$ $(\alpha_{hr}(G))$.*

Definition 1.3.7. *A broadcast is* hd-independent *if it is a minimal dominating h-independent broadcast. The* minimum (maximum) weight of a maximal hd-independent broadcast *on a graph G is $i_{hd}(G)$ $(\alpha_{hd}(G))$.*

In the figures below, notice that a maximal h-independent broadcast need not be hr- or hd-independent (1.9) and that a maximal hr- or hd-independent broadcast need not be maximally h-independent (1.10). Further, an hr-independent broadcast is not necessarily dominating and hence is not always hd-independent (1.11).

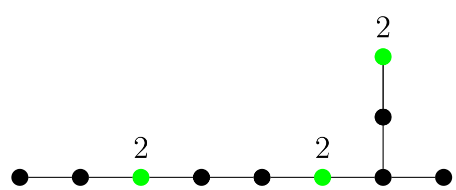


Figure 1.9: A maximal h-independent broadcast which is not minimal dominating and thus not hr- or hd-independent.

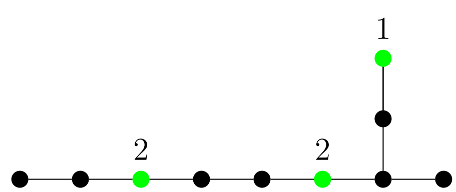


Figure 1.10: A maximal hr-,hd-independent broadcast which is not maximal h-independent.

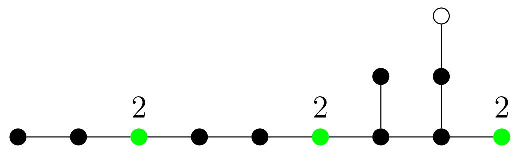


Figure 1.11: A broadcast which is maximal hr-independent but not maximal $hd-$ or maximal h-independent.

1.3.3 s, sr, sd-Independence: based on broadcast vertices forming an independent set

The third type of definition is based on the set of all broadcasting vertices forming an independent set.

Definition 1.3.8. ***7****. A broadcast f is* s-independent *if V_f^+ is an independent set. The* minimum (maximum) weight of a maximal s-independent broadcast *on a graph G is $i_s(G)$ ($\alpha_s(G)$).*

Definition 1.3.9. ***8****. A broadcast is* sr-independent *if it is an irredundant s-independent broadcast. The* minimum (maximum) weight of a maximal sr-independent broadcast *on a graph G is $i_{sr}(G)$ ($\alpha_{sr}(G)$).*

Definition 1.3.10. ***9****. A broadcast is* sd-independent *if it is a minimal dominating s-independent broadcast. The* minimum (maximum) weight of a maximal sd-independent broadcast *on a graph G is $i_{sd}(G)$ ($\alpha_{hd}(G)$).*

In Figure 1.12, we see that a maximal s-independent broadcast is not necessarily sr- or sd-independent. And, in Figure 1.13, we see that a maximal sr- or sd-independent broadcast is not necessarily maximally s-independent.

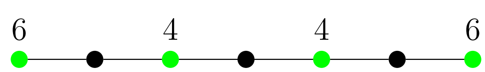


Figure 1.12: A maximal s-independent broadcast which is not minimal dominating and thus not sr-, sd-independent.

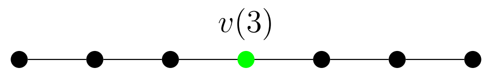


Figure 1.13: A maximal $sr-, sd$-independent broadcast. The broadcasting vertex set, $V^+ = \{v\}$, does not form a maximal independent set. Hence it is not a maximal s-independent broadcast.

In Figure 1.14, we see the relationships between the independence parameters.

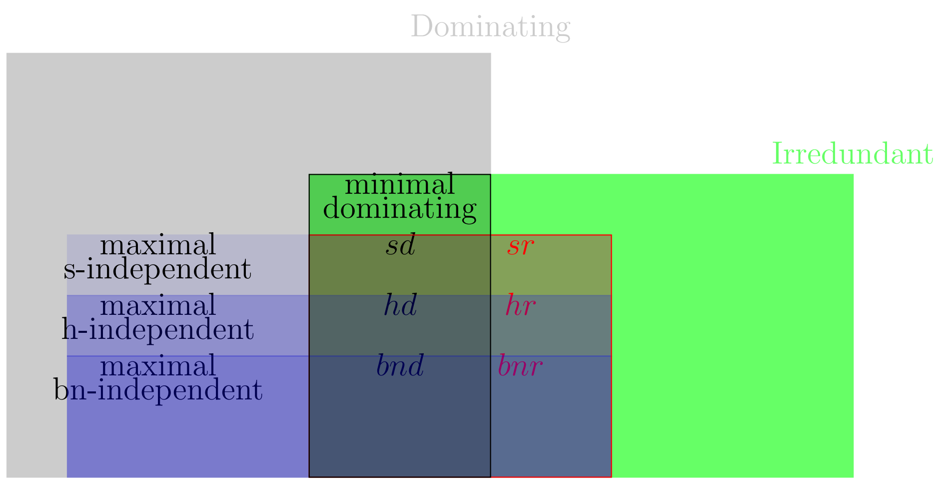


Figure 1.14: Venn Diagram of independent broadcasts for different independence parameters.

Remark 1.3.11. *Every minimal dominating broadcast is irredundant and dominating (see comments directly after Proposition 1.2.7). So for a graph G, the set of all bnd-independent broadcasts is contained in the set of all bnr-independent broadcasts with an analogous result for definitions based on hearing or set independence.*

Remark 1.3.12. *Recall that boundary independence implies hearing independence which implies broadcasting vertex set independence. So for a graph G, the set of all bn-independent broadcasts is contained in the set of all h-independent broadcasts which is contained in the set of all s-independent broadcasts with a similar result holding when irredundance or minimal domination is added.*

1.4 Overview

Some research [10, 5, 8, 2, 3] has been done on h-independence and hd-independence. Our goal is to further these results and to investigate the minimum maximal and the maximum weight of broadcasts meeting our new definitions for independent broadcasts.

In Chapter 2, we present background information on dominating and irredundant broadcasts and on dominating and independent sets. We show that all our minimum maximal independent broadcast parameters which require irredundance or minimal domination are bounded above by the minimum weight of a dominating broadcast, $\gamma_b(G)$, with equality for $i_{sd}(G)$, $i_{hd}(G)$, $i_{bnd}(G)$ on all graphs G and for $i_{bn}(G')$ and $i_h(G')$ on all radial graphs G'. For parameters which require irredundance, we show that $ir_b(G)$, the minimum weight of a maximal irredundant broadcast, forms an important bound:

$$ir_b(G) \leq i_{sr}(G) \leq i_{hr}(G) \leq i_{bnr}(G) \leq \gamma_b(G) \leq \frac{5}{4} ir_b(G).$$

We show that it is possible that:

$$i_{sr}(G) \leq i_{hr}(G) \leq i_{bnr}(T) < \gamma_b(T).$$

For bn-independence, we show that:

$$i_{bn}(G) \leq \lceil \frac{4\gamma_b(G)}{3} \rceil.$$

Upper bounds on all maximum independent parameters are found with the exception of $\alpha_s(G)$. We observe that if a broadcast of a graph G is restricted to a subgraph of G, then it maintains $h-$, $s-$ and $bn-$ independence. Hence we note the importance of results on trees for studying $h-$, $s-$ and $bn-$independent broadcasts in general. We determine that for all boundary-type independence, the set of edges covered by each broadcasting vertex together with the uncovered edges forms a partition on $E(G)$. Partitioning $E(G)$ yields the result that for any broadcast f which has boundary type independence, $\sigma(f) \leq m - \sum_{v \in V_f^+} \deg(v) + |V_f^+|$. In particular, we find that $\alpha_{bn}(G) \leq \alpha_{bn}(T_n) \leq n - 1$, where T_n is any spanning tree of G, and characterize the trees which meet this bound as paths and spiders. We investigate the structure of bn- and bnr-broadcasts on trees and determine that for any tree T, leaves only hear leaves and that there is always an $\alpha_{bn}(\alpha_{bnr})$-broadcast in which the only broadcast vertices which are broadcasting with a strength greater than 1 are leaves. Using $\diamond$ to indicate that two values have no fixed order, we determine the inequality chains for the bn-type independent parameters to be:

$$\gamma_b(G) = i_{bnd}(G) \leq \alpha_{bnd}(G) \leq \Gamma_b(G),$$

$$i_{bnr}(G) \leq \gamma_b(G) \leq \alpha_{bnr}(G) \diamond \Gamma_b(G),$$

and

$$\gamma_b(G) \leq i_{bn}(G) \leq \alpha_{bn}(G) \diamond \Gamma_b(G).$$

For maximum hearing independence we note Dunbar et al.'s generalization of Erwin's bound $\alpha_h(G) \geq \mu(\text{diam}(G) - 1)$ with equality if G is a path. We report Bessy and Rautenbach's bound for $\alpha_h(G)$, which implies that $\alpha_h(G) < 4\alpha(G)$. Adapting their proof techniques [3], we show that $\alpha_h(G) < 2\alpha_{bn}(G)$ and $\alpha_h(G) < 3\alpha_{bnr}(G)$. Using our bound on $\alpha_{bn}(G)$, we give a new bound for h-independence. On any graph G with order $n \geq 2$, $\alpha_h(G) < 2(n-1)$. Although bn-independence must share the complexity of h-independence and $\alpha(G)$, in Chapter 6, we find a better upper bound for $\alpha_{bn}(G)$ and thus also for $\alpha_h(G)$. Finally, by using these relations along with Dunbar et al's lower bound for $\alpha_h(G)$, we note that $\alpha_{bn}(G) > \frac{1}{2}\mu(G)(\text{diam}(G) - 1)$.

In Chapter 3, we revisit all the parameters and look for results specific to paths. For the minimums, we determine that for any integer $n \geq 4$, $i_{bn}(P_n) = i_h(P_n) = \lceil \frac{2n}{5} \rceil$ and for all n, $i_{sr}(P_n) = \lceil \frac{n}{3} \rceil$. We give an example construction of an i_s-broadcast on P_n. For the maximums, we generate an α_s-broadcast on P_n and we recall Erwin's result that $\alpha_h = 2(n-2)$. All other maximum independence values, for a path, P_n, take on the value $n - 1$.

In Chapter 4, we give specific results for grids. We notice that, since grid graphs are radial, all lower broadcast independence parameters which do not require irredundance are equal to $\text{rad}(G_{m,n})$. We conjecture that $i_{sr}(G_{m,n}) = \text{rad}(G_{m,n})$, in which case the lower broadcast independence number for all our parameters would be $\text{rad}(G_{m,n})$. For the maximums, we present Bouchemakh and Zemir's results for hearing independence on grids and notice that $\alpha_h(G_{m,n})$ meets Erwin's bound for hearing independence, namely $\alpha_h \leq \max\{\alpha(G_{m,n}), 2(\text{diam}(G_{m,n}) - 1)\}$. We adapt Bouchemakh and Zemir's techniques to show that:

$$\alpha_{bnd}(G_{m,n}) = \alpha_{bnr}(G_{m,n}) = \alpha_{bn}(G_{m,n}) = \alpha(G_{m,n})$$

and that this bound applies to bipartite graphs in general.

In Chapter 5, we examine the relationships between $\alpha_{bnr}(G)$, $\alpha_{bn}(G)$ and $\Gamma_b(G)$

in greater detail. We show that, while $\alpha_{bnr}(G) - \Gamma_b(G)$ and thus $\alpha_{bn}(G) - \Gamma_b(G)$ are unbounded:

$$\alpha_{bnr}(G)/\Gamma_b(G) \leq \alpha_{bn}(G)/\Gamma_b(G) \leq 2.$$

We show, by example, that $\Gamma_b(G) - \alpha_{bn}(G)$ and $\Gamma_b(G) - \alpha_{bnr}(G)$ are unbounded for graphs in general. We give a second example showing that $\Gamma_b(T) - \alpha_{bnr}(T)$ is also unbounded for trees. Finally, we show that for bipartite graphs with $n \geq 2$,

$$\Gamma_b(G)/\alpha_{bn}(G) \leq \Gamma_b(G)/\alpha_{bnr}(G) < 2$$

and we give an example to show that $\Gamma_b(G)/\alpha_{bn}(G)$ and $\Gamma_b(G)/\alpha_{bnr}(G)$ are both unbounded for graphs in general.

In Chapter 6, we revisit trees getting our most important results on the maximum boundary independence parameter on trees. We further our observations on the structure of bn-independent broadcasts on trees and as a result we are able to improve the upper bound on $\alpha_{bn}(T)$ and express this bound in terms of the size of T and the structure of the vertices in $B(T) = \{v : v \in V(T) \text{ and } \deg(v) \geq 3\}$. Specifically, we show that:

$$\alpha_{bn}(T) \leq n - b(T) + \rho(T)$$

where $\rho(T)$ is a well-defined subset of $B(T)$. We determine a class of graphs which meet this bound as well as examples of trees which fall below the bound. In fact, we show a construction of a bn-independent broadcast on all trees which provides a lower bound for $\alpha_{bn}(T)$. We conjecture a better upper bound for $\alpha_{bn}(T)$. We present bounds and constructions of α_{bn}-broadcasts for trees with $|B(T)| = 2$ and for trees in which the paths connecting the vertices in $B(T)$ induce the generalized spider. For the latter we present an algorithm for generating an α_{bn}-broadcast. We present some preliminary results on understanding the structure of bnr-broadcasts on trees, leaving an interesting open problem regarding $\alpha_{bnr}(T_n)$. Finally, we note that our bound for $\alpha_{bn}(T)$ implies that $\alpha_h(G) < 2\min\{n - b(T) + \rho(T) : T \text{ is a spanning tree of } G\}$.

Chapter 2

Existing and Preliminary Results and Observations

2.1 Known Results

2.1.1 Dominating broadcasts

Erwin was the first to consider the broadcast domination problem [10]. Since we are interested in independent and dominating broadcasts, we include some of his initial results as well as those of researchers who have furthered his work. We look for ways to apply these results to our new independence parameters and treat them as a starting place to forward the research on the existing definitions.

Note 2.1.1. [10] *For every graph G, $\gamma_b(G) \leq min\{\text{rad}(G), \gamma(G)\}$.*

If $\gamma_b(G) = \text{rad}(G)$, then G is a *radial* graph; a *radial* tree is a radial graph which is a tree. The problem of characterizing radial trees was first addressed by Dunbar et all in [7], further studied in [8] and finally resolved by Herke and Mynhardt in [15] with a geometrical interpretation of the characterization. Since some of our parameters lie between $\gamma_b(G)$ and $\text{rad}(G)$, this characterization will be useful.

Used by Herke and Mynhardt [15] to find the characterization of radial trees is the idea of a *split-P* set which is a technique useful for decomposing (splitting) a tree into components which can each be dominated with a radial broadcast. If it is not possible to form such components, then the tree itself is radial. In a tree T where

P is a *diametrical* (longest) path, a non-empty subset M of edges of P is a *split-P* set if the end vertices of its edges all have degree 2 in T and if each component T' of $T - M$ has a diametrical path consisting of $T' \cap P$ which is of even positive length. If M is a *split-P-set* for some diametrical path P in a tree T, then it is a *split-set* of T. For all trees of even diameter, $M = \emptyset$ is a *split-P* set. And if a tree T of odd diameter has no nonempty split sets, then $M = \emptyset$ is a split-set of T.

If a tree has a nonempty split-set M, then each of the components of $T - M$ is a radial tree with even diameter so if we broadcast from the centre of each component with the strength of its radius, then we have a dominating broadcast. Notice that this broadcast does not cover the edges of M. For an example see Figure 2.1.

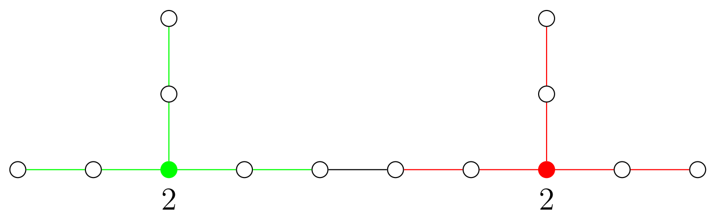


Figure 2.1: A tree T with a diametrical path of length 9 and a split-set M consisting of the single edge in black. The two components of $T - M$ are shown in red and green. By examining cases, one can show that this broadcast is the best possible. Hence $\gamma_b(T) = 4$ while $\text{rad}(T) = 5$.

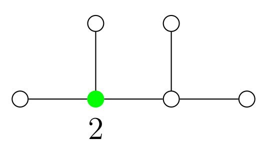


Figure 2.2: A tree T with a diametrical path of length 3 and no nonempty split-set. Hence $\gamma_b(T) = \text{rad}(T) = 2$.

Theorem 2.1.2. [15] *A tree is radial if and only if it has no nonempty split-set.*

Proposition 2.1.3. [14] *If G is connected, then $\gamma_b(G) = min\{\gamma_b(T) : T$ is a spanning tree of $G\}$.*

So any upper bounds for minimal broadcasts on trees on n vertices apply to graphs on n vertices, for example:

Corollary 2.1.4. [15] *If G is a connected graph such that no spanning subtree of G has a nonempty split-set, then G is radial.*

In his thesis, Erwin [10] makes observations about the location of dominating vertices in a $\gamma_b(G)$-broadcast. If $v \in V(G)$ and $\deg(v) = 1$, then we say that v is an *end vertex* of G. If a graph G has a dominating broadcast f with an end vertex v with $f(v) > 1$, then let u be any neighbour of v and create a new dominating broadcast g with $g(v) = 0$, $g(u) = f(v) - 1$ and $g(x) = f(x)$ otherwise. Notice that $\sigma(g) < \sigma(f)$. This leads to the following results:

Theorem 2.1.5. [10] *Part 1: Let f be a γ_b-broadcast on a graph G with order at least 3.*

i) If v is an endvertex of G and $v \in V^+$, then $f(v) = 1$.

ii) If G contains an endvertex v, there is a γ_b-broadcast f such that $v \notin V_f^+(G)$.

Part 2:

i) If T' is a subtree of a tree T, then $\gamma_b(T') \leq \gamma_b(T)$.

ii) If T' is a tree obtained by adding new leaves to vertices already adjacent to leaves in the tree T, then $\gamma_b(T') = \gamma_b(T)$.

Part 3: If G is a nontrivial connected graph, then

$$\gamma_b(G) \geq \left\lceil \frac{\operatorname{diam}(G) + 1}{3} \right\rceil .$$

An *efficient broadcast* f is a broadcast in which every vertex hears exactly one vertex in V_f^+.

Theorem 2.1.6. [8] *Every graph has a γ_b-broadcast which is efficient.*

If a broadcast f is efficient we can form a *ball graph* $B_G(f)$ which is the graph obtained by contracting the vertices in $N_f(u)$ to a single vertex for all $u \in V_f^+$. Figure 2.3 shows an efficient broadcast f on a tree T. The ball graph for this broadcast would consist of the three coloured vertices or, $B_T(f) = P_3$.

A *very efficient* γ_b- broadcast f on a graph G is a broadcast in which every vertex of V_f^+ lies on a path or a cycle.

A *very efficient* γ_b- broadcast f on a tree T is an efficient broadcast in which all vertices of V_f^+ lie on some diametrical path P of T and which does not overdominate the end vertices of P unless T is a bicentral radial tree [14].

Lokshtanov and Heggernes note that:

Theorem 2.1.7. [12] *Every graph G has a* very efficient γ_b-broadcast f*.*

Corollary 2.1.8. [14] *For any tree T, let f be a very efficient γ_b-broadcast with r broadcast vertices and let M be a split-set of maximum cardinality m. In this case,*

$$\gamma_b(T) = \sigma(f) = \mathrm{rad}(T) - \lfloor \frac{r}{2} \rfloor = \mathrm{rad}(T) - \lceil \frac{m}{2} \rceil.$$

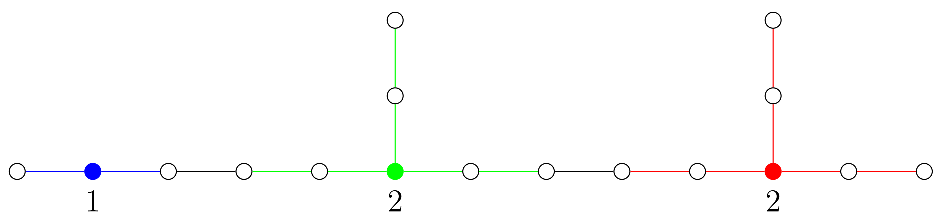


Figure 2.3: A tree T with a *very efficient* broadcast, a diametrical path of length 12, $\mathrm{rad}(T) = 6$ and a split-set M consisting of the two edges in black, $m = 2$. The three components of $T - M$ are shown in blue, red and green. There are 3 broadcasting vertices hence $r = 3$ and $\gamma_b(T) \leq \mathrm{rad}(T) - \lfloor \frac{r}{2} \rfloor = \mathrm{rad}(T) - \lceil \frac{m}{2} \rceil = 5$.

2.1.2 Irredundant broadcasts

Recall that although a minimal dominating broadcast is always irredundant, a *maximal irredundant* broadcast it not always dominating. We define U_f to be *the set of all vertices non-dominated by f*. Using the fact that when f is maximally irredundant no member of U_f can be added to V_f^+ without losing irredundancy, Mynhardt and Roux [17] show the following results:

Proposition 2.1.9. [17] *If an irredundant broadcast f is maximal irredundant, then for each $u \in U_f$ there exists $v \in V_f^+$ such that $PB_f(v) \subseteq N(u)$. In particular, each vertex in U_f is a distance $f(v) + 1$ from some vertex $v \in V_f^+$.*

Theorem 2.1.10. [17] *For any graph G, $\gamma_b(G) \leq \frac{5}{4} ir_b(G)$.*

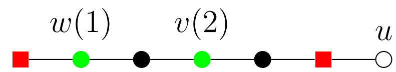


Figure 2.4: A maximal irredundant broadcast f on a tree T which is not dominating, $\{u\} = U_f$, $d(u,v) = f(v) + 1 = 3$. The red squares show the respective private boundary sets of w and v. The vertex adjacent to u is the only vertex in $PB_f(v)$. Note that T is radial and $\gamma_b(T) = 3 \leq \frac{5}{4} ir_b(G) = \frac{5}{4}(3)$ as in Mynhardt and Roux's Theorem 2.1.10.

2.2 Minimum Maximal Independent Broadcasts

The characteristic function of a maximal independent set satisfies all of our definitions of independence and is maximal irredundant and minimal dominating. Hence it is a maximal bnr, bnd, hr, hd, sr, sd-independent broadcast. Hence

$$i_{bnr}(G), i_{bnd}(G), i_{sd}(G), i_{sr}(G), i_{hd}(G), i_{hr}(G) \leq i(G).$$

All of our definitions for an independent broadcast f require the broadcasting vertices V_f^+ to form an independent set. Hence if there is a maximal independent broadcast f such that $V_f^+ = V_f^1$, then V_f^+ is the already defined independent dominating set and the following known results apply.

Lemma 2.2.1. [18] *A set of vertices in a graph is an independent dominating set if and only if it is a maximal independent set.*

Theorem 2.2.2. [18] *Every claw-free graph G has an independent dominating set of size $\gamma(G)$.*

To form a maximal s-independent broadcast the only restriction is that the broadcasting vertices form an independent set. Hence $f(v) = e(v)$ for every $v \in V_f^+$ and this definition produces high weight broadcasts with a lot of redundancy:

$$i_s(G) = \min\left\{\sum_{v\in I} e(v) : I \text{ is a maximal independent set of } G\right\}.$$

2.2.1 Independent broadcasts can be maximal without being dominating

As stated earlier, a maximal irredundant broadcast need not be dominating.

Proposition 2.2.3. *For any graph G,*

$$ir_b(G) \leq i_{sr}(G) \leq i_{hr}(G) \leq i_{bnr}(G) \leq \gamma_b(G) \leq \frac{5}{4} ir_b(G).$$

Proof. From Theorem 2.1.6, every graph has a minimum weight dominating broadcast which is efficient. Such a broadcast cannot be extended without losing its irredundance, so it is a maximal $bnr-, hr-, sr$-independent broadcast. If a broadcast f is maximal sr-independent, then either V_f^+ forms a maximal independent set or f is maximally irredundant or both. If V_f^+ is a maximal independent set, then f is a dominating broadcast and thus is maximally irredundant. Hence any maximal sr-independent broadcast is maximal irredundant and $ir_b(G) \leq i_{sr}(G)$. As stated in Remark 1.3.12, a maximal bnr-independent broadcast is a maximal hr-independent broadcast which in turn is a sr-independent broadcast. By Theorem 2.1.10, $\gamma_b(G) \leq \frac{5}{4} ir_b(G)$. Hence $ir_b(G) \leq i_{sr}(G) \leq i_{hr}(G) \leq i_{bnr}(G) \leq \gamma_b(G) \leq \frac{5}{4} ir_b(G)$. □

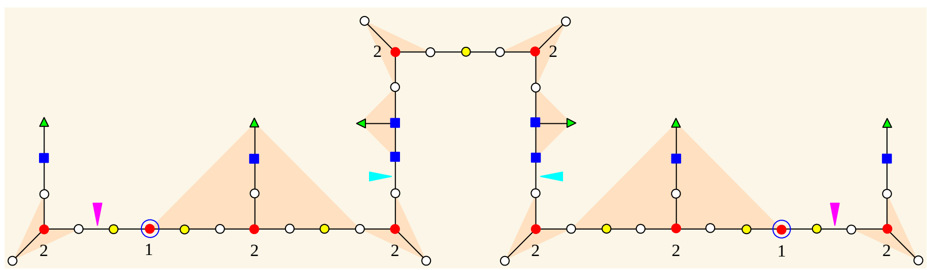


Figure 2.5: A tree with $i_{hr}(T), i_{sr}(T), i_{bnr}(T) < \gamma_b(T)$.

In Figure 2.5, as given in [17], we see a broadcast which is $s-, h-$ and bn-independent. The red vertices are broadcasting, the green triangles represent non-dominated vertices, the blue squares (and circles) are private boundaries and the yellow vertices represent overlap. Since every extension of the above broadcast eliminates the private boundary set of some vertex, the broadcast is maximally irredundant. Therefore $i_{sr}(T), i_{hr}(T), i_{bnr}(T) \leq 18$. The graph is not radial, as is seen by the small pointer triangles which show two different possible maximal split-sets. Using either split-set and a radial broadcast on each resulting component we get a minimum weight dominating broadcast [15]. Since both split-sets have size $m = 2$, Herke's Formula from Corollary 2.1.8 gives us $\gamma_b(T) = \text{rad}(T) - \lceil \frac{m}{2} \rceil = 19$ and we have an example of strict inequality in our bound: $i_{hr}(T), i_{sr}(T), i_{bnr}(T) \leq 18 < 19 = \gamma_b(T)$.

2.2.2 Independent broadcasts which are dominating when maximal

In the case of independent broadcast definitions which require minimal domination of the graph, all the minimum maximal parameters equal the broadcast number.

Proposition 2.2.4. *For any graph G, $\gamma_b(G) = i_{bnd}(G) = i_{hd}(G) = i_{sd}(G)$.*

Proof. Given a graph G, any maximal bnd, hd or sd-broadcast on G is by definition minimally dominating. Hence $\gamma_b(G) \leq i_{bnd}(G), i_{hd}(G), i_{sd}(G)$. By Theorem 2.1.6, there exists an efficient $\gamma_b(G)$-broadcast. Since any efficient minimal dominating broadcast is maximally $bnd-, hd-, sd$-independent, $\gamma_b(G) \geq i_{bnd}(G), i_{hd}(G), i_{sd}(G)$. □

Erwin [10] observes that any dominating broadcast which is not h-independent can be reduced to a dominating h-independent broadcast. Erwin furthers the results of Theorem 2.1.5, noting the following for hearing independent broadcasts:

Corollary 2.2.5. [10] *If G is a graph with order at least 3, then there exists an h-independent $\gamma_b(G)$- broadcast f on G such that no endvertex is in the f-dominating set.*

Although the broadcast in Erwin's corollary is h-independent , it is not necessarily maximal h-independent.

Proposition 2.2.6. *For any graph G, $\gamma_b(G) \leq i_{bn}(G), i_h(G) \leq \mathrm{rad}(G)$.*

Proof. Any maximal bn- or h-independent broadcast is dominating because there would be no reason not to put a 1 on any non-dominated vertex. Hence $\gamma_b(G) \leq i_{bn}(G), i_h(G)$. The definition for bn- or h-independence is met by a radial broadcast, thus giving the upper bound. □

Remark 2.2.7. *If G is radial, then Proposition 2.2.6 implies that $i_{bn}(G) = i_h(G) = \gamma_b(G)$.*

Recall that Heggernes and Lokshtanov [12] have shown that any graph G has a *very efficient optimal dominating broadcast* f on G, an efficient broadcast f such that the domination graph $B_G(f)$ is either a path or a cycle. We use this fact, along with the following well known result, to find an upper bound for $i_{bn}(G)$.

Theorem 2.2.8. *If the graph G is a path P_n or a cycle C_n, then $i(G) = \lceil \frac{n}{3} \rceil$.*

Proof. Suppose $i(G) < \lceil \frac{n}{3} \rceil$. Let I be a maximal independent set with $|I| = i(G)$. Partition G into $\lceil \frac{n}{3} \rceil - 1$ successive disjoint subpaths each consisting of 3 vertices, say a_i, b_i, c_i, and one subpath P' of length $k \pmod 3$ where $1 \leq k \leq 3$. Every subpath of length 3 must contain at least one vertex from I or b_i is not dominated. Due to the size of I, every subpath of length 3 contains exactly one vertex from I and there are no vertices in $I \cap P'$. Hence $k \equiv 1, 2 \pmod 3$ and P' consists of a single vertex $a \notin I$ or two vertices $a, b \notin I$. To dominate a, the vertex $c_{\lceil \frac{n}{3} \rceil - 1} \in I$. To dominate $a_{\lceil \frac{n}{3} \rceil - 1}$, the vertex $c_{\lceil \frac{n}{3} \rceil - 2} \in I$. This pattern continues until we conclude that $c_1 \in I$. Hence $b_1 \notin I$. In all cases $N(a_1) \subset \{b_1, a, b\}$, hence a_1 is non-dominated, I is not maximal and $|I| \geq \lceil \frac{n}{3} \rceil$.

If $k \equiv 0 \pmod 3$ let $I = \{b_i : 1 \leq i \leq \frac{n}{3}\}$ and if $n \equiv 1, 2 \pmod 3$ let $I = \{b_i : 1 \leq i \leq \lceil \frac{n}{3} \rceil - 1\} \cup \{a\}$. Notice that I is independent and $|I| = \lceil \frac{n}{3} \rceil$. Since b_i dominates a_i, b_i, c_i and a dominates a and b (if it exists), I is dominating. Hence $i(G) \leq |I| = \lceil \frac{n}{3} \rceil$ and the result follows. □

Proposition 2.2.9. *For any graph G and any very efficient $\gamma_b(G)$-broadcast f, $i_{bn}(G) \leq \gamma_b(G) + \left\lceil \frac{|V_f^+|}{3} \right\rceil$.*

Proof. By Theorem 2.1.7, we choose f to be a very efficient optimal broadcast on G and note that $B_G(f)$ is a path or a cycle. Let D be a minimum independent dominating set of $B_G(f)$. By Theorem 2.2.8, $|D| = i(B_G(f)) = \left\lceil \frac{|V_f^+|}{3} \right\rceil$. Let g be the broadcast on G obtained by increasing $f(v)$ to $f(v) + 1$ for each $v \in D$ and leaving $f(v)$ unchanged otherwise. If $x \in N_g(v) \cap N_g(u)$, then u, v are adjacent in $B_G(f)$ and since D is an independent dominating set either $g(u) = f(u)$ or $g(v) = f(v)$. Hence, without loss of generality, $x \in B_g(u) \cap B_f(v)$ and f is bn-independent. Also, note that since f is dominating and D is a dominating set there is no $v \in V_g^+$ such that $B_g(v) - PB_g(v) = \emptyset$. Hence g is a maximal bn-independent broadcast. □

Corollary 2.2.10. *For any graph G, $i_{bn}(G) \leq \left\lceil \frac{4\gamma_b(G)}{3} \right\rceil$.*

Proof. By Proposition 2.2.9, there is a minimal dominating broadcast f such that $i_{bn}(G) \leq \gamma_b(G) + \left\lceil \frac{|V_f^+|}{3} \right\rceil$. Notice that $|V_f^+| \leq \sigma(f) = \gamma_b(G)$ and the result follows. □

Erwin [10] illustrates (see Figure 2.6) a difference between h-independent and dominating *broadcasts* and independent and dominating *sets* with the example of the double star $S_{t,t}$ which is obtained from P_2 by adding t pendant vertices at each vertex. If $t > 1$, then $\gamma(S_{t,t}) = 2$ (use the vertices of P_2) and $i(S_{t,t}) = t+1$ (use all end vertices of one star and the central vertex of the other). However, $i_h(S_{t,t}) = \gamma_b(S_{t,t}) = 2$ (broadcast from either vertex of P_2 with strength 2).

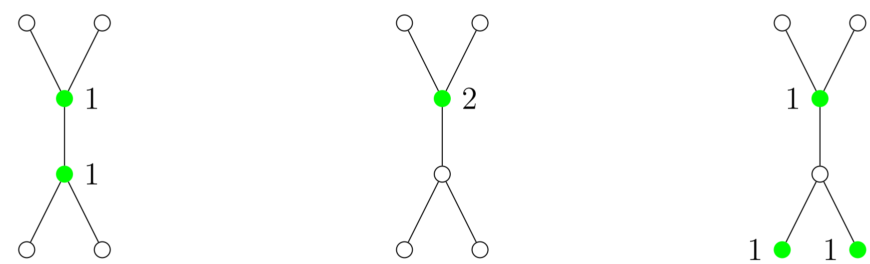


Figure 2.6: A tree $S_{2,2}$ is shown with $\gamma(S_{t,t}) = 2$ (left), $i_h(S_{2,2}) = \gamma_b(S_{2,2}) = 2$ (middle) and $i(S_{2,2}) = 2 + 1$ (right).

2.3 Maximum Independent Broadcasts

To obtain a trivial lower bound for all the upper independence parameters, note that broadcasting with a strength of $\operatorname{diam}(G)$ from a peripheral vertex produces an independent broadcast of each type. Also, the characteristic function of a maximum independent set is an independent broadcast of each type. Hence

$$\alpha_{(bn,bnr,bnd,h,hr,hd,s,sr,sd)}(G) \geq \max\{\operatorname{diam}(G), \alpha(G)\}$$

for all graphs G.

2.3.1 Set independence

Again, an s-independent broadcast is different from all the others. We note that

$$\alpha_s(G) = \max\left\{\sum_{v \in I} e(v) : I \text{ is an independent set of } G\right\}.$$

2.3.2 Independent broadcasts which might be maximal without dominating

Definitions 1.3.2, 1.3.6, 1.3.9 imply that

$$\alpha_{bnr}(G) \le \alpha_{hr}(G) \le \alpha_{sr}(G) \le IR_b, \tag{2.1}$$

where IR_b is the size of a maximum irredundant broadcast.

2.3.3 Types of independent broadcasts which are minimal dominating when maximal

By definition, any α_{bnd}-, α_{hd}-, or α_{sd}-broadcast is minimal dominating. And by Remark 1.3.12, $\alpha_{bnd}(G) \le \alpha_{hd}(G) \le \alpha_{sd}(G)$. Hence

$$\alpha_{bnd}(G) \le \alpha_{hd}(G) \le \alpha_{sd}(G) \le \Gamma_b(G), \tag{2.2}$$

where $\Gamma_b(G)$ is weight of a maximum minimal dominating broadcast on G.

So far the inequality chains for sr and sd-independence are:

$$ir_b(G) \le \gamma_b(G) = i_{sd}(G) \le \gamma(G) \le i(G) \le \alpha(G) \le \alpha_{sd}(G) \le \Gamma_b(G) \le IR_b(G)$$

$$ir_b(G) \le i_{sr}(G) \le \gamma_b(G) \le \gamma(G) \le i(G) \le \alpha(G) \le \alpha_{sr}(G) \le IR_b(G).$$

2.3.4 Boundary independence

Remark 2.3.1. *Given a non-dominating bn-independent broadcast f on a graph G, recall that U_f is the set of vertices which are not dominated by f. For some $u \in U_f$, create a new broadcast $g_u = (f - \{(u,0)\}) \cup \{(u,1)\}$. Notice that g_u is a bn-independent broadcast such that $\sigma(g_u) > \sigma(f)$. In this case, we say that f has been extended (to g_u). This process can continue until we have a broadcast g with $U_g = \emptyset$. We say that f has been extended to produce a dominating bn-independent broadcast.*

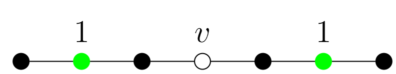


Figure 2.7: A bn-broadcast f which leaves v non-dominated.

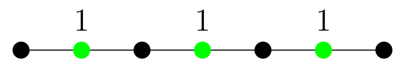


Figure 2.8: The *bn*-broadcast f from above has been extended to produce a dominating broadcast g_v.

Theorem 2.3.2. *A bn-independent broadcast f on a graph G with $V_f^+ = \{v\}$ is maximal bn-independent if and only if $f(v) = e(v)$. A bn-independent broadcast f on a graph G with $|V_f^+| > 1$ is maximal bn-independent if and only if it is dominating and $B_f(v) - PB_f(v) \neq \emptyset$ for each vertex $v \in V^+$.*

Proof. Consider a maximal *bn*-independent broadcast f. By remark 2.3.1, f is dominating. Suppose $V_f^+ = \{v\}$. To dominate the graph $f(v) = e(v)$. Suppose $|V_f^+| \geq 2$. If there is a vertex v such that $B_f(v) - PB_f(v) = \emptyset$, then $f' = (f - \{(v, f(v))\}) \cup \{(v, f(v)+1)\}$ is a *bn*-independent broadcast with $\sigma(f) < \sigma(f')$. Hence $B_f(v) - PB_f(v) \neq \emptyset$ for each vertex $v \in V^+$.

Conversely, suppose f is a dominating *bn*-independent broadcast. If v is the only broadcasting vertex, then $f(v) = e(v)$. Therefore, f is maximal. Otherwise, suppose that $B_f(v) - PB_f(v) \neq \emptyset$ for each vertex $v \in V_f^+$. Consider any $v \in V(G)$ and define $f' = (f - \{(v, f(v))\}) \cup \{(v, f(v)+1)\}$. If $v \in V_f^+$, let $u \in B_f(v) - PB_f(v)$ and let $w \in V_f^+ - \{v\}$ be a vertex such that $u \in B_f(w)$, then $u \in (N_{f'}[v] \cap N_{f'}[w]) - B_{f'}(v)$, hence f' is not *bn*-independent. Similarly, if $f(v) = 0$, then $v \in N_f[w]$ for some $w \in V_f^+$. And $v \notin B_{f'}(v)$ but $v \in N_{f'}[v] \cap N_{f'}[w]$, a contradiction to *bn*-independence. Therefore, f is *maximal bn*-independent. $\square$

Remark 2.3.3. *If f is a bn-, bnr-, or bnd-independent broadcast on a graph G, then each edge of G is covered by at most one vertex $v \in V_f^+$. Suppose for a contradiction that an edge uv is covered by two f-broadcasting vertices say x, y. By the definition of covered, $\{u, v\} \nsubseteq B_f(x)$ but $\{u, v\} \subseteq N_f(x) \cap N_f(y)$. This violates the bn-independence of f.*

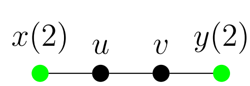


Figure 2.9: A broadcast f with the edge uv covered by x and y. Note that $v \in B_f(x) \cap (N_f(y) - B_f(y))$. Hence f is not *bn*-independent.

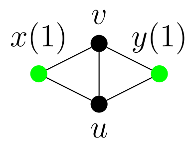


Figure 2.10: A broadcast f. The edge uv is not covered by x or y. Note that $\{u, v\} \subseteq B_f(x) \cap B_f(y)$.

Remark 2.3.3 leads to observations which apply to all three boundary independent definitions. In these cases we use α_{BN} to represent ($\alpha_{bn}, \alpha_{bnr}$, and α_{bnd}) broadcast values and refer to BN-independence instead of (bn, bnr and bnd)-independence. We indicate that the result applies to only two of the definitions by using parenthesis, for example $\alpha_{bn(bnr)}$ is short for two separate broadcast identifiers: α_{bn} and α_{bnr}.

Proposition 2.3.4. *For any graph G of size m and any BN-independent broadcast f on G, $\sigma(f) \leq m - \Sigma_{v \in V_f^+} \deg(v) + |V_f^+|$.*

Proof. In a BN-independent broadcast, every edge is covered by at most one broadcast vertex. Since $f(v) \leq e(v)$, every broadcast vertex v covers at least $f(v) + \deg(v) - 1$ edges. So, by counting covered edges,

$$\Sigma_{v \in V_f^+}(f(v) + \deg(v) - 1) \leq m,$$

and

$$\Sigma_{v \in V_f^+} f(v) + \Sigma_{v \in V_f^+} \deg(v) - \Sigma_{v \in V_f^+} 1 \leq m,$$

and

$$\sigma(f) + \Sigma_{v \in V_f^+} \deg(v) - |V_f^+| \leq m,$$

thus

$$\sigma(f) \leq m - \Sigma_{v \in V_f^+} \deg(v) + |V_f^+|. \quad \square$$

Corollary 2.3.5. *For any tree T of order $n \geq 2$, $\alpha_{bn}(T) \leq n - 1$.*

Proof. For trees of order n, $m = n - 1$. And since T is connected, $\Sigma_{v \in V^+} \deg(v) \geq |V^+|$. $\square$

To characterize trees which satisfy equality in Corollary 2.3.5, we need the following result.

Proposition 2.3.6. *For a tree T of order $n \geq 2$, $\alpha_{bn}(T) = n-1$ if and only if there is a bn-independent broadcast such that all edges are covered, every broadcasting vertex is also a leaf and for all $v \in V_f^+(T)$, v covers $f(v)$ edges.*

Proof. Let f be an α_{bn}- broadcast on a tree T. From Proposition 2.3.4, $\sigma(f)$ is bounded above by the number of covered edges, with equality if and only if $\Sigma_{v \in V_f^+} \deg(v) = |V_f^+|$. This occurs if and only if every broadcast vertex is a leaf. Given that the broadcasting vertices are all leaves, the maximum of $n-1$ is reached if and only if all edges are covered. Since no two broadcasting vertices cover the same edge (Remark 2.3.3), all edges are covered if and only if the edge-sets $E(v)$ covered by each $v \in V_f^+(T)$ form a partition on $E(T)$. Finally, by observations in Proposition 2.3.4, $|E(v)| \geq f(v)$. Assuming all previous conditions and by the pigeon hole principle, $\sigma(f) = n-1$ if and only if $|E(v)| = f(v)$ for all $v \in V_f^+(T)$. $\square$

For $k \geq 3$ and $n_i \geq 1$ for $1 \leq i \leq k$, let the generalized spider, $S = S(n_1, n_2, ..., n_k)$, be the tree which has exactly one vertex b with $\deg(b) = k$ and for which the k components of $S(n_1, n_2, ..., n_k)) - \{b\}$ are paths of length $(n_1 - 1), (n_2 - 1), ..., (n_k - 1)$ respectively. We call $S(n, n, ..., n)$ the *spider* $S(n^k)$. See Figure 2.11 for an example of $S(2, 2, 2, 2, 2, 2)$ or, equivalently $S(2^6)$. Also note that the star $K_{1,k} = S(1^k)$, $k \geq 3$.

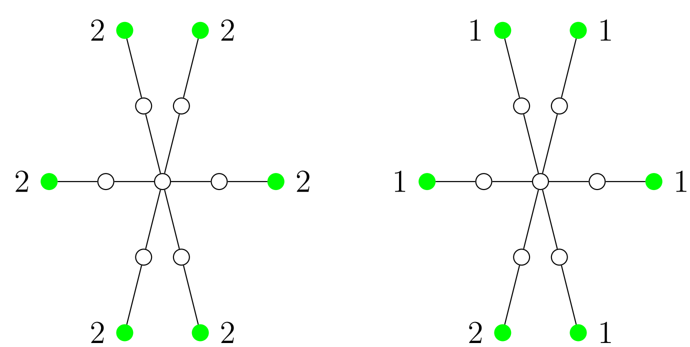


Figure 2.11: A spider $S = S(2, 2, 2, 2, 2, 2) = S(2^6)$ is shown with a $\alpha_{bn}(S) = |E(S)| = 12$ (left) and $\Gamma_b(S) = \alpha_{bnr}(S) = \alpha_{bnd}(S) = m - (k-1) = 7$, where m is the size of S.

Proposition 2.3.6 allows us to characterize the trees for which equality holds in the bound given in Corollary 2.3.5 for bn-independent broadcasts.

Corollary 2.3.7. *For a tree of order $n \geq 2$, $\alpha_{bn}(T) = n - 1 \iff T = P_n$ or $T = S(n_1, n_2, ..., n_k)$, a generalized spider.*

Proof. From Proposition 2.3.6, $\sigma(f) = n-1$ if and only if, for all $v \in V_f^+$, v covers $f(v)$ edges. This holds if and only if for all $v \in V_f^+$, v is a leaf and the subgraph induced by $N_f(v)$ is a path of length $f(v)$. Since T is connected and f is bn-independent, the subpaths all have exactly one vertex in common in $V(T)$, namely their non-broadcasting leaf. Thus we have the characterization: $\alpha_{bn}(T) = n-1 \iff T = P_n$ or $T = S(n_1, n_2, ..., n_k)$, the generalized spider. □

The same characterization does not always apply to *bnr* or *bnd*. The irredundance required by these broadcasts implies that each broadcasting vertex must have a nonempty private boundary. For $K_{1,k} = S(1, 1, ..., 1)$, let f broadcast from all leaves with strength 1, $\sigma(f) = k = n-1 = \alpha_{bn}(K_{1,k})$ and since every broadcasting vertex is in its own private boundary f is *bnr*(*bnd*)-independent. Hence f is a α_{BN}-broadcast.

However, in the generalized spider, where l_i is a leaf and b is the central vertex, if $d(l_i, b) > 1$ and l_i dominates b, then $H(b) = l_i$. And in an irredundant bn-independent broadcast $H(b) \leq 1$. We give an example of a *bnr*(*bnd*)-broadcast for all spiders $S = S(n_1, n_2, ..., n_k)$, in which $UE_f \neq \emptyset$. This broadcast forms a lower bound for $\alpha_{bnr(bnd)}(S)$ and we leave an open problem as to when this broadcast is an $\alpha_{bnr(bnd)}$-broadcast.

Proposition 2.3.8. *Let S be a spider $S(n_1, n_2, ..., n_k)$ with $k \geq 3$, b the vertex of degree k, $O = \{l : deg(l) = 1$ and $d(l, b) = 1\}$ and n the order. Then*

$$\alpha_{bnd(bnr)}(S) \geq n - 1 - k + \max\{|O|, 1\}.$$

Proof 1. Let $L(S) = \{l_1, l_2, ..., l_k\}$ be the leaves of S. If $O \neq \emptyset$, define a broadcast f with $f(l_i) = 1$ if $l_i \in O$, $f(l_j) = n_j - 1$ if $l_j \in L(S) - O$ and $f(x) = 0$ otherwise. If $l_i \in O$ then $l_i \in PB_f(l_i)$ and if $l_j \in L(S) - O$ then the vertex v_j adjacent to b on the $l_j - b$ path is in the private boundary of l_j. Hence f is irredundant. If $O = \emptyset$, define a broadcast f' with $f'(l_1) = n_1$, $f'(l_j) = n_j - 1$ for all $2 \leq j \leq k$ and $f'(x) = 0$ otherwise. For $2 \leq j \leq k$, the vertex v_j adjacent to b on the $l_j - b$ path is in the private boundary of l_j and $PB_{f'}(l_1) = \{b\}$. Hence f' is irredundant. Since, in both cases, no vertex overdominates b, f and f' are both bn-independent. Further, in both f and f', b is dominated by at least one leaf and every leaf l dominates itself as well as the internal vertices of the $b - l$ path. Thus $f$and f' are dominating. Hence f and f' are, in their respective cases, maximal *bnr*(*bnd*)-broadcasts. By counting edges, the respective weights are $\sigma(f) = n - 1 - (k - |O|)$ and $\sigma(f') = n - 1 - (k - 1)$. □

It seems that the broadcast in Proposition 2.3.8 is a $\alpha_{bnr(bnd)}$-broadcast for many spiders. However, for $S = S(n_1, n_2, 1)$ with $n_1, n_2 \geq 2$ a broadcast from an antipodal leaf l with $f(l) = diam(S)$ is a $bnr(bnd)$-broadcast with $\sigma(f) = n - 2$, a weight greater than the broadcast described in Proposition 2.3.8. The following remark may be useful in determining whether or not $S(n_1, n_2, 1)$ is the only exception.

Remark 2.3.9. *If for all $v \in V_f^{++}$ there exists $v' \in PB_f(v)$ such that v' is not a leaf, then $|UE_f| \geq \left\lceil \frac{V_f^{++}|}{2} \right\rceil$.*

Question 2.3.10. *Determine which spiders meet lower bound of Proposition 2.3.8 and, for those that do not, find $\alpha_{bnr(bnd)}(S(n_1, n_2, ..., n_k))$.*

Proposition 2.3.11. *For any graph G of order $n \geq 2$,*

$$\alpha_{bn}(G) \leq \min\{\alpha_{bn}(T) : T \textit{ is a spanning tree of } G\} \leq n - 1.$$

Proof 1. Spanning subgraphs induced by removal of edges maintain boundary-independence. Therefore, if T is a spanning tree of G, then every α_{bn}-broadcast on G is a bn-independent broadcast on T so by definition $\alpha_{bn}(G) \leq \alpha_{bn}(T)$. □

We include an induction proof that for $n \geq 2$, $\alpha_{bn}(G) \leq n - 1$:

Proof 2. Observe that the statement is true for $n = 2$. Let $n > 2$ and assume that the statement is true for all graphs of order $n - 1$. Suppose that there is a graph G with order $n > 2$ and $\alpha_{bn}(G) > n - 1$. Let f be a bn-independent broadcast on G with $\sigma(f) > n - 1$. If there is a vertex $v \in V(G)$ such that $v \in B_f(w)$ for some $w \in V_f^+$and v is not a cut-vertex, then either $V_f^+ = \{w\}$ and $\sigma(f) = e(v) \leq \text{diam}(G) \leq n - 1$, or f is also a bn-independent broadcast on the graph $G - v$. Either case violates our hypothesis, so all boundary vertices are cut-vertices. Choose any $w \in V_f^+$ and any $v \in B_f(w)$ and consider the $k \geq 2$ components $G_1, G_2...G_k$ of $G - v$. Let $\sigma(f_m) = \Sigma_{v \in V(G_m)} f(v)$. Since $v \in B_f(w)$, each f_m is a maximal bn-independent broadcast on the graph made from reconnecting v to G_m in the obvious way. By our induction hypothesis $\sigma(f_m) \leq (|V(G_m)| + 1) - 1$. Therefore $\sigma(f) = f_1 + f_2 + ... + f_k \leq |V(G_1)| + |V(G_2)| + ... + |V(G_k)| = n - 1$. □

It will be useful to understand and restrict the structure of a maximal *bn*- independent broadcast. The following lemmas are used to show that every tree has a maximum *bn*-independent broadcast whose broadcast vertices are either leaves or they are broadcasting with a strength of 1 or both.

Lemma 2.3.12. *For any tree T and any α_{bn}-broadcast of T, no leaf of T hears a broadcast from any non-leaf vertex.*

Proof. By the arguments of Corollary 2.3.7, our statement is true for any path. Hence we assume that T is not a path. Suppose T has an α_{bn}-broadcast f in which a leaf hears a non-leaf v. Either v overdominates all leaves in $N_f(v)$ or there is at least one leaf u such that $d(u, v) = f(v)$.
Case 1: There exists a nonleaf vertex $v \in V_f^+$ and a leaf u such that $d(u, v) = f(v)$. Let w be the vertex on the uv path which is closest to u and has $\deg(w) > 2$; if no such vertex exists, then let $w = v$. Since v is not a leaf, $\deg(w) \geq 2$ in both cases. Create a new broadcast, g_1, with

$$g_1(x) = \begin{cases} f(v) - d(u, w) & : \text{if } x = v \\ d(u, w) & : \text{if } x = u \\ f(x) & : \text{if } x \neq u, v. \end{cases}$$

Notice that $N_{g_1}(v) \cup N_{g_1}(u) \subseteq N_f(v)$, and either $f(v) = 0$ or $N_{g_1}(v) \cap N_{g_1}(u) = \{w\}$. Hence g_1 is *bn*-independent and has the same weight as f. If $f(v) > 0$, then, since $\deg(w) > 2$, w now has a non-dominated neighbour. If $f(v) = 0$, then $v = w$ and since v is not a leaf it has at least one non-dominated neighbour. We can define a new *bn*-independent broadcast by letting $g_1'(u) = g_1(u)+1$ and $g_1'(x) = g_1(x)$ otherwise. In either case g_1 can be extended to produce a larger weight *bn*-independent broadcast, contradicting the maximality of our $\alpha_{bn}(T)$-broadcasts.
Case 2: Suppose v overdominates all leaves in $N_f(v)$. There are two sub-cases.

Case 2a: Suppose there exist $a, b \in B_f(v)$ such that $a = v_0, v_1, ...v_{2f(v)} = b$ is a path P of length $2f(v)$ in $N_f(v)$. Define a new broadcast g_2 by

$$g_2(x) = \begin{cases} 0 & : \text{ if } x = v_i \text{ and } i \equiv 0 \pmod 2 \\ 1 & : \text{ if } x = v_i \text{ and } i \equiv 1 \pmod 2 \\ f(x) & : \text{otherwise.} \end{cases}$$

Notice that $g_2(v_0) = g_2(v_{2f(v)}) = 0$, $\bigcup_{i=0}^{2f(v)} N_{g_2}[v_i] \subseteq N_f(v)$ and no two adjacent vertices of P are both broadcasting. Hence g_2 is bn-independent. Also note that g_2 has the same weight as f. Hence, $\sigma(g_2) = \alpha_{bn}(T)$ and is maximal. Since g_2 is maximal, it is dominating and every leaf in $N_f(v)$ hears a v_i in g_2. Thus the broadcast now satisfies our first set of conditions so we can follow case 1 to obtain a contradiction.

Case 2b: Finally, if v overdominates all leaves in $N_f(v)$ and there is no path of length $2f(v)$ in $N_f(v)$, then there is exactly one edge e incident with v which is on all $v - w$ paths for $w \in B_f(v)$. Let u be a neighbour of v that is not incident with e. This choice is possible because $\deg(v) \geq 2$. Create a new broadcast, g_3, with

$$g_3(x) = \begin{cases} f(v) + 1 & : \text{ if } x = u \\ 0 & : \text{ if } x = v \\ f(x) & : \text{ if } x \neq u, v. \end{cases}$$

Note that $N_{g_3}(u) \subseteq N_f(v)$, hence g_3 is bn-independent and $\sigma(g_3) > \sigma(f)$ which again contradicts the maximality of f. Since we have exhausted all possibilities for f, the result follows. □

For $\alpha_{bnr}(T)$-broadcasts, the arguments of Theorem 2.3.12 are adapted in Theorem 2.3.13 to show that although there may be α_{bnr}-broadcasts of T in which a leaf hears a non-leaf, we can always find at least one α_{bnr}-broadcast in which no leaf hears a non-leaf.

Theorem 2.3.13. *Any tree T has an α_{bnr}-broadcast in which no leaf of T hears a broadcast from any non-leaf vertex.*

Proof. Among all α_{bnr}-broadcasts of T, let f be one such that the number of leaves that hear the broadcast from a non-leaf vertex is a minimum. Assume there is a non-leaf v which dominates at least one leaf, else our result is shown. Either there is a path of length $2f(v)$ in $N_f(v)$ or there is not.

Case 1: Suppose there exist $a, b \in B_f(v)$ such that $P : a = v_0, v_1, ...v_{2f(v)} = b$ is a path of length $2f(v)$ in $N_f(v)$. Define a new broadcast g by

$$g(x) = \begin{cases} 0 & : \text{ if } x = v_i \text{ and } i \equiv 0 \pmod 2 \\ 1 & : \text{ if } x = v_i \text{ and } i \equiv 1 \pmod 2 \\ f(x) & : \text{otherwise.} \end{cases}$$

Notice that g has the same weight as f. Since $g(v_0) = g(v_{2f(v)}) = 0$, $\bigcup_{i=0}^{2f(v)} N_g(i) \subseteq N_f(v)$. Also, $v_i \in PB_g(v_i)$ for all $i \equiv 1 \pmod 2$. Hence g is *bnr*-independent. Since $\alpha_{bnr}(T) = \sigma(f) = \sigma(g)$, g is maximal. Suppose that there is a leaf l which is dominated by v in f and is also dominated by g. Since f and g are *bnr*-independent a leaf can be dominated by at most one vertex. By the construction of g, l is dominated by some v_j in P with $j \equiv 1 \pmod 2$ and $g(v_j) = 1$. Define a new broadcast g' with $g'(v_j) = 0$, $g'(l) = 1$ and $g'(x) = g(x)$ otherwise. Since $l, v_j \in PB_{g'}(l)$, g' is *bnr*-independent. Notice that $\sigma(g') = \sigma(g)$, hence g' is also maximal *bnr*-independent. In either case, g or g' both dominate fewer leaves with nonleaves, hence the choice of f is contradicted and there is no such path.

Case 2: Suppose there is no path P of length $2f(v)$ in $N_f(v)$. Since v is not a leaf, $k = \deg(v) \geq 2$. Let $v_1, ..., v_k$ be the neighbours of v. Since $f(v) \leq e(v)$, there exists $y \in B_f(v)$, suppose without loss of generality that v_1 is on a $v - v_1 - y$ path. Of all paths in $N_f(v)$ which contain v but not v_1 let $P : v - t$ be the longest. If t is not overdominated, then the $t - y$ path has length $2f(v)$ and contradicts our choice of v. And if t is not a leaf, then P is not the longest path of its type. Hence t is a leaf which is overdominated by v. Further note that all paths from v not containing v_1 must also terminate in leaves which are overdominated by v. Define a new broadcast with $g(t) = f(v) + d(t, v)$, $g(v) = 0$ and $g(x) = f(x)$ otherwise. Notice that $N_g(t) = N_f(v)$ and g is a *bnr*-independent broadcast. Since $\sigma(g) > \sigma(f)$, g contradicts the maximality of f.

This covers all possible cases, hence we may assume that no such v exists and that f is the desired broadcast. $\square$

Theorem 2.3.14. *For all trees T there exists an $\alpha_{bn(bnr)}$-broadcast f such that for all $v \in V^+$, $f(v) = 1$ or $\deg(v) = 1$.*

Proof. Given any tree T which contradicts our statement, consider an $\alpha_{bn(bnr)}$-broadcast f in which no leaf hears a broadcast from a non-leaf and for which $|\{v : v \in V(T) \text{ and } \deg(v) > 1\} \cap V_f^{++}(T)|$ is a minimum. Such a choice is possible from Lemma 2.3.12 and Theorem 2.3.13. Let v be a non-leaf vertex in T with $f(v) > 1$. Since v does not broadcast to a leaf, there are at least two vertices in $B_f(v)$, such that v lies on the $u_1 - u_2$ path P. Label the vertices of $P : u_1 = v_0, v_1, \ldots, v_{2f(v)} = u_2$ and create a new broadcast:

$$g(x) = \begin{cases} 0 & : \text{if } x = v_i \text{ and } i \equiv 1 \pmod 2 \\ 1 & : \text{if } x = v_i \text{ and } i \equiv 0 \pmod 2 \\ f(x) & : \text{otherwise.} \end{cases}$$

Notice that $g(v_0) = g(v_{2f(v)}) = 0$, and $\bigcup_{i=0}^{2f(v)} N_g(i) \subseteq N_f(v)$. Also, $v_i \in PB_g(v_i)$ for all $i \equiv 1 \pmod 2$. Hence the new broadcast is still $bn(bnr)$-independent. Further, the weight of the broadcast is unchanged. Hence g is maximal $bn(bnr)$-independent. The number of non-leaf vertices in V_g^{++} is one less than the number of such vertices in V_f^{++} which contradicts the choice of f and the result follows. □

We make one more observation about the structure of α_{bn}-broadcasts on trees.

Lemma 2.3.15. *If f is an α_{bn}-broadcast on a tree T such that $|V_f^1|$ is maximum, then $PB_f(v) = \emptyset$ for all $v \in V^{++}$.*

Proof. Given a tree T, consider an α_{bn}-broadcast f on T such that $|V_f^1|$ is maximum. Either our statement is true or there exists $v \in V^{++}$ such that $u \in PB_f(v)$. Define a new broadcast g with:

$$g(x) = \begin{cases} f(x) - 1 & : x = v \\ 1 & : x = u \\ f(x) & : x \neq u, v. \end{cases}$$

The new broadcast g is bn-independent and has the same weight as f. If g can be extended, then it violates the maximality of f. Hence we conclude that g is an α_{bn} -broadcast. But $|V_g^1| > |V_f^1|$, which contradicts the choice of f. Therefore $PB_f(v) = \emptyset$ for all $v \in V_f^{++}$. □

We now establish some preliminary results on the inequality chains for our new parameters. The weight of a maximum boundary-independent broadcast is incomparable with the size of a maximum minimal dominating broadcast. First, to show that $\alpha_{bn}(S(2^k)) > \Gamma_b(S(2^k))$ we need the following result.

Proposition 2.3.16. *For $k \geq 3$, $\Gamma_b(S(2^k)) = n - k$.*

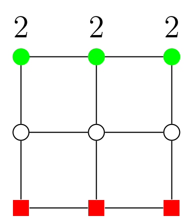
Proof. Notice that $n = 2k + 1$ and $n - k = k + 1$. Let $L(S) = \{l_1, l_2, ..., l_k\}$ be the leaves of S. For $1 \le t \le k$, and $l_t \in L(S)$, let x_t be the vertex adjacent to l_t. Let f be a Γ_b-broadcast on S.

Case 1: Suppose that there is a leaf l_i that hears a vertex $v_i \neq l_i$. If v_i is a leaf then v_i dominates the graph and $\sigma(f) = f(v_i) = diam(S) = 4 \le k+1$. If v_i is not a leaf then there exists a leaf l such that v_i is on the $l-b$ path and l also hears v_i. It is possible but not necessary that $l = l_i$. To maintain irredundancy, $f(l) = 0$. If $PB_f(v_i) - \{l, v_i\} \neq \emptyset$, then make a new broadcast g with $g(v_i) = 0$, $g(l) = d(l, v_i) + f(v_i)$ and $g(x) = f(x)$ otherwise. Notice that $N_g(l) = N_f(v_i)$ and $PB_g(l) = PB_f(v_i) - \{l\} \neq \emptyset$. Hence g is a minimal dominating broadcast with more weight than f which contradicts the choice of f. Hence $PB_f(v_i) \subseteq \{l, v_i\}$. If $v_i = b$, then since l_i hears v_i, $f(b) = 2$ and $PB_f(b) = L(S)$. Hence $v_i = x_i$ and since $PB_f(v_i) \subseteq \{l, v_i\}$, $f(v_i) = 1$ and $B_f(v_i) = \{l, b\}$. If $v_i \in PB_f(v_i)$, then make a new broadcast with $g(l) = 1$, $g(v_i) = 0$ and $g(x) = f(x)$ otherwise. Notice that $\sigma(f) = \sigma(g)$. Since $b \notin PB_f(v_i)$ and $PB_g(l) = \{l, v_i\}$ and all other neighbourhoods are unchanged, g is a Γ_b-broadcast. Using g in place of f allows us to postpone analyzing this case. Assuming that $PB_f(v_i) = \{l\}$, there exists v_j, $j \neq i$, such that $v_i \in N_f(v_j)$. In order for v_j to dominate v_i, $V(S) - N_f(v_j) \subseteq (L(S) - \{l\})$. Without loss of generality, there are 3 possible choices for v_j. If $v_j = b$, then $f(b) = 1$, $f(l_t) + f(x_t) \le 1$ for all $l_t \in L(S) - \{l\}$, $f(v_i) = 1$ and $f(x) = 0$ otherwise. Thus $\sigma(f) = k + 1$. If $v_j \neq b$ and v_j is not a leaf then $f(v_j) = 2$ and it overdominates a leaf $l_j \neq l$. To maintain irredundancy $f(l_j) = 0$ and there is a leaf $l' \neq l, l_j$ such that $f(l') = 0$, $f(l_t) + f(x_t) \le 1$ for all $l_t \in L(S) - \{l, l', l_j\}$, $f(v_i) = 1$ and $f(x) = 0$ otherwise. Thus, f is not a dominating broadcast contradicting the choice of f. Finally, if v_j is a leaf then $f(v_j) = 3$ and again, to maintain irredundancy, there is a leaf $l' \neq v_j, l$ such that $f(l') = 0$, $f(l_t) + f(x_t) \le 1$ for all $x \in L(S) - \{l, l', lv_j\}$, $f(v_i) = 1$ and $f(x) = 0$ otherwise. Again, f is not a dominating broadcast contradicting the choice of f.

Case 2: The only other possibility is that every leaf is broadcasting. To maintain irredundancy, at most one leaf can dominate b. If no leaf dominates b then $f(b) = 1$ and $f(l) = 1$ for all $l \in L(S)$ and $f(x) = 0$ otherwise. Thus $\sigma(f) = k + 1$. If a leaf l' dominates b and $f(l') = 2$, then $f(l) = 1$ for all $l \in L(S) - \{l'\}$ and $f(x) = 0$ otherwise. Thus $\sigma(f) = k+1$. If a leaf l' dominates b and $f(l') = 3$, then, to maintain irredundancy, there exists $l'' \neq l'$ such that $f(l'') = 0$ and $f(l_t) + f(x_t) = 1$ for all $l_t \in L(S) - \{l', l''\}$ and $f(x) = 0$ otherwise. Thus f is not a dominating broadcast. These are the only possibilities. In every case, the weight of the Γ_b-broadcast f is at most $k+1 = n-k$. And since a broadcast f exists for which $\sigma(f) = n-k$, the result is shown. $\square$

Proposition 2.3.17. $\alpha_{bn}(G) \diamond \Gamma_b(G)$.

Proof. A maximal bn-independent broadcast is dominating but not necessarily minimal dominating, so in some cases $\alpha_{bn}(G) > \Gamma_b(G)$. For example, as shown in Propositions 2.3.6 and 2.3.16, for $S(2^k)$ with $k \geq 3$, $\alpha_{bn}(S(2^k)) = n-1 > n-3 \geq n-k = \Gamma_b(S(2^k))$. If there is a minimal dominating α_{bn}-broadcast on a graph G, then $\alpha_{bn}(G) \leq \Gamma_b(G)$. For the path P_n, $\alpha_{bn}(P_n) = n-1 = \Gamma_b(P_n)$. For the grid $G_{3,3}$, $\alpha_{bn}(G_{3,3}) = 5 < 6 = \Gamma_b(G_{3,3})$. See Figure 2.12. $\square$



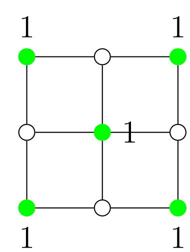


Figure 2.12: On the left, a Γ_b-broadcast on $G_{3,3}$. The three red squares are the respective private boundaries of the three broadcasting vertices. On the right, a maximum bn-independent broadcast. This grid is an example of $\Gamma_b(G_{3,3}) > \alpha_{bn}(G_{3,3})$.

The following proposition is used to show that the weight of a maximum irredundant-boundary-independent broadcast is also incomparable with the size of a maximum minimal dominating broadcast. To show that $\alpha_{bnr}(G) > \Gamma_b(G)$ we use the graph G_2 shown in Figure 2.13. In Chapter 5, we will examine the relationships between $\alpha_{bn}(G)$, $\alpha_{bnr}(G)$ and $\Gamma_b(G)$ in greater detail and will generalize this graph to get one of our results.

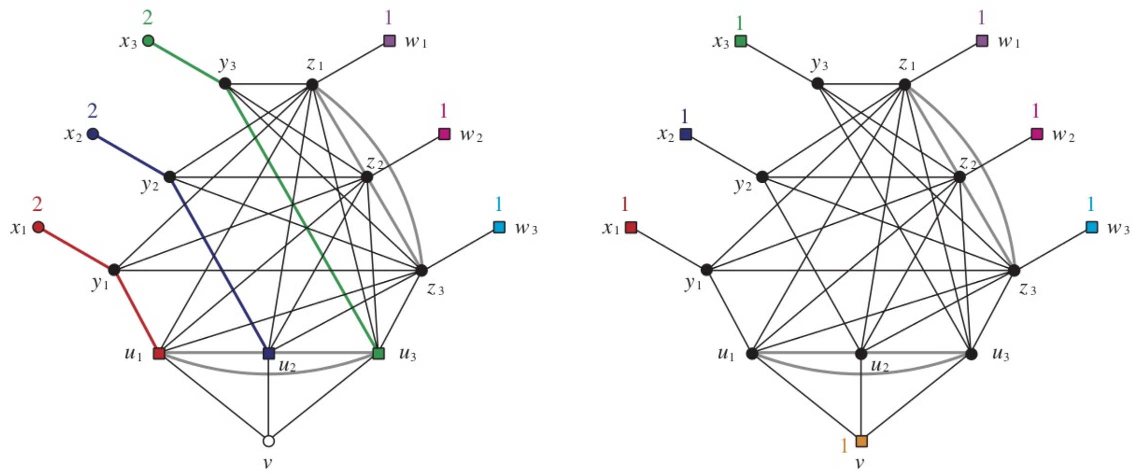


Figure 2.13: On the right, the graph G_2 with a non-dominating *bnr*-independent broadcast f; $\alpha_{\text{bnr}}(G_2) \geq \sigma(f) = 9$. On the left, a minimal dominating broadcast f'; in Proposition 2.3.19 we show that f' is a maximum minimum dominating broadcast thus $\Gamma_b(G_2) = \sigma(f') = 7$.

Proposition 2.3.18. *For the graph G_2, shown in Figure 2.13, $\alpha_{bnr}(G_2) \geq 9$.*

Proof. Define the broadcast f by

$$f(x) = \begin{cases} 2 & \text{if } x \in \{x_1, x_2, x_3\} \\ 1 & \text{if } x \in \{w_1, w_2, w_3\} \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = \{z_1, z_2, z_3\}$ and notice that $\sigma(f) = 9$ and

$$\left.\begin{array}{rcl} N_f(x_i) & = & \{x_i, y_i, u_i\} \cup Z \\ B_f(x_i) & = & \{u_i\} \cup Z \\ \text{PB}_f(x_i) & = & \{u_i\} \\ N_f(w_i) & = & \{w_i, z_i\} \\ \text{PB}_f(w_i) & = & \{w_i\} \end{array}\right\} \text{for } 1 \leq i \leq 3.$$

Thus we see that f is a bnr-independent broadcast, hence $\alpha_{\text{bnr}}(G_2) \geq 9$. □

The following proposition shows that $\Gamma_b(G_2) \leq 7$. Hence $\alpha_{\text{bnr}}(G_2) > \Gamma_b(G_2)$.

Proposition 2.3.19. *For the graph G_2 shown in Figure 2.13, $\Gamma_b(G_2) \leq 7$.*

Proof. For any minimal dominating broadcast f, due to symmetry, there are six possible ways to dominate the uncoloured vertex, v.

Case 1: The vertex v is dominated by v. To maintain irredundance, $f(u_1) = f(u_2) = f(u_3) = 0$. If $f(v) = 3$, then v dominates the graph, hence $\sigma(f) = 3$. If $f(v) = 2$, then to maintain irredundance there exists t such that either $y_t \in PB_f(v)$ or $z_k \in PB_f(v)$ or both. Hence either $f(z_1) = f(z_2) = f(z_3) = f(y_t) = f(x_t) = 0$ and for $i \neq t$, $f(x_i) + f(y_i) \leq 1$, or $f(y_1) = f(y_2) = f(y_3) = f(z_1) = f(z_2) = f(z_3) = f(w_t) = 0$ and $f(x_1), f(x_2), f(x_3) \leq 1$ and for $i \neq t$, $f(w_i) \leq 1$, or both. Hence $\sigma(f) \leq 5 + 2$. If $f(v) = 1$, then to maintain irredundance, for all $1 \leq j \leq 3$, $f(z_j) + f(w_j) \leq 1$ and $f(x_j) + f(y_j) \leq 1$. Hence $\sigma(f) \leq 2(3) + 1 = 7$.

Case 2: The vertex v is dominated, without loss of generality, by u_1. To maintain irredundance, $f(v) = 0$. Suppose $f(u_1) = 3$, then u_1 dominates G_2 and $\sigma(f) = 3$. Suppose $f(u_1) = 2$, then u_1 dominates $V(G_2) - \{x_2, x_3\}$. To maintain irredundancy, at most two other vertices can be broadcasting and $f(z_1) = f(z_2) = f(z_3) = f(x_1) = 0$. If, without loss of generality, $f(w_1) > 0$, then $f(w_1) = 3$ and u_1 and w_1 dominate the graph. If, without loss of generality, $f(x_2) > 0$, then either $f(x_2) = 1$, $f(x_3) + f(y_3) = 1$ and u_1, x_2 and x_3 (or y_3) dominate the graph, or $f(x_2) = 4$ and u_1 and x_2 dominate the graph. In all cases, $\sigma(f) \leq 6$.

If $f(u_1) = 1$, then, to maintain the irredundance of u_1, $f(x_1), f(y_2), f(y_3) \leq 2$, $f(x_2), f(x_3) \leq 3$ and for $1 \leq i \leq 3$, $f(w_i) \leq 2$. Notice that, to maintain a private boundary, $f(x_2), f(x_3) \neq 2$. If, without loss of generality, $f(x_2) = 3$, then x_2 and u_1 dominate $V(G_2) - \{x_1, x_3\}$ and to maintain irredundance, the only other possible broadcasting vertices are x_1, x_3, y_3 where $f(x_1), f(x_3) + f(y_3) \leq 1$. Hence, $\sigma(f) \leq 6$. Similarly, without loss of generality, if $f(y_2) = 2$, then $\sigma(f) \leq 5$. If $f(w_1) = 2$, then $PB_f(w_1) \subseteq \{y_2, y_3\}$. Hence, $f(x_2) + f(x_3) + f(y_2) + f(y_3) \leq 1$. And to maintain their irredundance, $f(x_1), f(w_2), f(w_3) \leq 1$. Hence, $\sigma(f) \leq 7$. The only remaining case is $V_f^+ = V_f^1$ and, in this case, $f(x_2) + f(y_2), f(x_3) + f(y_3) \leq 1$. Hence, $\sigma(f) \leq 7$.

Case 3-6: Suppose, without loss of generality, the vertex v is dominated by $s \in \{y_1, z_1, x_1, w_1\}$. Notice that $d(s, v) = e(s)$. Hence, the graph is dominated by s and $\sigma(f) = f(s) = e(s) \leq 3$.

This exhausts all possibilities, hence $\Gamma_b(G_2) \leq 7$. □

Proposition 2.3.20. $\alpha_{bnr}(G) \diamond \Gamma_b(G)$.

Proof. By definition, $\alpha_{bnr}(G) \leq \alpha_{bn}(G)$, so $\alpha_{bnr}(G_{3,3}) \leq \alpha_{bn}(G_{3,3}) < \Gamma_b(G_{3,3})$, as in

Proposition 2.3.17. For a BN- independent broadcast f on a graph G with n vertices we know that $\sigma(f) \leq n-1$ and a broadcast from a leaf of P_n with strength $n-1$ is bnr-independent, hence $\alpha_{bnr}(P_n) = n-1 = \Gamma_b(P_n)$. As shown in Proposition 2.3.19, $\alpha_{bnr}(G_2) > \Gamma_b(G_2)$. $\square$

So far the inequality chains for bn, bnr and bnd-independence are:

$$ir_b(G) \leq i_{bnd}(G) = \gamma_b(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \alpha_{bnd}(G) \leq \Gamma_b(G) \leq IR_b(G)$$

$$ir_b(G) \leq i_{bnr}(G) \leq \gamma_b(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \alpha_{bnr}(G) \diamond \Gamma_b(G) \leq IR_b(G)$$

$$\gamma_b(G) \leq i_{bn}(G) \leq \alpha_{bn}(G) \diamond \Gamma_b(G).$$

Also note that:

$$i_{BN}(G) \leq \text{rad}(G) \leq \text{diam}(G) \leq \alpha_{BN}(G).$$

If we only require bn-independence, then our maximal broadcasts all dominate but not necessarily minimally. If we add a condition of irredundance to get bnr-independence, then a dominating broadcast is minimal dominating but a maximal bnr-broadcast does not necessarily dominate. Hence, if we want our inequality chain to mimic the chain for independent and dominating sets, we need to force the condition of minimal domination. This is one reason for defining bnd-independent broadcasts.

2.3.5 Hearing independence

Recall that a broadcast on a graph G is h-independent if every broadcast vertex only hears itself, that is if $H(v) = \{v\}$ for all $v \in V^+$. Hence, $\alpha_h(G)$ is the maximum value of $\sigma(f)$ over all broadcasts f of G that satisfy $d(u,v) > \max\{f(u), f(v)\}$ for every pair of distinct vertices $u, v \in V_f^+$. This definition was introduced and studied by Erwin in his Ph.D. thesis [10]. Research has been furthered by Bouchemakh and Zemir [5], Dunbar, Erwin, Haynes, S.M. Hedetniemi, and S. T. Hedetniemi [8], Dunbar, S.M. Hedetniemi, and S. T. Hedetniemi [7] and recently by Ahmane, Bouchemakh and Sopena [2], and by Bessy and Rautenbach [3]. Some results from these papers follow. From Erwin [10]:

Theorem 2.3.21. [10] *Let f be an h-independent broadcast on a graph G. If $|V^+| = \{v\}$, then f is maximal h-independent if and only if $f(v) = e(v)$. If $|V^+| \geq 2$, then f*

is maximal h-independent if and only if it is dominating and for every $v \in V^+$, $f(v) = \min\{d(u,v) : u \in V^+ - \{v\}\} - 1$.

Erwin [10] further notes that if a connected graph $G \neq K_n$ has order $n > 2$, then any α_h-broadcast f on G has $|V_f^+| > 1$. Broadcasting from two antipodal vertices produces a broadcast with weight $2(\text{diam}(G)-1)$, giving us Erwin's first lower bound for $\alpha_h(G)$. From Theorem 2.3.21, we see that broadcasting vertices of a maximal h-independent broadcast can be assigned larger values if they are farther apart. Let $\mu(G)$ denote the size of the largest set of mutually antipodal vertices in G. Dunbar et al. refines Erwin's bound:

Proposition 2.3.22. [8] *For any connected graph G, $\alpha_h(G) \geq \mu(G)(\text{diam}(G)-1) \geq 2(\text{diam}(G)-1)$ and this bound is sharp.*

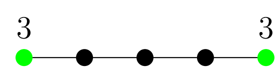


Figure 2.14: An $\alpha_h(P_5)$-broadcast f which meets Erwin's bound.

Erwin [10] was the first to show a graph which meets the bound: $\alpha_h(P_n) = 2(n-2)$. See Figure 2.14. We note that for $n \geq 1$, the spiders $S(n^k)$ with $k \geq 2$ also meet the bound: broadcast from each leaf with a strength of $2n-1$ for a total weight of $(2n-1)k = \mu(G)(\text{diam}(S(n^k)) - 1)$.

Erwin also notes the following properties of the set V_f^+ of an h-independent broadcast f:

Observation 2.3.23. [10] *Given an h-independent broadcast f on a graph G, no pair of vertices in V_f^+ hear each other. Hence, V_f^+ forms an independent set of $V(G)$.*

And he further observes that:

Observation 2.3.24. [10] *The characteristic set of a maximum independent set is h-independent. Hence $\alpha_h(G) \geq \alpha(G)$ where $\alpha(G)$ is the size of a maximum independent set of G.*

Theorem 2.3.25. [10] *Let f be an α_h-broadcast on a graph G and $I = V_f^+(G)$. Then I is an independent set and for every vertex $u \in V(G) - I$, there is a vertex $v \in I$ for which $d(u,v) < \min\{d(v,x) : x \in I - \{v\}\}$.*

Corollary 2.3.26. [10] *Let f be a maximal h-independent broadcast on a graph G. There exist two vertices $u, v \in V_f^+$ such that $f(v) = f(u)$.*

Again we note the value of studying trees, similar to Proposition 2.3.11 for $\alpha_{bn(bnr)}(T)$:

Proposition 2.3.27. *For any graph G, $\alpha_h(G) \leq \min\{\alpha_h(T) : T$ is a spanning subtree of $G\}$.*

Proof. Spanning subgraphs induced by removal of edges maintain hearing-independence. Therefore, if T is a spanning tree of G, then every α_h-broadcast on G is a h-independent broadcast on T so by definition $\alpha_h(G) \leq \alpha_h(T)$. $\square$

However, this bound can be far from tight. For example, for $n > 2$, $\alpha_h(K_n) = 1$ and, since trees are bipartite, $\min\{\alpha_h(T) : T \text{ is a spanning subtree of } K_n\} \geq \alpha(T_n) \geq \lceil \frac{n}{2} \rceil \geq 2$. Further, according to Ahmane, Bouchemakh and Sopena [2], finding $\alpha_h(T)$ for trees is still a difficult problem. In [2], they are able to determine $\alpha_h(T)$ for all graphs T in a subcategory of a subcategory of trees. A caterpillar is a tree which has a diametrical path such that every vertex not on the path is adjacent to a vertex on the path. Ahmane et al. [2] determine $\alpha_h(T)$ for all caterpillars T with no adjacent *trunks* (vertices of degree 2). Suppose such a caterpillar T has a diametrical path, $P = v_1, ..., v_m$. Consider the subpath $P' : v_2, ...v_{m-1}$. For $2 \leq i \leq m-1$, let λ_i be the number of vertices which are adjacent to v_i but not on P' . Let $\tau(T)$ be the number of trunks. Notice that, for $2 \leq k \leq m-1$, v_k is a trunk if and only if $\lambda_k = 0$. We give Ahmane et al's results for a specific subcase of caterpillars with no two adjacent trunks:

Corollary 2.3.28. [2] *If T is a caterpillar with no two adjacent trunks and such that, for $2 \leq i \leq |\operatorname{diam}(T)| - 1$, either $\lambda_i \geq 3$ or $\lambda_i = 0$, then*

$$\alpha_h(T) = \sum_{i=2}^{|\operatorname{diam}(T)|-1} \lambda_i + \tau(T) + 2.$$

For graphs in general, Bessy and Rautenbach [3] have an upper bound for $\alpha_h(G)$:

Theorem 2.3.29. [3] *If G is a connected graph such that G has diameter at least 3 or $\alpha(G) \geq 3$, and f is a maximum hearing independent broadcast on G, then*

$$\sigma(f) \leq 4\alpha(G) - 4\min\left\{1, \frac{2\alpha(G)}{f_{max} + 2}\right\}$$

where $f_{max} = \max\{f(x) : x \in V(G)\}$.

So we now have $\alpha(G) \leq \alpha_h(G) < 4\alpha(G)$. Bessy and Rautenbach [3] note that, since $\alpha_h(G)$ and $\alpha(G)$ are within a constant factor, $\alpha_h(G)$ inherits the computational hardness of $\alpha(G)$ and also any efficient constant factor approximation algorithms which exist for $\alpha(G)$.

The approach used by Bessy and Rautenbach [3] to achieve their bound is to start with an $\alpha_h(G)$-broadcast f and then to create a new broadcast g with $g(v) = \lfloor \frac{f(v)}{2} \rfloor$ for all $v \in V_f^+$. The h-independence of f guarantees that $N_g(v) \cap N_g(w) \subseteq B_g(v) \cap B_g(w)$ for all $v, w \in V_g^+$. Or equivalently, g is bn-independent. We adapt this approach to examine the relationships between $\alpha_h(G)$, $\alpha_{bn}(G)$ and $\alpha_{bnr}(G)$.

Since $\alpha_h(P_n) = 2(n-2)$ and $\alpha_{bn}(P_n) = n-1$, the difference $\alpha_h(G) - \alpha_{bn}(G)$ can be arbitrary. However, the ratio $\alpha_h(G)/\alpha_{bn}(G)$ is bounded, as we show next.

Theorem 2.3.30. *For any graph G, $\alpha_h(G) < 2\alpha_{bn}(G)$.*

Proof. Given an α_h-broadcast f on a graph G, notice that if $v, w \in V_f^+$ both cover the same edge, then $v, w \in V_f^{++}$. Define f' by $f'(x) = \left\lceil \frac{f(x)}{2} \right\rceil$ if $x \in V_f^{++}$ and $f'(x) = f(x)$ otherwise.

We claim that for $v, w \in V_f^{++}$, if at least one of $f(v)$ and $f(w)$ is even, then no vertex hears f' from both v and w, while if $f(v)$ and $f(w)$ are both odd, then $N_{f'}(v) \cap N_{f'}(w) \subseteq B_{f'}(v) \cap B_{f'}(w)$. This covers all possibilities, hence, proving our claim will show that f' is bn-independent. If there exists a vertex u such that $v, w \in H_{f'}(u) \cap V_f^{++}$, then $f'(v) \geq d(v,u)$, $f'(w) \geq d(w,u)$ and $d(w,v) \leq f'(w) + f'(v)$. If, without loss of generality, $f(w) < f(v)$, then

$$d(w,v) \leq \left\lceil \frac{f(w)}{2} \right\rceil + \left\lceil \frac{f(v)}{2} \right\rceil \leq f(v),$$

contradicting the h-independence of f. If $f(w) = f(v) \equiv 0 \pmod 2$, then

$$d(w,v) \leq \left\lceil \frac{f(v)}{2} \right\rceil + \left\lceil \frac{f(v)}{2} \right\rceil = f(v),$$

again contradicting the h-independence of f. Finally, if $f(w) = f(v) \equiv 1 \pmod 2$, then

$$d(w,v) \leq \left\lceil \frac{f(w)}{2} \right\rceil + \left\lceil \frac{f(v)}{2} \right\rceil \leq f(v) + 1.$$

Since f is h-independent, $d(w,v) = f(v) + 1 = 2f'(v) = 2f'(w)$ and $u \in B_{f'}(v) \cap B_{f'}(w)$. These are the only possible cases. Hence, for all $v, w \in V_{f'}^+$, $N_{f'}(v) \cap N_{f'}(w) \subseteq B_{f'}(v) \cap B_{f'}(w)$ and f' is bn-independent.

If $f(v)$ is odd for at least one $v \in V_f^{++}$, then $\alpha_{\text{bn}}(G) \geq \sigma(f') > \frac{1}{2}\sigma(f)$. If $f(v)$ is even for all $v \in V_f^{++}$, then f' is not maximal bn-independent (at least one $f'(v)$ can be increased without edges hearing the broadcast from more than one vertex), and $\alpha_{\text{bn}}(G) > \sigma(f') \geq \frac{1}{2}\sigma(f)$. If $V_f^{+} = V_f^{1}$, then f is a *bn*-independent broadcast and $\alpha_h(G) = \sigma(f) \leq \alpha_{bn}(G)$. But all *bn*-independent broadcasts are also *h*-independent broadcasts by definition. Hence $\alpha_{bn}(G) \leq \alpha_h(G)$. Thus, in this case, $\alpha_{bn}(G) = \alpha_h(G)$. Therefore, in all cases, $\alpha_h(G)/\alpha_{\text{bn}}(G) < 2$. □

Recall that, for $n, k \geq 2$, $\alpha_h(S(n^k)) = k(2n-1)$ and $\alpha_{bn}(S(n^k)) = kn$. Hence, this bound is asymptotically the best possible. Using our results for $\alpha_{bn}(G)$, we note a new bound for maximum hearing independent broadcasts.

Corollary 2.3.31. *For any graph G of order $n \geq 2$, $\alpha_h(G) < 2(n-1)$.*

Proof. By Theorem 2.3.5, for a graph G of order $n \geq 2$, $\alpha_{bn}(G) \leq n-1$. The result now follows from Theorem 2.3.30. □

We now show that our new bound and the bound from Theorem 2.3.29 are both useful. For example, for $K_{1,k}$ where $k > 2$, $\alpha(K_{1,k}) = k$ and $n = k+1$. Hence, with f broadcasting with a strength of 1 from each leaf, the bound from Theorem 2.3.29 gives:

$$\alpha_h(K_{1,k}) \leq 4\alpha(K_{1,k}) - 4\min\left\{1, \frac{2\alpha(K_{1,k})}{f_{max}+2}\right\} = 4(k) - 4\min\left\{1, \frac{2(k)}{3}\right\} = 4k-4$$

and the bound from Corollary 2.3.31 gives:

$$\alpha_h(K_{1,k}) \leq 2(n-1) = 2k.$$

However, in the case of $K_{3,3,3}$, a complete 3-partite graph, a maximum independent set consists of one of the partite sets and with f the characteristic function of this set, the bound from Theorem 2.3.29 gives:

$$\alpha_h(K_{3,3,3}) \leq 4\alpha(K_{3,3,3}) - 4\min\left\{1, \frac{2\alpha(K_{3,3\ldots3})}{f_{max}+2}\right\} = 4(3) - 4\min\left\{1, \frac{2(3)}{3}\right\} = 8$$

and the bound from Corollary 2.3.31 gives:

$$\alpha_h(K_{3,3,3}) \leq 2(9-1) = 16.$$

We further adapt the broadcast f' in the proof of Theorem 2.3.30 to consider the relationship between $\alpha_h(G)$ and $\alpha_{bnr}(G)$.

Theorem 2.3.32. *For any graph G, $\alpha_h(G) < 3\alpha_{bnr}(G)$.*

Proof. Let f be an α_h-broadcast and let f' be the associated broadcast described in the proof of Theorem 2.3.30. We define an *bnr*-independent broadcast f''. If, in f', every vertex hears at most one broadcasting vertex, then f' is bnr-independent, $f'' = f'$, and $\sigma(f) = \alpha_h(G) \leq 2\sigma(f'') \leq 2\alpha_{bnr(G)}$. Assume therefore that $\mathrm{PB}_{f'}(v) = \emptyset$ for some $v \in V_{f'}^+$. By the arguments above it follows that $f(v)$ is odd and $f(v) \geq 3$. Let Z be the set of all vertices $v \in V_f^{++}$ such that $f(v)$ is odd and choose any vertex $z \in Z$. Define the broadcast f'' by

$$f''(x) = \begin{cases} \left\lceil \frac{f(x)}{2} \right\rceil & \text{if } x = z \\ \\ \left\lfloor \frac{f(x)}{2} \right\rfloor & \text{if } x \in V_f^{++} - \{z\} \\ f(x) & \text{otherwise.} \end{cases}$$

Then $N_{f''}(v) \cap N_{f''}(w) = \emptyset$ for all $v \in V_f^{++}$ and $w \in V_f^+$, hence f'' is bnr-independent. Moreover, $\sigma(f'') > \sigma(f) - \frac{2}{3}\sigma(f) = \frac{1}{3}\sigma(f)$. Hence $\alpha_{\mathrm{bnr}}(G) \geq \sigma(f'') > \frac{1}{3}\sigma(f) = \frac{1}{3}\alpha_h(G)$, i.e., $\alpha_h(G) < 3\alpha_{\mathrm{bnr}}(G)$. □

For the spider $S(2^k)$, $k \geq 3$, $\alpha_h(S(2^k)) = 3k$ and $\alpha_{\mathrm{bnr}}(S(2^k)) = k + 1$, hence the ratio $\alpha_h/\alpha_{\mathrm{bnr}} < 3$ is asymptotically best possible.

Hence, any upper bounds for *bn* or *bnr*-independent broadcasts can be used to provide an upper bound for $\alpha_h(G)$. Conversely, lower bounds for $\alpha_h(G)$ provide lower bounds for $\alpha_{bn}(G)$.

Corollary 2.3.33. *Given a graph G of order $n > 2$, $\alpha_{bn}(G) > \frac{1}{2}\mu(G)(\mathrm{diam}(G) - 1)$.*

Proof. Apply Proposition 2.3.22 and Theorem 2.3.30. □

However, as noted above, hearing independence is a difficult problem. So far the inequality chains for a h, hr and hd-independent broadcast on a graph G of order n are:

$$ir_b(G) \leq \gamma_b(G) = i_{hd}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \alpha_{hd}(G) \leq \Gamma_b(G) \leq IR_b(G)$$

$$ir_b(G) \leq i_{hr}(G) \leq \gamma_b(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \alpha_{hr}(G) \leq IR_b(G)$$

$$\gamma_b(G) \leq i_h(G) \leq \alpha_h(G) \leq 2\alpha_{bn}(G) \leq 2(n-1).$$

2.4 Chapter Summary

We presented and used existing results on broadcasts and irredundance to get new bounds for our parameters. Since efficient broadcasts exist for all graphs G we showed that $i_{bnd}(G) = i_{hd}(G) = i_{sd}(G) = \gamma_b(G) \leq \min\{\text{rad}(G), \gamma(G)\}$. Since maximal *bn*- and $h-$ independent broadcasts are dominating, $\gamma_b(G) \leq i_{bn}(G), i_h(G) \leq \text{rad}(G)$ with equality in all for radial graphs. Using the ball graph of a very efficient broadcast f on G, we were able to show that $i_{bn}(G) \leq \lceil \frac{4\gamma_b(G)}{3} \rceil$. Since irredundant broadcasts are not necessarily dominating and dominating broadcasts are irredundant, $i_{sr}(G), i_{hr}(G), i_{sr}(G) \leq \gamma_b(G)$ and in Figure 2.5 we gave an example where the inequality is strict. All the inequalities requiring irredundance are bounded below by $ir_b(G)$.

For the maximums, parameters which are irredundant are bounded above by $IR_b(G)$ and parameters that are minimal dominating are bounded above by $\Gamma_b(G)$. For BN-independent broadcasts f we used the fact that any edge is covered by at most one broadcasting vertex to show that $\sigma(f) \leq m - \sum_{v_f^+} \deg(v) + |V_f^+|$. Removal of edges maintains *bn*-independence, hence for a graph G of order n, $\alpha_{bn}(G) \leq \min\{\alpha_{bn}(T) : T \textit{ is a spanning tree of } G\} \leq n-1$. The only graphs which meet this bound are spiders and paths. We focused on the structure of maximal BN-independent broadcasts on trees and determined that in any $\alpha_{bn}(T)$-broadcasts no leaf hears a nonleaf and, although this in not the case for *bnr*-independence, there always is an $\alpha_{bnr}(T)$-broadcasts such that no leaf hears a nonleaf. For all trees there exists an $\alpha_{bn(bnr)}(T)$-broadcast such that $f(v) = 1$ or $\deg(v) = 1$ or both. For $\alpha_{bn}(T)$-broadcasts, if the number of vertices broadcasting with a strength of 1 is maximized, then $PB_f(v) = \emptyset$ for all $v \in V_f^+$. We gave specific examples showing that both $\alpha_{bn}(G)$ and $\alpha_{bnr}(G)$ are incomparable with $\Gamma_b(G)$. And we will examine these relationships further in Chapter 5.

For maximum hearing independence we noted Dunbar et al.'s generalization of Erwin's bound $\alpha_h(G) \geq \mu(\text{diam}(G) - 1\}$. We observed that removal of edges preserves h-independence. Hence $\alpha_h(G) \leq \min\{\alpha_h(T) : T \textit{ is a spanning tree of } G\}$. In Theorem 2.3.29, we reported Bessy and Rautenbach's bound for $\alpha_h(G)$. Adapting their proof techniques [3], we showed that $\alpha_h(G) < 2\alpha_{bn}$ and $\alpha_h(G) < 3\alpha_{bnr}$. Using our bound on $\alpha_{bn}(G)$, we gave a new bound for h-independence. On any graph G with order $n \geq 2$, $\alpha_h(G) < 2(n-1)$. Although bn-independence must share the complexity of h-independence, in Chapter 6, we find a better upper bound for $\alpha_{bn}(T)$ on trees and our results there will allow us to improve this bound for hearing independence (Section 6.3). Finally, by using these relations along with Dunbar et al's lower bound for $\alpha_h(G)$, we showed that $\alpha_{bn}(G) > \frac{1}{2}\mu(G)(\text{diam}(G) - 1)$.

Chapter 3

Paths

Paths are one of the simplest types of trees allowing us to demonstrate techniques that will later be used and expanded upon to get results for trees in general. Surprisingly, some of our parameters prove difficult to compute even on this simple category of trees. However, we are able to determine exact formulas, in terms of path length, for all nine independence parameters.

3.1 The Minimums

The results from Section 2.1 yield, for each $n \geq 1$,

$$\gamma_b(P_n) = i_{bnd}(P_n) = i_{hd}(P_n) = i_{sd}(P_n) = \left\lceil \frac{n}{3} \right\rceil$$

and

$$i_{sr}(P_n) \leq i_{hr}(P_n) \leq i_{bnr}(P_n) \leq \gamma_b(P_n) = \left\lceil \frac{n}{3} \right\rceil . \tag{3.1}$$

Again from previous results, for $n \geq 2$,

$$\left\lceil \frac{n}{3} \right\rceil = \gamma_b(P_n) \leq i_{bn}(P_n), i_h(P_n) \leq rad(P_n) = \left\lfloor \frac{n}{2} \right\rfloor .$$

If $\lceil \frac{n}{3} \rceil = \lfloor \frac{n}{2} \rfloor$, then P_n is radial. So for $n = 2, 3, 4, 5, 7$, $i_{bn}(P_n) = i_h(P_n) = \lfloor \frac{n}{2} \rfloor$. For $n = 6$, we have a case of $2 = \gamma_b(P_6) < i_{bn}(P_6) = i_h(P_6) = \text{rad}(P_6) = 3$. In Figure 3.1 we give a $\gamma_b(P_6)$-broadcast. Below it we give a broadcast extension and note that it is still bn-independent.

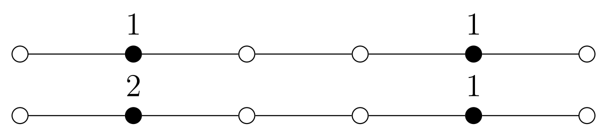


Figure 3.1: A minimum dominating broadcast on P_6 (above) is extended to a maximal bn-independent broadcast on P_6 (below).

3.1.1 Minimum bn-independent broadcasts on paths: $i_{bn}(P_n)$

Note that $i_{bn}(P_3) = 1$. The following lemmas show that for $n \neq 3$, $i_{bn}(P_n) = \lceil \frac{2n}{5} \rceil$. Part of the proof involves Lemma 3.1.1 which shows that there is always an $i_{bn}(P_n)$-broadcast f such that $V_f^{++} = \emptyset$. Further, if $V_f^{+} = V_f^{1}$, then a maximal bn-independent broadcast is also maximal h-independent. Hence $i_h(P_n) \leq i_{bn}(P_n)$ and we also get an upper bound for $i_h(P_n)$.

Lemma 3.1.1. *Every path has an i_{bn}-broadcast f where $f(v) = 1$ for all $v \in V^+$.*

Proof. If there is a path P_n such every i_{bn}-broadcast has at least one $v \in V^{++}$, then of all such broadcasts let f be one for which $|V_f^{++}|$ is a minimum. Let $u \in V_f^{++}$. Notice that $N_f(u)$ is a path. Since $f(u) \leq e(u)$, there is also a vertex v such that $v \in B_f(u)$ and v is at one end of the path dominated by u. Let v' denote the vertex at the other end of the path dominated by u. Since f is bn-independent, $f(v) = f(v') = 0$. Label the $v - v'$ subpath P: $v = v_1, ..., v_{|N_f(u)|} = v'$. Since P is a path, $|N_f(u)| \leq 2f(u) + 1$. Define a new broadcast g as follows:

$$g(x) = \begin{cases} 0 & : \text{ if } x = v_i \text{ and } i \equiv 1 \pmod 2 \\ 1 & : \text{ if } x = v_i \text{ and } i \equiv 0 \pmod 2 \\ f(x) & : \text{otherwise.} \end{cases}$$

If $|N_f(u)|$ is odd, $g(v') = g(v) = 0$. If $|N_f(u)|$ is even, then $|N_f(u)| < 2f(u) + 1$ and u overdominates a leaf, say l. In this case, $v' = l$. In both cases, $PB_g(v) \cup PB_g(v') \subseteq PB_f(u)$ and $PB_g(V(P_n) - N_f(u)) = PB_f(V(P_n) - N_f(u))$. Also, g forms a maximal bn-independent broadcast on $N_f(u)$. Hence g is maximal bn-independent. Notice $|V_g^{++}| = |V_f^{++}| - 1$ and $\sigma(g) \leq \sigma(f) - f(u) + \lfloor \frac{2f(u)+1}{2} \rfloor = \sigma(f)$. But since f is an i_{bn}-broadcast, $\sigma(f) = \sigma(g)$. Thus g contradicts the minimality of V_f^{++}. $\square$

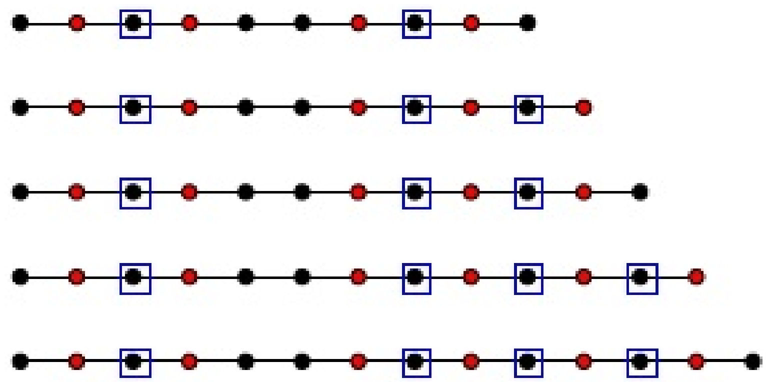


Figure 3.2: Maximal bn-independent broadcasts on paths P_n for $n = 0, 1, 2, 3, 4 \pmod 5$. The red vertices are all broadcasting with strength 1. The blue squares show non-private boundaries indicating that the broadcast is maximally bn-independent.

Lemma 3.1.2. *For a path P_n with $n \neq 3$, $i_{bn}(P_n) \leq \lceil \frac{2n}{5} \rceil$.*

Proof. In Figure 3.2, the only broadcasting vertices are red and they are all broadcasting with a strength of 1. The squares show where two broadcasts overlap which indicates that every broadcast g with $g(v) = f(v) + 1$ for any $v \in V_f^+$ will no longer be bn-independent. Since the graph is dominated, every broadcast g with $g(v) = f(v) + 1$ for $v \in V(T) - V^+$ will no longer be bn-independent. Hence f is a maximal bn-independent broadcast. The result generalizes for all paths P_{5k+r} with $5k + r \neq 3$ where $k \geq 0$ and $0 \leq r < 5$. A maximal bn-independent broadcast on $P_{5k+r} : v_1, ..., v_{5k+r}$ is given by

$$f(x) = \begin{cases} 1 & : \text{ if } x = v_j \text{ and } j \equiv 2, 4 \pmod 5 \text{ and } j < 5k \\ 1 & : \text{ if } x = v_{5k+1} \text{ or } x = v_{5k+3} \\ 0 & : \text{ otherwise.} \end{cases}$$

Notice that $\sigma(f) = \lceil \frac{2n}{5} \rceil$. Since f is maximal bn-independent, $i_{bn}(P_n) \leq \lceil \frac{2n}{5} \rceil$. □

Lemma 3.1.3. *For any path P_n with $n \neq 3$, $i_{bn}(P_n) \geq \lceil \frac{2n}{5} \rceil$.*

Proof. Suppose $n = 5k + r$, where $r = 0, 1, 2, 3, 4$. By Lemma 3.1.1, we may choose an i_{bn}-broadcast f on P_n with $V_f^+ = V_f^1$. First we show that every subpath of length

5 has at least two f-broadcasting vertices. Suppose P_n has a subpath a, b, c, d, e with only one f-broadcast vertex. Then this vertex is c, otherwise one of b, c, d is non-dominated, contradicting the maximality of f. Define a new broadcast g with $g(c) = 2$ and $g(x) = f(x)$ otherwise. Since $N_g(c) = \{a, b, c, d, e\}$ and all other broadcast values are unchanged, g is bn-independent. Notice that $\sigma(g) > \sigma(f)$ which violates the maximality of f. Hence each subpath of 5 vertices has two or more broadcast vertices and $i_{bn}(P_n) \geq 2k$. There are 5 cases, depending on r. If $r = 0$, then by the argument above we are done.

If $r = 1$ or 2, but $\sigma(f) = 2k$, then partition P_n into paths of length 5 from left to right. Each such subpath a, b, c, d, e has exactly 2 broadcast vertices. We claim that $f(b) = f(d) = 1$. For a contradiction, starting with the leftmost subpath, suppose $f(a) = 1$. Then either $f(c) = 1$ or f is not maximal, because $(f - \{(a, 1)\} \cup \{(a, 2)\}$ is bn-independent. Hence $f(c) = 1$. If $f(e) = 1$, then this first section has 3 broadcasting vertices which contradicts the weight of f. Hence $f(e) = 0$ and the vertex e must be dominated by the first vertex in the next subpath, say a_1, b_1, c_1, d_1, e_1. Hence $f(a_1) = 1$. As before, to insure that f is maximal, $f(c_1) = 1$ and thus $f(e_1) = 0$. Hence e_1 is dominated by the first vertex of the next subpath. This pattern repeats until we reach the end of the subpaths of length 5 and find the last 2 ($r = 1$) or 3 ($r = 2$) vertices of P_n non-dominated. Hence we assume that $f(b) = f(d) = 1$ for all of our subpaths. This still leaves the last 1 or 2 vertices of P_n non-dominated. Hence f is not maximal bn-independent and the weight of an i_{bn}-broadcast on these paths is at least $2k + 1 = \lceil \frac{2n}{5} \rceil$, in this case.

If $r = 3$ or 4, but $\sigma(f) = 2k + 1$, then use the same partitions. Again $f(b) = f(d) = 1$ for all subpaths of length 5. Hence the broadcast f restricted to these partitions has a weight of $2k$ and we need to dominate the last 3 ($r = 3$) or 4 ($r = 4$) vertices of P_n with a weight of 1. But a broadcast of strength 1 can cover at most 3 vertices. Hence for $r = 4$, f is not dominating and thus not maximal bn-independent. If $r = 3$, then to dominate the last three vertices, say a, b, c, $f(b) = 1$. Define a new broadcast g with $g(b) = 2$ and $g(x) = f(x)$ otherwise. Since $f(e) = 0$ for all of the subpaths of length 5, g is bn-independent and contradicts the maximality of f. Hence an i_{bn}-broadcast on these paths has a weight of at least $2k + 2 = \lceil \frac{2n}{5} \rceil$, in this case. In all cases with $n \neq 3$, $i_{bn}(P_n) \geq \lceil \frac{2n}{5} \rceil$. $\square$

Theorem 3.1.4. *For any integer $n \neq 3$, $i_{bn}(P_n) = \lceil \frac{2n}{5} \rceil$.*

Proof. The result follows from Lemmas 3.1.1, 3.1.2 and 3.1.3. $\square$

3.1.2 Minimum h-independent broadcasts on paths: $i_h(P_n)$

We now consider h-independent broadcasts on paths and show that $i_h(P_n) = i_{bn}(P_n) = \lceil \frac{2n}{5} \rceil$. To accomplish this goal, we show that there is always a $i_h(P_n)$-broadcast f with $V_f = V_f^1$ and observe that any such broadcast is also boundary independent and thus $i_h(P_n) \geq i_{bn}(P_n)$. We have already shown that there is a $i_{bn}(P_n)$-broadcast with $V_f = V_f^1$ and such a broadcast is also hearing independent; thus $i_h(P_n) \leq i_{bn}(P_n)$ and our result will follow.

To show that there is always a $i_h(P_n)$-broadcast f with $V_f = V_f^1$, we establish the existence of an h-independent broadcast f with $V_f = V_f^1$ and $\sigma(f) \leq \lfloor \frac{n}{2} \rfloor$. We then note that when a broadcast f covers all edges of P_n, $\sigma(f) \geq \lfloor \frac{n}{2} \rfloor$. Thus, if there is no $i_h(P_n)$-broadcast f with $V_f = V_f^1$, then all $i_h(P_n)$-broadcasts f must leave some edges uncovered. The uncovered edges are used in Lemma 3.1.8 to partition P_n and to show that the minimum size of V_f^{++} over all $i_h(P_n)$-broadcasts f is zero.

Observation 3.1.5. *For $P_n : v_1, ..., v_n$, the set $X_{0 \pmod 2} = \{v_i \in V(P_n) : i \equiv 0 \pmod 2\}$ is a maximal independent (hence dominating) set of cardinality $\lfloor \frac{n}{2} \rfloor$.*

Lemma 3.1.6. *The characteristic function f of the maximal independent set $X = X_{0 \pmod 2}$ of P_n is a maximal h-independent broadcast.*

Proof. The result is obvious if $n \leq 3$, hence assume $n \geq 4$. Certainly, f is h-independent. Since X is a dominating set, $X \cup \{v\}$ is not independent for each $v \in V(P_n) - X$. Hence no broadcast f' such that $V_f^+ \subsetneq V_{f'}^+$ is h-independent. Moreover, for each $v_i \in X$, at least one of v_{i-2} and v_{i+2} belongs to X, hence no broadcast f'' such that $V_{f''}^+ = V_f^+$ and $V_{f''}^{++} \neq \varnothing$ is h-independent. It follows that f is maximal h-independent. $\square$

Lemma 3.1.7. *If every edge of the path P_n is covered by a broadcast f, then $\sigma(f) \geq \lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$.*

Proof. Every broadcasting vertex v can cover at most $2f(v)$ edges. If every edge is covered by f, then $\sum_{v \in V^+} 2f(v) \geq n - 1$. $\square$

Lemma 3.1.8. *Every path P_n has an i_h-broadcast f such that $V_f^+ = V_f^1$.*

Proof. Among all i_h-broadcasts of P_n, let f be one such that $|V_f^{++}|$ is minimum. If $V_f^{++} = \varnothing$, we are done, hence assume $V_f^{++} \neq \varnothing$. Since f is maximal h-independent, f is dominating (Theorem 2.3.21).

Suppose $EU_f = \varnothing$. Then, by Lemma 3.1.7, $\sigma(f) \geq \lfloor \frac{n}{2} \rfloor$. But the characteristic function f' of $X_{0 \pmod 2}$ is a maximal h-independent broadcast with $\sigma(f') = \lfloor \frac{n}{2} \rfloor$. Hence, either $\sigma(f') < \sigma(f)$ contradicting the minimality of an i_n-broadcast or $\sigma(f') = \sigma(f)$ and since $|V_{f'}^{++}| = 0$, f' violates the minimality of $|V_f^{++}|$. So assume $EU_f \neq \varnothing$. Let $Q_1, ..., Q_r,\ r \geq 2$, be the components of $P_n - EU_f$ and let i be an index such that Q_i contains a vertex in V_f^{++}. By symmetry we may assume $i \geq 2$. Say Q_i is the path $v_j, v_{j+1}, ..., v_{j+t}$ for some $t \geq 4$. Since Q_i is a proper subpath of P_n and $v_{j-1}v_j \in EU_f$, we may assume that $j \geq 3$. Since f is dominating, some $v_k \in V_f^+$ with $k < j-1$ broadcasts to v_{j-1}. Let $f_1 = (f - \{(v_k, f(v_k))\}) \cup \{(v_k, f(v_k)+1)\}$. Then no broadcast vertex of Q_i hears f_1 from v_k. But f is a maximal h-independent broadcast, hence

$$\text{some } v_{k'} \in V_f^+, \text{ where } k' < k, \text{ hears } f_1 \text{ from } v_k. \tag{3.2}$$

Similarly, if $j + t < n$, then some $v_\ell \in V_f^+$ broadcasts to v_{j+t+1}, and for $f_2 = (f - \{(v_\ell, f(v_\ell))\}) \cup \{(v_\ell, f(v_\ell) + 1)\}$,

$$\text{some } v_{\ell'} \in V_f^+, \text{ where } \ell' > \ell, \text{ hears } f_2 \text{ from } v_\ell. \tag{3.3}$$

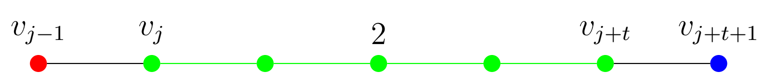


Figure 3.3: A Q_i subpath of P_n with $r \geq 2$. The vertices v_{j-1} and v_{j+t+1}, if the latter exists, are dominated by broadcast vertices, say v_k and v_l, whose broadcast value cannot be increased without violating hearing independence due to unseen broadcast vertices.

Therefore f_1 and f_2 (if the latter function is defined) are not h-independent. Denote the restriction of f to Q_i by $f \upharpoonright Q_i$. Since $EU_f \cap E(Q_i) = \varnothing$, Lemma 3.1.7 implies that

$$\sigma(f \upharpoonright Q_i) \geq \left\lceil \frac{|E(Q_i)|}{2} \right\rceil = \left\lceil \frac{t}{2} \right\rceil = \left\lfloor \frac{|V(Q_i)|}{2} \right\rfloor. \tag{3.4}$$

Define the broadcast g by

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in V(P_n) - V(Q_i) \\ 1 & \text{if} \quad x \in V(Q_i) \text{ and } x = v_{j+a}, \text{ where } a \equiv 1 \pmod 2 \\ 0 & \text{if} \quad x \in V(Q_i) \text{ and } x = v_{j+a}, \text{ where } a \equiv 0 \pmod 2. \end{cases}$$

Then $\sigma(g \restriction Q_i) = \left\lfloor \frac{|V(Q_i)|}{2} \right\rfloor \leq \sigma(f \restriction Q_i)$ by (3.4), hence $\sigma(g) \leq \sigma(f)$. If $|V(Q_i)|$ is odd, that is, if t is even, then no vertex in $V_g^+ \cap V(Q_i)$ broadcasts to vertices not in Q_i, and if $|V(Q_i)|$ is even, then t is odd and v_{j+t} broadcasts to $v_{j+t+1} \notin V_g^+$, hence g is h-independent. By Lemma 3.1.6, $g \restriction Q_i$ is maximal h-independent. By (3.2) and (3.3), the functions $(f_1 - f \restriction Q_i) \cup g \restriction Q_i$ and $(f_2 - f \restriction Q_i) \cup g \restriction Q_i$ (if the latter function is defined) are not h-independent. If g' is any broadcast of the form $(g - \{(x, g(x))\}) \cup \{(x, g(x) + 1)\}$, where $x \in V(P_n) - V(Q_i) - \{v_k, v_\ell\}$, then the maximality of f implies that g' is not h-independent. Therefore g is a maximal h-independent broadcast on P_n such that $\sigma(g) \leq \sigma(f)$ and $|V_g^{++}| < |V_f^{++}|$, contradicting the choice of f. □

Theorem 3.1.9. *For a path P_n with $n \neq 3$, $i_h(P_n) = \lceil \frac{2n}{5} \rceil$.*

Proof. Given a path P_n, consider a maximal i_h-broadcast f with $V_f^+ = V_f^1$. It is also a maximal bn-independent broadcast so $i_{bn}(P_n) \leq i_h(P_n)$. Every maximal bn-independent broadcast f with $V_f^+ = V_f^1$ is also a maximal h-independent broadcast, hence $i_h(P_n) \leq i_{bn}(P_n)$. The result now follows from Theorem 3.1.4. □

3.1.3 Minimum s-independent broadcasts on paths: $i_s(P_n)$

For i-independence, the broadcast is not maximal until V^+ forms a maximal independent set and each vertex $v \in V^+$ broadcasts with a strength equal to $e(v)$.

$$i_s = \min\{\sum_{v \in I_n} e(v) : I_n \text{ is a maximal independent set of } G.\}$$

This is always greater than $\text{rad}(G)$, except for P_n with $n \leq 3$ where the values are equal.

Theorem 3.1.10. *For a path P_n with $n \geq 7$,*

$$i_s(P_n) = \begin{cases} (n-2) + ... + (n-2) & \textit{if } n \equiv 0 \pmod 6 \\ (n-2) + ... + (\frac{n-1}{2}) + (\frac{n-1}{2} + 2) + ... + (n-2) & \textit{if } n \equiv 1 \pmod 6 \\ (n-2) + ... + (\frac{n}{2}) + (\frac{n}{2} + 2) + ... + (n-2) & \textit{if } n \equiv 2 \pmod 6 \\ (n-2) + ... + (\frac{n-1}{2}) + ... + (n-2) & \textit{if } n \equiv 3 \pmod 6 \\ (n-2) + ... + (\frac{n}{2}) + (\frac{n}{2} + 1) + ... + (n-2) & \textit{if } n \equiv 4 \pmod 6 \\ (n-2) + ... + (\frac{n+1}{2}) + (\frac{n+1}{2}) + ... + (n-2) & \textit{if } n \equiv 5 \pmod 6), \end{cases}$$

where the terms of each sum increase (or decrease) by 3 unless otherwise shown.

Proof. For s-independence, the broadcast is not maximal until V^+ forms a maximal independent set. A minimum maximal independent set on P_n has size $\lceil \frac{n}{3} \rceil$. The result in the statement comes from direct calculations using

$$i_s(P_n) = \min\{\sum_{v \in I_n} e(v) : I_n \text{ is a minimum maximal independent set of } P_n\}$$

and choosing each minimum maximal independent set with its vertices as close to the centre as possible to minimize eccentricity and thus the weight of the corresponding broadcast. See Figure 3.4 for an example. □

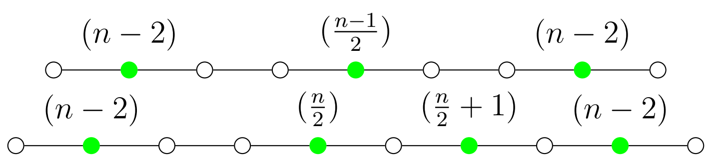


Figure 3.4: An i_s-broadcast on a path P_n with $n \equiv 3 \pmod 6$ (top) and an i_s-broadcast on a path P_n with $n \equiv 4 \pmod 6$.

3.1.4 Minimum sr-independent broadcasts on paths: $i_{sr}(P_n)$

Recall that every path P_n has a minimum dominating broadcast f which is efficient and $\sigma(f) = \lceil \frac{n}{3} \rceil$. Such a broadcast is the characteristic function of a minimum maximal independent set. Thus it is maximal sr-independent. Hence, as mentioned in equation 3.1, $i_{sr}(P_n) \le \gamma_b(P_n) = \lceil \frac{n}{3} \rceil$. If all sr-independent broadcasts are dominating, then $i_{sr}(P_n) = \gamma_b(P_n) = \lceil \frac{n}{3} \rceil$ and by equation 3.1, $i_{sr}(P_n) = i_{hr}(P_n) = i_{bnr}(P_n) = \gamma_b(P_n) = \lceil \frac{n}{3} \rceil$. However, definitions with irredundance may lead to maximal broadcasts which are not dominating. The following lemma is used to show that on any path P_n, even if they are nondominating, there are no sr-independent broadcasts f with $\sigma(f) < \lceil \frac{n}{3} \rceil$. Hence we have $i_{sr}(P_n) = \lceil \frac{n}{3} \rceil$.

Lemma 3.1.11. *For any $n \geq 1$, $i_{sr}(P_n) \geq \lceil \frac{n}{3} \rceil$.*

Proof: Let $P_n = v_1, ..., v_n$. The result is easy to verify for small values of n. Suppose the statement of the lemma is false and let n be the smallest integer such that $i_{sr}(P_n) < \lceil \frac{n}{3} \rceil$. Since $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$, no $i_{sr}(P_n)$-broadcast is dominating. Let $\mathcal{F}$ be the set of all $i_{sr}(P_n)$-broadcasts on P_n and partition $\mathcal{F}$ into subsets $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1$ consists of the broadcasts such that only leaves are undominated, and $\mathcal{F}_2 = \mathcal{F} - \mathcal{F}_1$. If $\mathcal{F}_2 \neq \varnothing$, choose $f \in \mathcal{F}_2$, otherwise let $f \in \mathcal{F}_1$. We consider two cases, depending on the choice of f.

Case 1: $f \in \mathcal{F}_1$. We may assume that v_1 is not f-dominated, otherwise we can relable P_n so that this is the case. By the maximality of f, there exists an index $k \geq 3$ and a vertex $v_k \in V_f^+$ such that $\text{PB}_f(v_k) = \{v_2\}$, otherwise $(f - \{(v_1, 0)\}) \cup \{(v_1, 1)\}$ is an s-independent irredundant broadcast, which is impossible. Note that v_k does not broadcast to v_n. If it does, then either $v_n \in \text{PB}_f(v_k)$ or, if v_k overdominates v_n, then $V_f^+ = \{v_k\}$. In the later case, define the broadcast f' as $f'(v_{k-1}) = f(v_k)$ and $f'(x) = 0$ otherwise. Note that $\sigma(f) = \sigma(f')$ and thus f' is a dominating $i_{sr}(P_n)$-broadcast, a contradiction. Hence v_k does not dominate v_n. Define the broadcast g_1 on P_n by

$$g_1(x) = \begin{cases} f(v_k) - 1 & \text{if } x = v_k \\ 1 & \text{if } x = v_1 \\ f(x) & \text{otherwise.} \end{cases}$$

Then $V_{g_1}^+$ is independent and $\sigma(g_1) = \sigma(f)$. Note that $\text{PB}_{g_1}(v_1) = \{v_1, v_2\}$. If $g_1(v_k) > 0$, then $v_3 \in \text{PB}_{g_1}(v_k)$, and if $g_1(v_k) = 0$, then $k = 3$ and, since $v_k \notin PB_f(v_k)$, there exists an index j such that $v_3 \in \text{PB}_{g_1}(v_j)$. For all other vertices $x \in V_{g_1}^+$, $\text{PB}_f(x) \subseteq \text{PB}_{g_1}(x)$. Therefore g_1 is an s-independent irredundant broadcast. If g_1 is dominating, then it is maximal irredundant and hence a dominating $i_{sr}(P_n)$-broadcast, which is impossible. Hence g_1 is not dominating. Since $\text{PB}_f(v_k) = \{v_2\}$, the only vertex of P_n that is f-dominated but not g_1-dominated by v_k is v_2. Therefore, since $f \in \mathcal{F}_1$, v_n is the only vertex of P_n not dominated by g_1. This implies that v_n is not f-dominated.

Arguing as for v_1 and v_k, there exists an index $l \leq n - 2$, $l \neq k$, and a vertex

$v_l \in V_f^+$ such that $\mathrm{PB}_f(v_l) = \{v_{n-1}\}$. Define the broadcast g_2 on P_n by

$$g_2(x) = \begin{cases} g_1(v_l) - 1 & \text{if } x = v_l \\ 1 & \text{if } x = v_n \\ g_1(x) & \text{otherwise.} \end{cases}$$

As in the case of g_1, g_2 is an s-independent irredundant broadcast such that $\sigma(g_2) = \sigma(f)$. We deduce as before that g_2 is not dominating. Since $\mathrm{PB}_f(v_l) = \{v_{n-1}\}$ and g_2 dominates v_1 and v_n, there must be a vertex $v_a \in N_f(v_k) \cap N_f(v_\ell)$, which is not g_2-dominated. This implies that $V_f^+ = \{v_k, v_l\}$. Since $\mathrm{PB}_f(v_k) = \{v_2\}$ and $\mathrm{PB}_f(v_l) = \{v_{n-1}\}$, and, moreover, V_f^+ is independent, it is not possible that $f(v_k) = f(v_l) = 1$. Hence, either only $N_f(v_k) \cap N_f(v_l) = \{v_a\}$ or, without loss of generality, $f(v_l) = 1$, $f(v_k) > 1$ and $v_l \in B_f(v_k)$. In the first case, v_k covers exactly $2f(v_k)$ edges, and the same holds for v_l and exactly two edges of P_n are uncovered, that is, $|UE_f| = 2$, while no edge is covered twice. Hence P_n has length $n - 1 = 2(f(v_k) + f(v_l) + 1) \geq 10$. See Figure 3.5.

This implies that $\sigma(f) = \frac{n-3}{2}$. And if $n \geq 11$ then $\sigma(f) = \frac{n-3}{2} \geq \lceil \frac{n}{3} \rceil$, contradicting our choice of P_n.

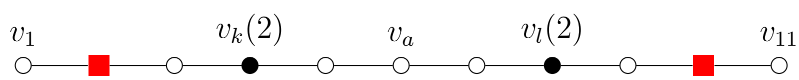


Figure 3.5: A maximal sr-independent broadcast f on P_{11} with two non-dominated leaf vertices v_1 and v_{11} separated by the subpath Q which is dominated by two overlapping broadcast vertices, v_k and v_l with $f(v_k), f(v_l) \geq 2$. The red squares indicate the private boundaries of the broadcasting vertices. For paths of this type, the length $P_n = n - 1 = 2(f(v_k) + f(w)) + 2$ and $n \geq 11$.

In the second case, without loss of generality $f(v_l) = 1$ and $f(v_k) \geq 2$. Hence v_k covers exactly $2f(v_k)$ edges, and the same holds for v_l, exactly two edges of P_n are uncovered, that is, $|UE_f| = 2$, and exactly one edge is covered twice. See Figure 3.6. Hence P_n has length $n-1 = 2f(v_k)+f(v_l)-1+2 \geq 7$. This implies that $\sigma(f) = \frac{n-2}{2}$. And if $n \geq 8$ then $\sigma(f) = \frac{n-2}{2} \geq \lceil \frac{n}{3} \rceil$, again contradicting our choice of P_n. Since these are the only possibilities, we assume $\mathcal{F}_2 \neq \emptyset$.

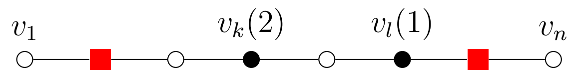


Figure 3.6: A sr-independent broadcast f on P_8 with a two non-dominated leaf vertices v_1 and v_8 separated by the subpath Q which is dominated by two overlapping broadcast vertices, v_k and $v_l(v_a)$ with $f(v_k) = 2$ and $f(v_l) = 1$. The red squares indicate the private boundaries of the broadcasting vertices. For paths of this type, the length $P_n = n - 1 = 2(f(v_k) + f(w)) + 1$ and $n \geq 8$.

Case 2: $f \in \mathcal{F}_2$. Since v_m is not f-dominated, $V_f^+ \cup \{v_m\}$ is independent. By the maximality of f there exists a vertex $v_k \in V_f^+$ such that $\mathrm{PB}_f(v_k) = \{v_{m-1}\}$ or $\mathrm{PB}_f(v_k) = \{v_{m+1}\}$. Since $1 < m < n$, we may assume without loss of generality that $k < m$ and $\mathrm{PB}_f(v_k) = \{v_{m-1}\}$. Let H_1 and H_2 be the components of $P_n - v_m v_{m+1}$, where H_1 is the component that contains v_1. Then $H_1 \cong P_m : v_1, ..., v_m$ and $H_2 \cong P_{n-m} : v_{m+1}, ..., v_n$. See Figure 3.7. For $i = 1, 2$, let f_i be the restriction of f to H_i.

Note that $\mathrm{PB}_{f_1}(v_k) = \{v_{m-1}\}$. Therefore $(f_1 - \{(v_m, 0)\}) \cup \{(v_m, 1)\}$ is not an irredundant broadcast on H_1. If f_1 can be extended to an s-independent irredundant broadcast on H_1 in some other way, then f can be similarly extended. We conclude that f_1 is a maximal s-independent irredundant broadcast on H_1.

Suppose $f' = (f_2 - \{(v_{m+1}, 0)\}) \cup \{(v_{m+1}, 1)\}$ is an s-independent irredundant broadcast on H_2. Then $v_{m+1} \in \mathrm{PB}_{f'}(v_{m+1})$ and $\mathrm{PB}_f(v_x) \neq \{v_{m+2}\}$ for any $v_x \in V_f^+$. But then $f_1 \cup f'$ is an extension of f to an s-independent irredundant broadcast on P_n, which is impossible. If f_2 can be extended to an s-independent irredundant broadcast on H_2 in some other way, then f can be similarly extended. Therefore f_2 is a maximal s-independent irredundant broadcast on H_2.

By the choice of n as the smallest integer such that $i_{sr}(P_n) < \lceil \frac{n}{3} \rceil$, we deduce that $\sigma(f_1) \geq \lceil \frac{m}{3} \rceil$ and $\sigma(f_2) \geq \lceil \frac{n-m}{3} \rceil$. But now

$$\left\lceil \frac{n}{3} \right\rceil > \sigma(f) = \sigma(f_1) + \sigma(f_2) \geq \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-m}{3} \right\rceil \geq \left\lceil \frac{n}{3} \right\rceil,$$

which is impossible. ■

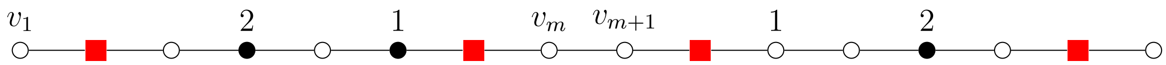


Figure 3.7: A sr-independent broadcast f on P_{16} with a non-dominated non-leaf vertex v_m. Notice that when the broadcast f is restricted to each component of $P_n - v_m v_{m+1}$, it forms, respectively, f_1 and f_2, maximal sr-independent broadcasts.

3.2 The Maximums

From Corollary 2.3.5, we have

$$\alpha_{bn}(P_n) = \alpha_{bnd}(P_n) = \alpha_{bnr}(P_n) = n - 1.$$

A maximum weight independent broadcast is obtained in each case by broadcasting from a leaf with strength $n - 1$.

Theorem 3.2.1. *If a broadcast f on P_n is irredundant, then* $\sum_{v \in V^+} f(v) \leq n - 1$.

Proof. Consider an irredundant broadcast f on P_n. If f is a broadcast from a single leaf, then we meet our bound. If $|V_f^+| \geq 2$, then, starting from one end of the path, group the broadcasting vertices in consecutive pairs. If $|V_f^+|$ is odd, then the last broadcasting vertex forms a group of size 1. To maintain irredundance, there is an uncovered edge between each consecutive group. Counting edges gives $\sigma(f) \leq n - 1 - \left(\left\lceil \frac{|V_f^+|}{2} \right\rceil - 1 \right) \leq n - 1$. □

Corollary 3.2.2. *For a path P_n, $\alpha_{sr}(P_n) = \alpha_{sd}(P_n) = \alpha_{hr}(P_n) = \alpha_{hd}(P_n) = n - 1$.*

From Erwin [10], we have

$$\alpha_h(P_n) = 2(n - 2).$$

A maximum weight h-broadcast can be obtained by broadcasting from each leaf with strength $n - 2$.

Theorem 3.2.3. *For a path P_n with $n \geq 7$,*

$$\alpha_s(P_n) = \begin{cases} (n-1) + ... + (\frac{n-1}{2}) + ... + (n-1) & \textit{if } n \equiv 1 \pmod 2 \\ 2[(n-1) + (n-3) + ... + (\frac{n}{2}+1)] & \textit{if } n \equiv 0 \pmod 4 \\ (n-1) + ... + (\frac{n}{2}) + ... + (n-1) & \textit{if } n \equiv 2 \pmod 4, \end{cases}$$

where the terms of each sum increase (or decrease) by 2 unless otherwise shown.

Proof. For s-independence, the broadcast is not maximal until V^+ forms a maximal independent set and each vertex broadcasts with strength $e(v)$. A maximum independent set on P_n has size $\lceil \frac{n}{2} \rceil$. The result above comes from direct calculations using

$$\alpha_s(P_n) = \max\{\sum_{v \in I_n} e(v) : I_n \text{ is a maximum independent set of } P_n\}$$

and choosing each maximum independent set with its vertices as far from the centre as possible which maximizes the eccentricity of each broadcasting vertex and the resulting weight. See Figure 3.8 for an example. $\square$

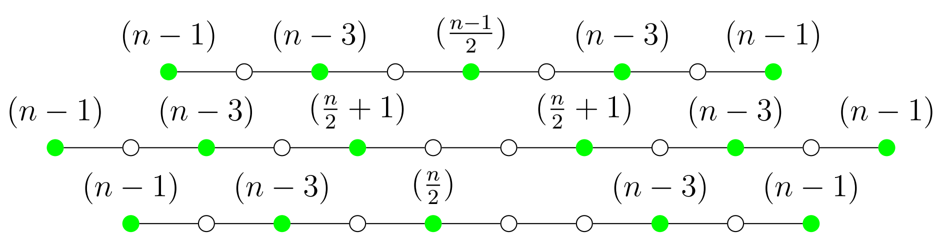


Figure 3.8: An α_s-broadcast on a path P_n with $n \equiv 1 \pmod 2$ (top), an α_s-broadcast on a path P_n with $n \equiv 0 \pmod 4$ (middle) and an α_s-broadcast on a path P_n with $n \equiv 2 \pmod 4$ (bottom).

3.3 Summary

For s-independence, the broadcasting vertices form an independent set and for maximality, every vertex in the set broadcasts with a strength equal to its eccentricity. We gave exact formulas for $i_s(P_n)$ and $\alpha_s(P_n)$ in Theorem 3.1.10 and Theorem 3.2.3.

By examining the edges covered and uncovered by an independent broadcast, and the existence of independent broadcasts f with $V_f^+ = V_f^1$, we determined exact values for all other independence parameters for paths in terms of their size. For the minimums, for $n \geq 1$,

$$i_{sr,hr,bnr,bnd,hd,sd}(P_n) = \lceil \frac{n}{3} \rceil$$

and, for all $n \neq 3$,

$$i_{bn,h}(P_n) = \lceil \frac{2n}{5} \rceil.$$

For the maximums, $\alpha_h(P_n) = 2(n-1)$, Erwin's bound. All other maximum independent broadcast types are met by a broadcast of strength diam(G) from a leaf. Or equivalently, for $n > 1$,

$$\alpha_{sr,hr,bnr,bnd,hd,sd,bn}(P_n) = n - 1.$$

Chapter 4

Grid Graphs

Determining the domination number for grids was a major problem in domination theory until Chang's conjecture [6], that, for $16 \leq m \leq n$, the domination number of $G_{m,n}$ equals $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$, was proven by D. Gonçalves, A. Pinlou, M. Rao and S. Thomassé [11]. Therefore, it is an important class of graphs to consider for other domination parameters. Also, Bouchemakh and Zemir [5] have worked on the hearing independence parameter for grids, making it one of the few subclasses of graphs for which any work on independent broadcasts has been done prior to this dissertation.

4.1 Grid Notation and Definitions

The Cartesian product of two graphs, G_1 and G_2, is denoted by $G_1 \square G_2$. The vertices of $G = G_1 \square G_2$ are $V(G) = \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Two vertices (u_i, v_i) and (u_j, v_j) of G are adjacent if $u_i = u_j$ and v_i is adjacent to v_j in G_2, or if $v_i = v_j$ and u_i is adjacent to u_j in G_1. The grid graph, $G_{m,n}$ is the Cartesian product of the paths P_n and P_m. Label the vertices of $G_{m,n} = P_m \square P_n$ as $v_{i,j}$ where $1 \leq i \leq m$ and $1 \leq j \leq n$, and $v_{i,j}$ is adjacent to $v_{k,l}$ if $|i - k| + |j - l| = 1$. Or equivalently, if $P_m : x_1, ..., x_m$ and $P_n : y_1, ..., y_n$ then $v_{i,j} = (x_i, y_j)$ as described for the cartesian product. The graph $G_{m,n}$ can be drawn as a grid with m rows and n columns or vertices. (See Figure 4.1).

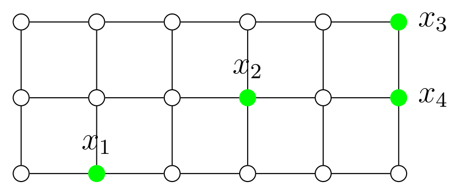


Figure 4.1: A $G_{3,6}$ grid with $C(x_1) = 2$, $R(x_1) = 3$, $C(x_3) = C(x_4) = 6$, $R(x_3) = 1$ and $R(x_4) = 2$.

Since graphs have no specific orientation in space we specify $1 \leq m \leq n$ to avoid having two different labels for the same grid. In this grid drawing, we can think of $v_{i,j}$ as referring to a vertex in the ith row and the jth column. A subgrid of $G_{m,n}$ is a subgraph of $G_{m,n}$ formed by choosing m_1, m_2, n_1, n_2 such that $1 \leq m_1 \leq m_2 \leq m$ and $1 \leq n_1 \leq n_2 \leq n$. The subgrid is induced by the vertices $v_{i,j}$ such that $m_1 \leq i \leq m_2$ and $n_1 \leq j \leq n_2$ and is isomorphic to $G_{(m_2-m_1+1),(n_2-n_1+1)}$. For the purpose of discussing the location of the broadcasting vertices of a grid, given a broadcast f on $G_{m,n}$, let $C(v_{i,j}) = j$ or equivalently the column in which $v_{i,j}$ is located. Similarly let $R(v_{i,j}) = i$. Label the broadcasting vertices $x_1, ...x_{|V_f^+|}$ such that $C(x_1) \leq C(x_2) \leq ... \leq C(x_{|V_f^+|})$. If there are k broadcasting vertices in the same column then label them such that $R(x_i) < R(x_{i+1}) <, ..., < R(x_{i+k})$. For example, the first column in which a broadcasting vertex appears would be given by $C(x_1)$, the last column by $C(x_{|V^+|})$. Informally, broadcasting vertices are labeled according to column first, then rows are used to break ties. In Figure 4.1, we see a $G_{3,6}$ grid with $C(x_1) = 2$, $R(x_1) = 3$, $C(x_3) = C(x_4) = 6$ and $R(x_3) = 1$ and $R(x_4) = 2$.

4.2 The Minimums

Grid graphs are radial [5]. Hence the general results on minimum independent broadcasts from Section 2.1 become:

$$\gamma_b(G_{m,n}) = i_{bn}(G_{m,n}) = i_{bnd}(G_{m,n}) = i_h(G_{m,n}) = i_{hd}(G_{m,n}) = i_{sd}(G_{m,n}) = \text{rad}(G_{m,n})$$

for all $2 \leq m \leq n$.

It is well known [5] that

$$\mathrm{rad}(G_{m,n}) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \text{if } m = 2 \text{ or } 3 \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor & \text{otherwise.} \end{cases}$$

$$\boxed{i_{bnr}(G_{m,n}), i_{hr}(G_{m,n}), \text{ and } i_{sr}(G_{m,n}).}$$

Any radial broadcast is maximally sr, hr, bnr-independent so $i_{sr}(G_{m,n}) \leq i_{hr}(G_{m,n}) \leq i_{bnr}(G_{m,n}) \leq \mathrm{rad}(G_{m,n})$. If the following conjecture holds then $i_{sr}(G_{m,n}) \geq \gamma_b(G_{m,n}) = \mathrm{rad}(G_{m,n})$ and equality holds throughout.

Conjecture 4.2.1. *Given a grid graph $G_{m,n}$, there is a dominating i_{sr}-broadcast.*

4.3 The Maximums

$$\boxed{\alpha_h(G_{m,n})}$$

Recall, from the initial results on maximums (Section 2.2), that the characteristic function of a maximum independent set corresponds to an h-independent broadcast. It is well known [5] that $\alpha(G_{m,n}) = \lceil \frac{mn}{2} \rceil$. Recall also (Section 2.2.5) that $\alpha_h(G) \geq \mu(G)(\mathrm{diam}(G) - 1)$ where $\mu(G)$ is the size of the largest set of pairwise antipodal vertices or equivalently, $\mu(G) = |\max\{X : X \subseteq V(G) \text{ and for any } u, v \in X, d(u, v) = \mathrm{diam}(G)\}|$. Notice for $G_{m,n}$, $\mu(G_{m,n}) = 2$ and $\mathrm{diam}(G_{m,n}) = m + n - 2$. Hence:

$$\alpha_h(G_{m,n}) \geq \max\{2(\mathrm{diam}(G_{m,n}) - 1), \alpha(G_{m,n})\} = \max\{2(m + n - 3), \lceil \frac{mn}{2} \rceil\}. \quad (4.1)$$

Bouchemakh and Zemir's paper [5] on h-independence in grid graphs determines $\alpha_h(G_{m,n})$ for all grid graphs. We will observe that Bouchemakh and Zemir's results meet the bound above for all grids except $G_{5,5}$. We will also adapt some of their methods for use on other definitions of independent broadcasts. To this end, some theorems and brief proofs of the paper are presented.

Analogously to Erwin's proof [10] that $\alpha_h(P_n) = 2(\mathrm{diam}(P_n) - 1)$, Bouchemakh and Zemir [5] examine the existence of broadcast vertices in the first and last columns of α_h-broadcasts for grids and conclude that for every α_h-broadcast f on $G_{m,n}$ with $|V_f^+| \geq 2$, $C(x_1) = 1$ and $C(x_{|V_f^+|}) = n$ (Proposition 4.3.1).

Proposition 4.3.1. [5] *Let f be an α_h-broadcast on $G_{m,n}$ and $|V_f^+| \geq 2$. Then there is at least one broadcast vertex in the first column and at least one broadcast vertex in the last column.*

Proof. Suppose that $C(x_1) > 1$. Let $C(x_1) = k$ and $x_1 = v_{i,k}$. Define a new broadcast g with $g(v_{i,1}) = f(x_1) + k - 1$, $g(v_{i,k}) = 0$ and $g(x) = f(x)$ for all other vertices. The resulting broadcast has greater weight than f. By the choice of $v_{i,k}$ there are no broadcasting vertices $v_{p,q}$ with $q < k$. For any $v_{p,q}$, with $q \geq k$, $d(v_{i,1}, v_{p,q}) \geq d(v_{i,k}, v_{p,q})$ and since f is h-independent, $H_g(v_{i,1}) \cap V_g^+ = \{v_{i,1}\}$. Further, $N_g(v_{i,1}) = N_f(v_{i,k}$ hence g is also h-independent and contradicts the maximality of f. See Figure 4.2. Hence $C(x_1) = 1$. A similar argument shows that $C(x_{|V_f^+|}) = n$. □

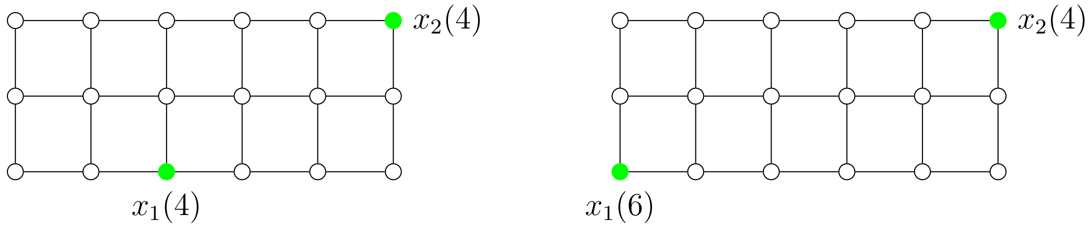


Figure 4.2: On the left, a h-independent f-broadcast on $G_{3,6}$. The broadcasting set V_f^+ consists of two vertices and the first vertex x_1 is not in column 1. On the right, x_1 is moved to column 1 and a larger weight h-independent broadcast g results.

Bouchemakh and Zemir [5] add the following result.

Lemma 4.3.2. [5] *Let f be an h-independent broadcast on G. If* $\operatorname{diam}(G) \geq 2$ *and* $|V_f^+| \leq 2$ *then* $\sigma(f) \leq 2(\operatorname{diam}(G) - 1)$.

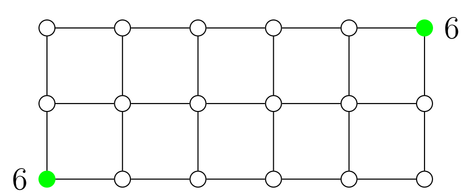


Figure 4.3: An α_h-broadcast on $G_{3,6}$. The broadcasting set V_f^+ consists of two antipodal broadcast vertices with $\sigma(f) = f(x_1) + f(x_2) = 12 = 2(n + m - 3) = 2(6 + 3 - 3)$.

Remark 4.3.3. [5] *For an α_h-broadcast f on a grid graph $G_{m,n}$ with* $\operatorname{diam}(G) \geq 2$*, either $\alpha_h(G_{m,n}) = 2(m+n-3)$ or $|V_f^+| > 2$.*

Let $\mathcal{G}$ be the set of all subgraphs of G. Let F be the set of all broadcasts on the graph G and for any $f \in F$ and $G' \in \mathcal{G}$, let $f' = f \upharpoonright G'$, the broadcast f restricted to G'. If, for every graph in $\mathcal{G}$, f' has the same type of independence on G' as f has on G, then we say that the subgraphs of G *inherit this independence from G*. For example, Bouchemakh and Zemir [5] observe that if f is an h-independent broadcast on $G_{m,n}$ and G is a subgraph of $G_{m,n}$ then f' is h-independent. Or equivalently, subgraphs of $G_{m,n}$ inherit h-independence from $G_{m,n}$. Note that although f' will adhere to the principle of hearing independence it may violate the definition of a broadcast, namely it is possible that $f'(v) > e(v)$ in G'. For small m, Bouchemakh and Zemir [5] use the fact, Proposition 4.3.1, that in every α_h-broadcast of $G_{m,n}$ with $|V^+| > 2$, there is a broadcast vertex $x_{|V_f^+|}$ in the nth column to locate the position of this vertex by a case analysis based only on its row. For each case, once the location of $x_{V_f^+}$ is known the graph is divided into smaller subgraphs. Each subgrid inherits the hearing-independence of the larger graph and induction is used to get upper bounds on $\alpha_h(G_{m,n})$. When the upper and lower bounds meet results follow. We give a sketch of one case from their proof that $\alpha_h(G_{3,n}) = 2n$. All other cases are similar.

Proof sketch. [5] Suppose that for all $3 \leq n < k$, we have $\alpha_h(G_{3,n}) = 2n$. Base cases can easily be shown by brute force. Let f be an α_h-broadcast on $G_{3,n}$ with $n = k$. From Remark 4.3.3, either $|V_f^+| > 2$ or our claim is shown. Hence we assume the former. By Proposition 4.3.1, $C(x_{|V_f^+|}) = n$. In an h-independent broadcast V_f^+ forms an independent set and since $m = 3$ there are at most two broadcasting vertices in the nth column. First suppose that there are two such vertices. Then, to maintain h-independence, $f(v_{1,n}) = f(v_{3,n}) = 1$. If there is a vertex x such that $C(x) = n-1$ then $x = v_{2,n-1}$ and $f(v_{2,n-1}) = 1$. See Figure 4.4. Consider the two subgraphs $G_{m,n-2}$ induced by the vertices in the first $n-2$ columns, and G' induced by the vertices in the last two columns. By the observations made on the last two columns, for $f' = f \upharpoonright G'$, $\sigma(f') = 3$. Note that since f is h-independent, the function f, when restricted to $G_{m,n-2}$, is still a legitimate broadcast. And since $G_{m,n-2}$ inherits h-independence,

$$\sum_{v \in V(G_{m,n-2})} f(v) \leq 2(n-2)$$

by our induction hypothesis. Hence

$$\sigma(f) = \sum_{v \in V(G_{m,n-2})} f(v) + \sum_{v \in V(G')} f(v) \leq 2(n-2) + 3 = 2n - 1.$$

But from Dunbar et al.'s bound (Proposition 2.3.22) we know that for $G_{3,n}$, $\alpha_h \geq 2n$, so this choice of broadcasting vertices in the last two columns contradicts the choice of f. The remaining cases are similar. □

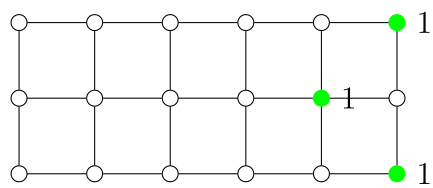


Figure 4.4: The first case in Bouchemakh's proof. An α_h-broadcast on $G_{3,6}$ with two broadcasting vertices in the last column. If the $(n-1)$th column contains a broadcasting vertex it must be in row 2. By induction, the broadcast restricted to the smaller subgrid consisting of the first 4 columns has a maximum weight $2(4) = 8$. Thus the entire broadcast has a maximum weight of $11 < 2(6)$ and is not a α_h-broadcast.

Using this approach, Bouchemakh and Zemir [5] prove that

$$\alpha_h(G_{2,n}) = 2(n-1),$$
$$\alpha_h(G_{3,n}) = 2n,$$
$$\alpha_h(G_{4,n}) = 2(n+1).$$

Or, equivalently, for $G_{m,n}$ with $2 \leq m \leq 4 \leq n$, we have $\alpha_h(G_{m,n}) = 2(m+n-3)$ so $\alpha_h(G_{m,n})$ equals Erwin's bound $\alpha_h(G) \leq 2(\text{diam}(G) - 1)$ for $m \leq 4 \leq n$.

Remark 4.3.4. *Bouchemakh and Zemir* [5] *are also able to prove that in all the above cases every α_h-broadcast f on $G_{m,n}$ has $|V_f^+| = 2$ and V_f^+ consists of two antipodal vertices each broadcasting with a strength of* $\text{diam}(G_{m,n}) - 1$.

As the grid size increases something interesting happens. First Bouchemakh and

Zemir [5] show by brute force that

$$\alpha_h(G_{5,5}) = 15 > \max\{2(\text{diam}(G_{5,5}) - 1), \alpha(G_{5,5})\} = 14.$$

See Figure 4.5.

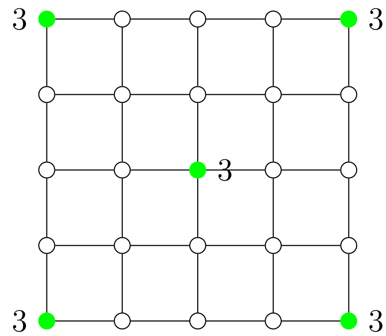


Figure 4.5: An α_h-broadcast on $G_{5,5}$. The weight is $\alpha_h(G_{5,5}) = 15 > \max\{2(\text{diam}(G_{5,5}) - 1), \lceil \frac{mn}{2} \rceil\} = 14$.

Bouchemakh and Zemir [5] also show that

$$\alpha_h(G_{5,6}) = 16 = 2(\text{diam}(G_{5,6}) - 1) = \max\{2(\text{diam}(G_{5,6}) - 1), \alpha(G_{5,6})\}$$

but unlike on the smaller grids, an α_h-broadcast f can also be achieved on $G_{5,6}$ with $|V_f^+| > 2$. See Figure 4.6.

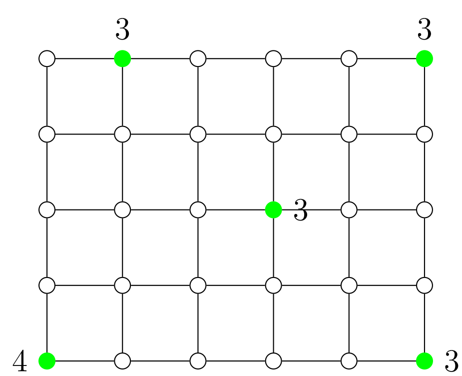


Figure 4.6: An α_h-broadcast on $G_{5,6}$. The weight is $\alpha_h(G_{5,6}) = 16 = \max\{2(\text{diam}(G_{5,6}) - 1), \lceil \frac{mn}{2} \rceil\} = 16$.

When $m \geq 5, n \geq 7$, Bouchemakh and Zemir [5] use the same techniques as for the smaller grids to determine that every α_h-broadcast is achieved by a broadcast f such that $|V_f^{++}| = 0$. Hence every α_h-broadcast on these larger grids corresponds to the characteristic function of a maximum independent set.

Theorem 4.3.5. [Bouchemakh and Zemir] [5] *For every pair of integers m and n, $5 \leq m \leq n$, $(m,n) \neq (5,5),(5,6)$, $\alpha_h(G_{m,n}) = \lceil \frac{mn}{2} \rceil$.*

Interpreting Bouchemakh and Zemir's results through our initial bound

$$\alpha_h(G) \geq \max\{\mu(G)(\text{diam}(G) - 1), \alpha(G)\},$$

which for grids is

$$\alpha_h(G_{m,n}) \geq \max\{2(m+n-3), \lceil \frac{mn}{2} \rceil\},$$

and observing that

$$2(m+n-3) > \lceil \frac{mn}{2} \rceil \text{ for } 1 \leq m \leq 4 \text{ and } m \leq n,$$

while

$$2(m+n-3) < \lceil \frac{mn}{2} \rceil \text{ for } 5 \leq m \text{ and } 7 \leq n,$$

and recalling that Bouchemakh and Zemir showed that

$$\alpha_h(G_{5,5}) = 15 > \max\{2(\text{diam}(G_{5,5}) - 1), \alpha(G_{5,5})\}$$

and that

$$\alpha_h(G_{5,6}) = 16 = 2(\text{diam}(G_{5,6}) - 1) = \max\{\mu(\text{diam}(G_{5,6}) - 1), \alpha(G_{5,6})\},$$

we conclude that

$$\alpha_h(G_{m,n}) = \max\{2(\text{diam}(G_{m,n}) - 1), \alpha(G_{m,n})\} \text{ for all } 1 \leq m \leq n, (m,n) \neq (5,5).$$

If $1 \leq m \leq 4$ then every α_h-broadcast f on $G_{m,n}$ has $|V_f^+| = 2$, $C(x_1) = 1$, $C(x_2) = n$, and $f(x_1) = f(x_2) = \text{diam}(G_{m,n}) - 1$. If $m \geq 5$ and $n \neq 5, 6$ then every α_h-broadcast f on $G_{m,n}$ has $|V_f^+| = |V_f^1| = \lceil \frac{mn}{2} \rceil$, and $f(x) = 1$ for all $x \in V_f^+$. If $(m,n) \in \{(5,5),(5,6)\}$, then $G_{m,n}$ has an α_h-broadcast as shown in Figures 4.5 and 4.6.

$\boxed{\alpha_{bn}(G_{m,n}), \alpha_{bnr}(G_{m,n}) \text{ and } \alpha_{bnd}(G_{m,n})}$

We show that for $2 \leq m \leq n$,

$$\alpha_{bn}(G_{m,n}) = \alpha_{bnr}(G_{m,n}) = \alpha_{bnd}(G_{m,n}) = \lceil \frac{mn}{2} \rceil.$$

For α_{bn}-, α_{bnr}- and $\alpha_{bnd}-$broadcasts in grid graphs, recall that

$$\lceil \frac{mn}{2} \rceil = \alpha(G_{m,n}) \leq \alpha_{bnd}(G_{m,n}) \leq \alpha_{bnr}(G_{m,n}) \leq \alpha_{bn}(G_{m,n}) \leq \alpha_h(G_{m,n}). \quad (4.2)$$

So for $n \geq m \geq 5$ and $(m,n) \neq (5,5), (5,6)$, Bouchemakh and Zemir 's results yield: $\alpha_{bn}(G_{m,n}) = \alpha_{bnr}(G_{m,n}) = \alpha_{bnd}(G_{m,n}) = \alpha_h(G_{m,n}) = \lceil \frac{mn}{2} \rceil$.

To show the same results for $m = 2, 3, 4$ and $(m,n) \in \{(5,5), (5,6)\}$, we show that $\alpha_{bn}(G_{m,n}) \leq \alpha(G_{m,n})$ for all grids.

Lemma 4.3.6. *Every grid graph $G_{m,n}$, $2 \leq m \leq n$, has an $\alpha_{bn}-$broadcast f such that $f(v) = 1$ for all $v \in V_f^+$.*

Proof. Suppose $G_{m,n}$ has no α_{bn}-broadcast such that $|V^{++}| = 0$. Then out of all $\alpha_{bn}-$ broadcasts, consider one, say f, for which $|V_f^{++}|$ is a minimum. There is a vertex $v = v_{i,j} \in V_f^+$ with $f(v) = k > 1$. Since $f(v) \leq e(v) \leq \text{diam}(G)$ and $m \geq 2$, $N_f(v)$ contains two internally disjoint paths of length $f(v)$, hence $N_f(v)$ contains a path or a cycle of length $2f(v)$. Label the path or cycle Q: $v_0, v_1, ..., v_{f(v)} = v, v_{f(v)+1}, ..., v_{2f(v)}$. Define a new broadcast g with:

$$g(x) = \begin{cases} 1 & \text{if } x = v_j \text{ and } j \equiv 1 \pmod 2 \\ 0 & \text{if } x = v_j \text{ and } j \equiv 0 \pmod 2 \\ f(x) & \text{otherwise.} \end{cases}$$

Notice that either $g(x) = f(x)$ or $x \in V(Q) - \{v_0, v_{2f(v)}\}$ and $N_g(x) \subseteq N_f(v)$. The grid $G_{m,n}$ is bipartite, so if v_s and v_t are adjacent then $s \not\equiv t \pmod 2$, otherwise $G_{m,n}$ has an odd cycle, which is impossible. Hence g is a bn-independent broadcast. See Figure 4.7. The weight of g is the same as the weight of f and $|V_g^{++}| < |V_f^{++}|$. Either g is an α_{bn}-broadcast or it can be increased to form an $\alpha_{bn}-$ broadcast. Both possibilities contradict the choice of f, hence the result follows. $\square$

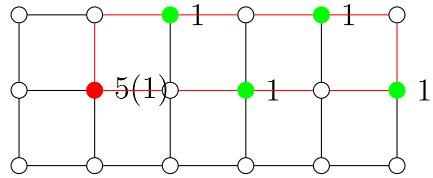


Figure 4.7: A bn-independent broadcast f on a grid has a vertex v with $f(v) = 5$. A cycle dominated by v is shown in red. A broadcast g replaces $f(v)$ with the 5 broadcasting vertices each of strength 1. Notice that the new broadcast neighbourhood is contained in $N_f(v)$.

Theorem 4.3.7. *For* $2 \leq m \leq n$, $\alpha_{bnd}(G_{m,n}) = \alpha_{bnr}(G_{m,n}) = \alpha_{bn}(G_{m,n}) = \lceil \frac{mn}{2} \rceil$.

Proof: Consider an $\alpha_{bn}-$ broadcast which is composed of all ones. Then V^+ is an independent set so $\alpha_{bn}(G_{m,n}) = |V^+| \leq \alpha(G)$. The result now follows from Equation 4.2. $\square$

The ideas in Lemma 4.3.6 and Theorem 4.3.7 suggest a result for 2-connected bipartite graphs in general.

Theorem 4.3.8. *For every 2-connected bipartite graph* G, $\alpha_{bnd}(G) = \alpha_{bnr}(G) = \alpha_{bn}(G) = \alpha(G) \geq \lceil \frac{|V(G)|}{2} \rceil$.

Proof. Let G be a 2-connected bipartite graph. For bipartite graphs, each partition forms an independent set thus the pigeon hole principle gives us the lower bound: $\alpha(G) \geq \lceil \frac{|V(G)|}{2} \rceil$. As for grids, we show that there is an α_{bn}-broadcast with $|V_f^{++}| = 0$. Let f be an α_{bn}-broadcast of G with the smallest number of vertices in V_f^{++}. Assume that V_f^{++} has at least one vertex v. Since G is bridgeless, $\deg v > 1$. Let $f(v) = k > 1$. Since $f(v) \leq e(v)$, there is a vertex u at distance k from v. If u is the only such vertex then, since G is 2-connected, u and v lie on a common cycle C of length $2k$. Let $C : v = v_0, v_1, ..., v_k = u, v_{k+1}, ..., v_{2k} = v$. Define g by $g(x) = 0$ if $x = v_i$ and $i \equiv k \pmod 2$, $g(x) = 1$ if $x = v_i$ and $i \not\equiv k \pmod 2$ and $g(x) = f(x)$ otherwise. If there are two vertices u, w at distance k from v then there is a path P of length $2k$. Let $P : u = v_0, v_1, ..., v_k = v, v_{k+1}, ..., v_{2k} = w$. Define g by $g(x) = 0$ if $x = v_i$ and $i \equiv 0 \pmod 2$, $g(x) = 1$ if $x = v_i$ and $i \equiv 1 \pmod 2$ and $g(x) = f(x)$ otherwise. In both cases, since G is bipartite, no two vertices v_i, v_j (on P or C) where $i \equiv j (\pmod 2)$ are adjacent. Also $N_g(v_i) \subseteq N_f(v)$. Hence g is bn-independent. Notice that $\sigma(g) = \sigma(f)$. Thus either g contradicts the minimality of $|V_f^{++}|$ or g

is not maximal h-independent and contradicts the maximality of the α_{bn}-broadcast f. Hence we assume that there is an α_{bn}-broadcast f with $|V_f^{++}| = 0$. Since V_f^+ forms an independent set, $\alpha_{bn}(G) = \sigma(f) \leq \alpha(G)$. We have already observed that the characteristic function of a maximal independent set is a maximal bn-, bnr, bnd-independent broadcast. Hence $\alpha(G) \leq \alpha_{bnd}(G) \leq \alpha_{bnr}(G) \leq \alpha_{bn}(G) = \alpha(G)$. □

4.4 Summary

For the minimum independence parameters, since grid graphs are radial, we showed, for all $2 \leq m \leq n$:

$$\gamma_b(G_{m,n}) = i_{bn}(G_{m,n}) = i_{bnd}(G_{m,n}) = i_h(G_{m,n}) = i_{hd}(G_{m,n}) = i_{sd}(G_{m,n}) = \text{rad}(G_{m,n})$$

where:

$$\text{rad}(G_{m,n}) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \text{if } m = 2 \text{ or } 3 \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor & \text{otherwise.} \end{cases}$$

And for the minimums which require irredundance, we obtained:

$$i_{sr}(G_{m,n}) \leq i_{hr}(G_{m,n}) \leq i_{bnr}(G_{m,n}) \leq \text{rad}(G_{m,n}).$$

If our conjecture that there exists a dominating $i_{sr}(G_{m,n})$-broadcast holds, then all minimum irredundant independence parameters will equal $\text{rad}(G_{m,n})$.

For the maximums, we presented Bouchemakh and Zemir's results for h-independence:

$$\alpha_h(G_{m,n}) = \max\{2(\text{diam}(G_{m,n}) - 1), \alpha(G_{m,n})\}$$

for all $1 \leq m \leq n, (m, n) \neq (5, 5)$ and

$$\alpha_h(G_{5,5}) = 15.$$

For boundary independence, for $G_{m,n}$ where $2 \leq m \leq n$, we obtained:

$$\alpha_{bnd}(G_{m,n}) = \alpha_{bn}(G_{m,n}) = \alpha_{bnr}(G_{m,n}) = \alpha(G) = \lceil \frac{mn}{2} \rceil.$$

Our boundary independence results generalized to 2-connected bipartite graphs. Given a 2-connected bipartite graph G:

$$\alpha_{bnr}(G) = \alpha_{bnd}(G) = \alpha_{bn}(G) = \alpha(G) \geq \lceil \frac{|V(G)|}{2} \rceil.$$

Chapter 5

Maximum Boundary Independence, Maximum Boundary Independence with Irredundance and Maximum Broadcast Domination Weight Comparison

Out of the nine definitions for independence studied, the maximum values for *bn*- and *bnr*-independent broadcasts are the most interesting and useful. As shown in Chapter 2, since an α_{bn} -broadcast might not be minimal dominating and an α_{bnr}-broadcast might not be dominating, both associated parameters are incomparable with $\Gamma_b(G)$. We now show that while the differences in both directions are unbounded, the ratios $\alpha_{bnr}(G)/\Gamma_b(G)$ and $\alpha_{bn}(G)/\Gamma_b(G)$ are bounded for any graph G. Moreover, while $\Gamma_b(G)/\alpha_{bn}(G)$ and $\Gamma_b(G)/\alpha_{bnr}(G)$ are unbounded for graphs in general, the former is bounded for bipartite graphs and the latter is bounded for trees.

5.1 The Differences

5.1.1 $\Gamma_b(G) - \alpha_{bn(bnr)}(G)$

We proceed to show that $\Gamma_b(G) - \alpha_{bn}(G)$ is unbounded and since $\alpha_{bn}(G) \geq \alpha_{bnr}(G)$ we will have also shown that $\Gamma_b(G) - \alpha_{bnr}(G)$ is unbounded for graphs in general.

Proposition 5.1.1. *For any integer $k \geq 3$, there exists a graph G_k such that $\Gamma_b(G_k) - \alpha_{bn}(G_k) \geq \lfloor \frac{k}{2} \rfloor$.*

Proof. Consider the grid $G_{3,n}$. From Theorem 4.3.7, we have $\alpha_{bn}(G_{3,n}) = \lceil \frac{3n}{2} \rceil$. Consider a broadcast f, such that $f(v_{1,i}) = 2$ for all $1 \leq i \leq n$ and $f(x) = 0$ otherwise. For $1 \leq i \leq n$, $v_{n,i} \in PB_f(v_{1,i})$. Hence, f is an irredundant dominating broadcast of weight $2n$ (see Figure 2.12 for an example with $n = 3$). Thus, $\Gamma_b(G_{3,n}) \geq 2n$ and $\Gamma_b(G_{3,n}) - \alpha_{bn}(G_{3,n}) \geq \lfloor \frac{n}{2} \rfloor$ and $G_k = G_{k,3}$. □

For any spanning tree T of a graph G, recall that $\alpha_{bn}(T) \geq \alpha_{bn}(G)$. Hence, it is possible that $\Gamma_b(T) - \alpha_{bn}(T)$ is bounded. We are unable to prove a bound or find a counterexample so we leave this as an open question.

Question 5.1.2. *Is $\Gamma_b(T) - \alpha_{bn}(T)$ bounded for all trees T?*

To show that $\Gamma_b(T) - \alpha_{bnr}(T)$ is unbounded for trees in general, we use the tree T_k described below and shown in Figure 5.1 for $k = 4$.

To form the tree T_k, take k copies of P_5 and add edges so that the subtree induced by the central vertices of the copies of P_5 is P_k. Let f be a broadcast such that, for each distinct P_5, $|V_f^+ \cap P_5| = 1$ and the broadcasting vertices are all leaves each broadcasting with of strength 4. Notice that f is dominating and irredundant. Again see Figure 5.1. Hence, $\Gamma_b(T_k) \geq \sigma(f) = 4k$.

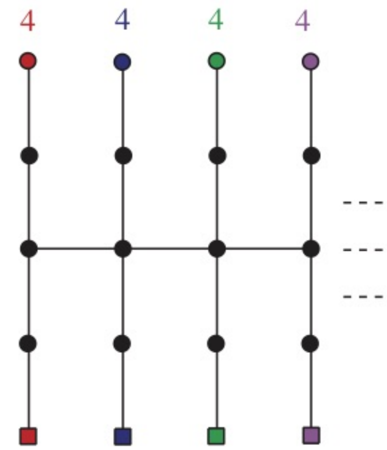


Figure 5.1: The tree T_4. Notice that each square is the private boundary of the broadcasting vertex of matching colour. Hence, the broadcast shown is a Γ_b-broadcast and $\Gamma_b(T_k) \geq 4k$. We show below that $\alpha_{bnr}(T_k) \leq 3k$. Thus $\Gamma_b(T_k) - \alpha_{bnr}(T_k) \geq k$ and T_k is an example for which this difference is unbounded.

Lemma 5.1.3. *For $k \geq 2$, $\alpha_{bnr}(T_k) \leq 3k$, where T_k is the tree described above and shown in Figure 5.1 (for $k = 4$).*

Proof.

Let $B = \{v : v \in V(T_k) \text{ and } \deg(v) \geq 3\}$. Label the vertices of the path induced by the vertices in B, $P_k : b_1, ..., b_k$. Label the k subgraphs generated by removing all edges of P_k as $P_5^1, ...P_5^k$ where P_i contains b_i. For $1 \leq i \leq k$, label vertices of $P_i^5 : l_i, v_i, b_i, v_i', l_i'$. Suppose the statement is false and let $k \geq 2$ be the smallest value such that $\alpha_{bnr}(T_k) > 3k$. By Theorem 2.3.13, we select an $\alpha_{bnr}(T_k)$-broadcast f such that no leaf hears a non-leaf. In such a broadcast, for $1 \leq i \leq k$, $f(v_i) = f(v_i') = 0$ and $f(b_i) \leq 1$. For $1 \leq i \leq k-1$, since f bn-independent, $f(b_i) + f(b_{i+1}) \leq 1$. Since f is irredundant, for $1 \leq i \leq k$, either $f(l_i) < 2$ or $f(l_i') < 2$ or both.

Of all such broadcasts, choose one such that $L = \{v : v \in V(T_k) \text{ and } f(v) \geq 4\}$ is a minimum. Let $M = \max\{f(v) : v \in V_f^+\}$.

Suppose $L = \emptyset$. If $M < 3$ then, by the preliminary comments, $f(P_5^i) \leq 3$ for all $1 \leq i \leq k$. Hence, by the pigeon hole principle, $\sigma(f) \leq 3k$, a contradiction. Suppose $M = 3$. If $f(l_1) = 3$ then $B_f(l_1) = \{v_1', b_2\}$ and, since f is maximal bnr-independent, $f(l_2) = f(l_2') = f(l_1') = 1$ and $f(b_1) = f(b_2) = 0$. Hence $f(P_5^1) + f(P_5^2) = 6$. By symmetry, if $f(l_k) = 3$ then $f(P_5^k) + (P_5^{k-1}) = 6$. If $f(l_j) = 3$ and $j \neq 1, k$ then $B_f(l_j) = \{b_{j-1}, v_j', b_{j+1}\}$ and $f(l_{j-1}) = f(l_{j-1}') = f(l_j') = f(l_{j+1}) = f(l_{j+1}') = 1$ and

$f(b_{j-1}) = f(b_j) = f(b_{j+1}) = 0$. Hence, $f(P_5^{j-1}) + f(P_5^j) + f(P_5^{j+1}) = 8$. Since $M \leq 3$, the only other possibility is that $f(l_t) \leq 2$ and thus $f(P_5^t) \leq 3$. Since $\sigma(f) = f(P_5^1) + ... + f(P_5^k)$, we have shown that $\sigma(f) \leq 3k$.

Suppose $L \neq \emptyset$ and $M = 4$. If $k = 2$ then without loss of generality, let $f(l_1) = 4$ then $B_f(l_1) = \{v_2', v_2, l_1'\}$ and, since f is *bnr*-independent, $f(l_2) = f(l_2') = 1$ and $f(b_1) = f(b_2) = f(l_1') = 0$. Hence $f(P_5^1) + f(P_5^2) = 6 \leq 3k$. If $k = 3$ and $f(l_2) = 4$ then $f(l_1) = f(l_1') = f(l_3) = f(l_3') = 1$ and $f(b_1) = f(b_2) = f(b_3) = f(l_1') = 0$ and $\sigma(f) = 8 \leq 3k$. If $f(l_j) = 4$ and either $k = 3$ and $j \neq 2$, or $k > 3$ then, without loss of generality, $\{v_{j+1}, v_{j+1}', b_{j+2}\} \subseteq B_f(l_j)$ and $f(l_{j+1}) = f(l_{j+1}') = 1$ and $f(b_j) = f(b_{j+1}) = f(b_{j+2}) = 0$. Create a new broadcast g with $g(l_j) = 3$, $g(l_j') = 1$ and $g(x) = f(x)$ otherwise. Notice that $b_{j+1} \in PB_g(l_j)$. Since $g(l_j') = 1$, $l_j' \in PB_g(l_j')$. Also, $N_g(l_j) \cup N_g(l_j') \subset N_f(l_j)$. Hence, g is a *bnr*-independent broadcast. Since $\sigma(f) = \sigma(g)$, either g can be extended and violates the maximality of f or, since it has fewer vertices broadcasting with strength 4, it violates the choice of f.

Assume that $f(v) = M \geq 5$. Without loss of generality, let l_j be a vertex such that $f(l_j) = M$. Since $f(l_j) \leq e(l_j)$ and by the structure of T_k, there are two leaves l_t, l_t' such that $d(l_t, l_j) = d(l_t', l_j) = M$. Assume, without loss of generality, $t > j$. (If $t < j$ then reverse the labeling on T_k.) Create a new broadcast with $g(l_i) = 2$, $g(l_i') = 1$ and $g(b_i) = 0$ for all $j \leq i \leq t$, and $g(x) = f(x)$ otherwise. Notice that $\cup_{i=j}^{t}(N_g(l_i) \cup N_g(l_i')) \subset N_f(l_j)$. For $j \leq i \leq t$, $b_i \in PB_g(l_i)$ and $l_i' \in PB_g(l_i')$. Hence g is a *bnr*-broadcast with $\sigma(g) = \sigma(f) - M + 3(t-j) = \sigma(f) - M + 3(M-3) = \sigma(f) + 2M - 9$. Since $M \geq 5$, $\sigma(g) > \sigma(f)$ and g violates the maximality of f. We have exhausted all possibilities for a counterexample, hence $\alpha_{bnr}(T_k) \leq 3k$. □

Lemma 5.1.4. *For $k \geq 2$, $\alpha_{bnr}(T_k) = 3k$, where T_k is the tree described above and shown in Figure 5.1 (for $k = 4$).*

Proof. Let f be a broadcast such that $f(l_i) = 1$ and $f(l_i') = 2$ for all $1 \leq i \leq k$ and $f(x) = 0$ otherwise. Notice that $l_i \in PB_f(l_i)$ and $b_i \in PB_f(l_i')$ for all $1 \leq i \leq k$. Also, $N_f(l_i) \cup N_f(l_i') = \emptyset$ and no b_i is overdominated. Hence, f is a *bnr*-independent broadcast with $\sigma(f) = 3k$ and by Lemma 5.1.3, our claim is shown. □

Since $\Gamma_b(T_k) \geq 4k$, the following result is an immediate consequence of Lemmas 5.1.3 and 5.1.4.

Theorem 5.1.5. *For any integer $k \geq 1$, there exists a tree T_k such that $\Gamma_b(T_k) - \alpha_{bnr}(T_k) \geq k$.*

5.1.2 $\alpha_{bnr(bn)}(G) - \Gamma_b(G)$

If a tree T has an α_{bnr}-broadcast which is dominating then $\alpha_{bnr}(T) \leq \Gamma_b(T)$. However, not all trees have such a broadcast and there exist trees such that $\alpha_{bnr}(T) > \Gamma_b(T)$. Figure 5.2 gives an example of a *bnr*-independent broadcast on a tree T which is not dominating. By using symmetry and examining a few cases, it can be shown that $\alpha_{bnr}(T) = 14$ and $\Gamma_b(T) = 13$.

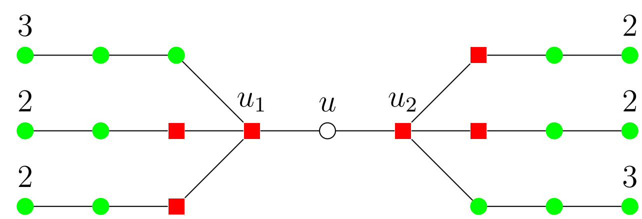


Figure 5.2: A Tree T with $\alpha_{bnr}(T) = 14 > 13 = \Gamma_b(T)$.

We use the tree T in Figure 5.2 as a basis to construct a bigger tree H_k to show that the difference $\alpha_{\text{bnr}} - \Gamma_b$ can be arbitrary for trees. Since $\alpha_{bn}(T) \geq \alpha_{bnr}(T)$ we will have shown the same result for $\alpha_{bn}(T) - \Gamma_b(T)$.

Theorem 5.1.6. *For any integer $k \geq 1$ there exists a tree H_k such that $\alpha_{\text{bnr}}(H_k) - \Gamma_b(H_k) \geq k$.*

Proof. Take $3k$ copies $T^1, ..., T^{3k}$ of the tree T in Figure 5.2 and label the central vertex and its neighbours of the i^{th} copy $u_1^i u^i u_2^i$. Let H_k be the tree formed by joining u^i to u^{i+1} for each $i = 1, ..., 3k-1$. Let f^i be the broadcast on T^i obtained by copying the broadcast illustrated in Figure 5.2 for each T^i, and $f = \bigcup_{i=1}^{3k} f^i$. Then f is a bnr-broadcast, hence $\alpha_{\text{bnr}}(H_k) \geq 42k$.

Define the broadcasts g^i on T^i and g on H_j as follows:

$$g^i(x) = \begin{cases} 1 & \text{if } x = u^i \\ 2 & \text{if } x \text{ is a leaf of } T^i \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} g^i(x) & \text{if } x \in V(T^i) \text{ and } i \equiv 2 \pmod 3 \\ f^i(x) & \text{otherwise.} \end{cases}$$

Since $\sigma(f^i) = 14$ and $\sigma(g^i) = 13$, $\sigma(g) = 28k + 13k = 41k$. It is also easy to see that g is a dominating broadcast. Suppose $i \equiv 2 \pmod 3$. Then $\mathrm{PB}_g(u^i) = \{u_1^i, u^i, u_2^i, u^{i-1}, u^{i+1}\}$, and if ℓ is a leaf, then $\mathrm{PB}_g(\ell)$ consists of the vertex at distance 2 from ℓ. Suppose $i \equiv 0$ or $1 \pmod 3$. If $g(\ell) = 3$, then $\mathrm{PB}_g(\ell) = \{u_j^i\}$ for $j \in \{1, 2\}$, as the case may be, and if $g(\ell) = 2$, then, as before, $\mathrm{PB}_g(\ell)$ consists of the vertex at distance 2 from ℓ. Hence g is a minimal dominating broadcast and we deduce that $\Gamma_b(H_k) \geq 41k$. We show that $\Gamma_b(H_k) = 41k$.

Consider any minimal dominating broadcast h on H_k and say $h^i = h \upharpoonright T^i$. If each u^i is dominated only by a vertex of T^i, then, as in the case of T, $\sigma(h^i) \leq 13$ for each i, so that $\sigma(h) \leq 39k < \Gamma_b(H_k)$. Hence assume a vertex u^i is dominated by a vertex $v \in V(T^j)$, $j \neq i$. Say v overdominates u^i by exactly t, $t \geq 0$. Upper bounds for $\sigma(h^i)$ and $\sigma(h^j)$ are recorded in the table below.

t	0	1	2	3	≥ 4
$\sigma(h^i)$	≤ 14	≤ 12	≤ 6	≤ 6	0
$\sigma(h^j)$	$\leq 12 + h(v)$	$\leq 6 + h(v)$	$\leq 6 + h(v)$	$h(v)$	$h(v)$

Suppose $t > 0$. Let j be the smallest index such that T^j contains a vertex v which overdominates a vertex u^i, $i \neq j$. By symmetry we may assume that $j \leq \lceil \frac{3k}{2} \rceil$.

- Suppose v overdominates u^{3k} by at least 4. Then v overdominates each u^i, $i = 1, ..., 3k$, by at least 4. Then, regardless of the value of j, v dominates H_k and $\sigma(h) = e(v) \leq \mathrm{diam}(H_k) = 3k + 7 < \Gamma_b(H_k)$.

- Suppose v overdominates u^{3k} by 3.
  - If k is odd and $j = \lceil \frac{3k}{2} \rceil$, then $h(v) \leq \lfloor \frac{3k}{2} \rfloor + 7$ and v dominates all of H_k except for 12 leaves, so $\sigma(h) \leq \lfloor \frac{3k}{2} \rfloor + 19 < \Gamma_b(H_k)$ for all k.
  - In all other cases, $h(v) \leq \mathrm{diam}(H_k) - 1$ and v dominates all of H_k except for 6 leaves, so $\sigma(h) \leq 3k + 12 < \Gamma_b(H_k)$ for all k.

- Similarly, if v overdominates u^{3k} by 2, we obtain that $\sigma(h) \leq \mathrm{diam}(H_k) - 2 + 24 = 3k + 29 < 41k$ for all k.

Hence we assume that there exists a smallest index $i \neq j$ such that v overdominates u^i by exactly 1. We consider two cases, depending on the value of i.

Case 1 $i < j$. Since $j \leq \lceil \frac{3k}{2} \rceil$, v also overdominates u^{2j-i} by exactly 1. Then $\sigma(h^i), \sigma(h^{2j-i}) \leq 12$, and, if these indices exist and are distinct from j, $\sigma(h^{i-1}), \sigma(h^{2j-i+1}) \leq 14$. Define the broadcast η on H_k by

$$\eta(x) = \begin{cases} g^l(x) & \text{if } x \in V(T^l) \text{ and } l \in \{i-1, 2j-i+1\} \\ f^l(x) & \text{if } x \in V(T^l) \text{ and } l \in \{i, 2j-1\} \\ h(x) - 1 & \text{if } x = v \\ h(x) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sigma(\eta) = \sigma(h) &- [\sigma(h^i) + \sigma(h^{2j-i}) + \sigma(h^{i-1}) + \sigma(h^{2j-i+1})] \\ &+ [\sigma(f^i) + \sigma(f^{2j-i}) + \sigma(g^{i-1}) + \sigma(g^{2j-i+1})]. \end{aligned}$$

Since $\sigma(f^i) - \sigma(h^i) \geq 2$ and $\sigma(h^{i-1}) - \sigma(g^{i-1}) \leq 1$ (and similarly for the other values), we see that $\sigma(\eta) > \sigma(h)$. Also, all broadcast vertices of η have nonempty private boundaries (note that $u^i \in \mathrm{PB}_\eta(v)$), so η is bnr-independent. All vertices u^l are dominated, so any non-dominated vertices lie on endpaths. Since v dominates these vertices in f, and $u^i \in \mathrm{PB}_\eta(v)$, these vertices can be dominated by extending η on an appropriate leaf to get a minimal dominating broadcast. We conclude that $\sigma(h) < \sigma(\eta) \leq \Gamma_b(H_k)$.

Case 2 $i > j$. By the choice of j and by symmetry, v overdominates each u^l, $l < i$, by at least 2. Now $\sigma(h^i) \leq 12$, $\sigma(h^{i-1}) \leq 6$, and, if $i < 3k$, then $\sigma(h^{i+1}) \leq 14$. Define the broadcast η' on H_k by

$$\eta'(x) = \begin{cases} g^{i+1}(x) & \text{if } x \in V(T^{i+1}) \\ f^i(x) & \text{if } x \in V(T^i) \\ h(x) - 1 & \text{if } x = v \\ h(x) & \text{otherwise.} \end{cases}$$

Then

$$\sigma(\eta') = \sigma(h) - [\sigma(h^i) + \sigma(h^{i+1})] + [\sigma(f^i) + \sigma(g^{i+1})].$$

As above, $\sigma(f^i) - \sigma(h^i) \geq 2$ and $\sigma(h^{i+1}) - \sigma(g^{i+1}) \leq 1$, and following the reasoning above we conclude that $\sigma(h) < \sigma(\eta') \leq \Gamma_b(H_k)$.

Therefore, if $v \in V(T^j)$ dominates a vertex u^i, $j \neq i$, we may assume that v does not overdominate u^i. This implies that $i = j \pm 1$. (Assume $j-1$ and $j+1$ both exist; the proof works the same if only one of them exists.) Then $\sigma(h^{j-1}), \sigma(h^{j+1}) \leq 14$ and, as for the tree T, $\sigma(h^j) \leq 13$. Consequently,

$$\sigma(h^l) \leq \begin{cases} 13 & \text{if some vertex of } T^l \text{ dominates a vertex of } T^i,\ i \neq l \\ 14 & \text{otherwise.} \end{cases}$$

Since each u^i is dominated and the subtree of T induced by $\{u^1, ..., u^{3k}\}$ is the path P_{3k}, there are at least $\gamma(P_{3k}) = k$ indices j such that some vertex of T^j dominates u^i, $i \neq j$. This implies that $\sigma(h) \leq 13k + 14 \cdot 2k = 41k$ and we conclude that $\Gamma_b(H_k) = 41k$, that is, $\alpha_{\text{bnr}}(H_k) - \Gamma_b(H_k) \geq k$. $\square$

Since $\alpha_{bn}(G) \leq \alpha_{bn}(T)$, if T is a spanning tree of G it is possible that $\alpha_{bn}(G) - \Gamma_b(G)$ is bounded for non-trees. Again, we use the unboundedness of $\alpha_{bnr}(G) - \Gamma_b(G)$ to show that this is not the case. To show that $\alpha_{bnr}(G) - \Gamma_b(G)$ is unbounded, we generalize the construction of the graph G_2 which is shown in Figure 2.13 and again, for reference, in Figure 5.3.

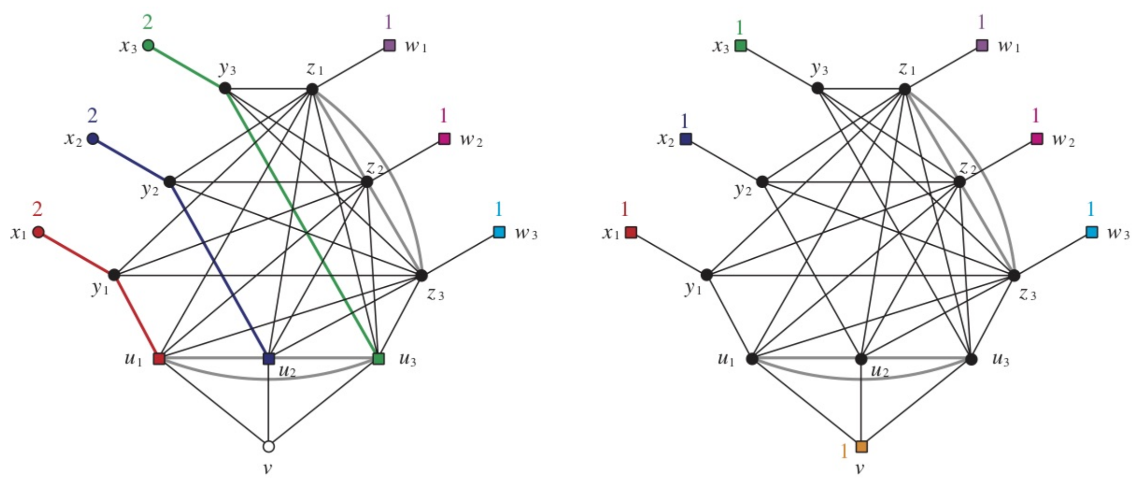


Figure 5.3: The graph G_2 with $\alpha_{\text{bnr}}(G_2) = 9$ and $\Gamma_b(G_2) = 7$. A non-dominating α_{bnr}-broadcast is shown on the left, and a Γ_b-broadcast on the right.

To generalize the construction, let the corona of G and K_1 be denoted by $G \circ K_1$.

For $k \geq 1$, construct the graph G_k as follows. Let $U = \{u_1, ..., u_{k+1}\}$, $W = \{w_1, ..., w_{k+1}\}$, $X = \{x_1, ..., x_{k+1}\}$, $Y = \{y_1, ..., y_{k+1}\}$, $Z = \{z_1, ..., z_{k+1}\}$ and $\{v\}$ be disjoint sets of vertices. Add edges so that

$$\begin{gathered}G_k[X] \cong G_k[Y] \cong G_k[W] \cong \overline{K_{k+1}},\\ G_k[U \cup \{v\}] \cong K_{k+2},\ G_k[Z] \cong K_{k+1},\ G_k[U \cup Z] \cong K_{2(k+1)},\\ G_k[\{y_i\} \cup Z] \cong K_{k+2} \text{ for each } i \in \{1, ..., k+1\},\\ G_k[Y \cup U] \cong G_k[W \cup Z] \cong K_{k+1} \circ K_1,\\ G_k[X \cup Y] \cong \overline{K_{k+1}} \Box K_2 \cong (k+1)K_2.\end{gathered}$$

Assume that the perfect matchings of $G_k[U \cup Y]$, $G_k[X \cup Y]$ and $G_k[W \cup Z]$ are $\{u_iy_i : i = 1, ..., k+1\}$, $\{x_iy_i : i = 1, ..., k+1\}$ and $\{w_iz_i : i = 1, ..., k+1\}$, respectively. The graph G_2 is illustrated in Figure 5.3.

Proposition 5.1.7. *Let G_k be the graph described above and shown in Figure 5.3 for $k = 2$. For any integer $k \geq 2$, $\alpha_{bnr}(G_k) \geq 3(k+1)$.*

Proof.

Define the broadcast f by

$$f(x) = \begin{cases} 2 & \text{if } x \in X \\ 1 & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma(f) = 3(k+1)$ and

$$\left.\begin{array}{rcl} N_f(x_i) & = & \{x_i, y_i, u_i\} \cup Z \\ B_f(x_i) & = & \{u_i\} \cup Z \\ \mathrm{PB}_f(x_i) & = & \{u_i\} \\ N_f(w_i) & = & \{w_i, z_i\} \\ \mathrm{PB}_f(w_i) & = & \{w_i\} \end{array}\right\} \text{for each } i \in \{1, ..., k+1.$$

Thus we see that f is a bnr-independent broadcast, hence $\alpha_{\mathrm{bnr}}(G_k) \geq 3(k+1)$. □

To see that $\Gamma_b(G_k) = 2k+3$, and hence that $\alpha_{\mathrm{bnr}}(G_k) - \Gamma_b(G_k) \geq k$, we use arguments similar to those for G in Proposition 2.3.19.

Proposition 5.1.8. *For $k \geq 1$, $\Gamma_b(G_k) \leq 2k+3$ where G_k is the graph constructed above (and shown in Figure 2.13 for $k = 2$).*

For any minimal dominating broadcast f, due to symmetry, there are six possible ways to dominate the uncoloured vertex, v.

Case 1: The vertex v is dominated by v. If $f(v) = 3$ then v dominates the graph, hence $\sigma(f) = 3$. If $f(v) = 2$ then to maintain irredundance there exists t such that either $y_t \in PB_f(v)$ or $z_t \in PB_f(v)$ or both. Hence either $f(Z) = 0$ and $f(y_t) + f(x_t) = 0$ or $f(Y) = 0$ and $f(z_t) + f(w_t) = 0$ or both. To further maintain irredundance, for all $1 \leq j \leq k+1$, $f(u_j) = 0$, $f(x_j)+f(y_j) \leq 1$ and $f(z_j)+f(w_j) \leq 1$. Hence $\sigma(f) \leq 2k + 1 + 2 = 2k + 3$. If $f(v) = 1$ then to maintain irredundance, for all $1 \leq j \leq k + 1$, $f(u_j) = 0$ and $f(z_j) + f(w_j) \leq 1$ and $f(x_j) + f(y_j) \leq 1$. Hence $\sigma(f) \leq 2(k + 1) + 1 = 2k + 3$.

Case 2: The vertex v is dominated by u_t. Suppose $f(u_t) = 3$. Then t dominates G_k and $\sigma(f) = 3$. Suppose $f(u_t) = 2$. Then t dominates $G_k - X + \{x_t\}$. To maintain irredundancy, $f(v) = 0$ and for all $1 \leq i \leq k + 1$, $f(z_i) = f(w_i) = 0$ and either $f(u_i) = 0$ or $f(u_i) = 2$. Let $U = \{u_i | f(u_i) = 2\}$. For all $1 \leq j \leq k+1$, if $u_i \in U$ then to maintain irredundance, $f(x_i) = 0$, and if $u_i \notin U$ then $f(x_j) + f(y_i) \leq 1$. Hence, $\sigma(f) \leq 2|U| + k + 1 - |U| \leq |U| + k + 1 \leq 2k + 2$.

Case 3-4: Suppose the vertex v is dominated by $s \in \{x_t, y_t\}$ for some t, $1 \leq t \leq k + 1$. In each case, s dominates $G_k - X + \{x_t\}$. To maintain irredundance, for $0 \leq i \leq k + 1$, $f(x_i) \leq 1$, $f(s) = e(s)$ and $f(x) = 0$ otherwise. Hence, $\sigma(f) \leq f(s) + k \leq 4 + k$.

Case 5-6: Suppose the vertex v is dominated by $s \in \{z_t, w_t\}$ for some t, $1 \leq t \leq k + 1$. Then since $d(s, v) = e(s)$, s dominates G_k. Hence, $\sigma(f) = f(s) = e(s) \leq 3$.

This exhausts all possibilities, hence $\Gamma_b(G_k) \leq 2k + 3$. □

Propositions 5.1.7 and 5.1.8 imply the following theorem:

Theorem 5.1.9. *For any integer $k \geq 2$ there exists a graph G_k such that $\alpha_{bnr}(G_k) - \Gamma_b(G_k) \geq k$.*

5.2 The Ratios

5.2.1 $\frac{\alpha_{bnr(bn)}(G)}{\Gamma_b(G)}$

We show that the ratios $\alpha_{bn}(G)/\Gamma_b(G)$, $\alpha_{bnr}(G)/\Gamma_b(G)$ and $\alpha_{bn}(G)/\alpha_{bnr}(G)$ are bounded.

Theorem 5.2.1. *For any graph G,*

$$\alpha_{bnr}(G)/\Gamma_b(G) \leq \alpha_{bn}(G)/\Gamma_b(G) < 2$$

and

$$\alpha_{bn}(G)/\alpha_{bnr}(G) < 2.$$

Proof. Given a graph G, if there exists a minimal dominating α_{bn}-broadcast then $\alpha_{bn}(G) \leq \Gamma_b(G)$. So we may assume that there is a graph G for which no such broadcast exists. Every α_{bn}-broadcast is dominating and an irredundant dominating broadcast is minimal dominating. Thus we may assume that none of the α_{bn}-broadcasts on G are irredundant. We examine what this means and determine a strategy for turning a non-irredundant dominating α_{bn}-broadcast into an irredundant dominating broadcast. Given any bn-independent broadcast f on G and any $v \in V_f^1$, $v \in PB_f(v)$. Hence if f is an α_{bn}-broadcast which is not irredundant there exists $v \in V_f^{++}$ such that $PB_f(v) = \emptyset$. Create a new broadcast f' with $f'(v) = f(v) - 1$ and $f'(x) = f(x)$ otherwise. Since $PB_f(v) = \emptyset$, f' is dominating. Since f is bn-independent $PB_{f'}(v) \neq \emptyset$. Hence f' is a dominating bn-independent broadcast. Notice that $V_f^+ = V_{f'}^+$ and $V_{f'}^+$ contains more vertices with non-empty private boundaries. If f' is not irredundant, this process can be repeated until we have a dominating irredundant broadcast f''. Hence $\sigma(f'') \leq \Gamma_b(G)$ and $\alpha_{bn}(G) = \sigma(f) \leq \sigma(f'') + |V_f^{++}| \leq \Gamma_b(G) + |V_f^{++}|$. Since f'' is irredundant and bn-independent, we also have $\alpha_{bn}(G) = \sigma(f) \leq \sigma(f'') + |V_f^{++}| \leq \alpha_{bnr}(G) + |V_f^{++}|$. Notice that in any broadcast $\sigma(f) \geq 2|V_f^{++}|$. Hence $\Gamma_b(G) \geq \sigma(f) - |V_f^{++}| \geq \frac{1}{2}\sigma(f) = \frac{1}{2}\alpha_{bn}(G)$. Thus $\alpha_{bn}(G)/\Gamma_b(G) \leq 2$. We can improve this bound slightlly. If $V_f^1 \neq \emptyset$ then $\sigma(f) > 2|V_f^{++}|$. Further if $V_f^1 = \emptyset$ then we can select any vertex from V_f^+, say v and let $g = \{f'' - (v, f''(v)) + (v, f(v)\}$. Notice that g is irredundant and thus can be used in the place of f''. In both cases, we now have $\alpha_{bn}(G)/\Gamma_b(G) < 2$. Similarly, $\alpha_{bn}(G)/\alpha_{bnr}(G) < 2$. And finally, since $\alpha_{bnr}(G) \leq \alpha_{bn}(G)$, $\alpha_{bnr}(G)/\Gamma_b(G) < 2$. □

As seen in Propositions 2.3.6 and 2.3.16, for the spider $S(2^k)$ where $k \geq 3$, $\alpha_{bn}(S(2^k)) = 2k$ and $\Gamma_b(S(2^k)) = k + 1$. Hence

$$\lim_{k\to\infty} \frac{\alpha_{\text{bn}}(S(2^k))}{\Gamma_b(S(2^k))} = \lim_{k\to\infty} \frac{2k}{k+1} = 2.$$

Hence, this bound is tight.

5.2.2 $\frac{\Gamma_b(G)}{\alpha_{bn(bnr)}(G)}$

Bouchemakh and Fergani [4] show the upper bound

$$\Gamma_b(G) \leq n - \delta(G),$$

where $\delta(G)$ is the minimum degree over $V(G)$. Recalling that $\alpha_{bnr}(G) \geq \alpha(G)$ we get

$$\frac{\Gamma_b(G)}{\alpha_{bn}(G)} \leq \frac{\Gamma_b(G)}{\alpha_{bnr}(G)} \leq \frac{n-\delta(G)}{\alpha(G)}.$$

If G is bipartite then $\alpha(G) \geq \frac{n}{2}$, hence we have the following result.

Proposition 5.2.2. *If G is bipartite graph of order $n \geq 1$, then $\Gamma_b(G)/\alpha_{bn}(G) \leq \Gamma_b(G)/\alpha_{bnr}(G) < 2$.*

Proof. Note that, for $n \geq 2$,

$$\frac{\Gamma_b(G)}{\alpha_{bn}(G)} \leq \frac{\Gamma_b(G)}{\alpha_{bnr}(G)} \leq \frac{n-\delta(G)}{\alpha(G)} \leq \frac{n-1}{\frac{n}{2}} < 2.$$

Also, if $n = 1$ then $\alpha_{bn}(G) = \alpha_{bnr}(G) = \Gamma_b(G) = 1$ and the result holds. □

Theorem 5.2.3. *For general graphs, the ratios $\Gamma_b(G)/\alpha_{\mathrm{bn}(G)}$ and $\Gamma_b(G)/\alpha_{\mathrm{bn}(G)}$ are unbounded.*

Proof. Consider $G_n \cong K_n \Box P_3$; let $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., x_n\}$, $Z = \{z_1, ..., z_n\}$ be the vertex sets of the copies of K_n, and $Q_i : x_i, y_i, z_i$ the copies of P_3. First we show that $\Gamma_b(G_n) = 2n$. Let f be the broadcast obtained by broadcasting with a cost of 2 from each x_i. Then $\mathrm{PB}_f(x_i) = \{z_i\}$ for each i and each Q_i is dominated by x_i. Hence f is a minimal dominating broadcast. We claim that f has the largest weight of all such broadcasts. Suppose, for a contradiction, that there is a broadcast g with $\sigma(g) > \sigma(f)$. By the pigeon hole principle, there is some Q_i such that $g(Q_i) > 2$. Since g is irredundant and $\mathrm{diam}(G) = 3$, there are only three possibilities.

Case 1: Suppose without loss of generality that $g(x_i) = 2$ and $g(z_i) = 1$. Since $N_g(x_i) = X \cup Y \cup \{z_i\}$ and $N_g(z_i) = Z \cup \{y_i\}$, x_i and z_i dominate G_n and $\sigma(g) = 3$.

Case 2: Suppose without loss of generality that $g(x_i) = g(y_i) = g(z_i) = 1$. Since $X \subset N_g(x_i)$, $Y \subset N_g(y_i)$ and $Z \subset N_g(z_i)$, x_i, y_i, and z_i dominate G_n and $\sigma(g) = 3$.

Case 3: Suppose $g(v) = 3$ for some $v \in Z \cup X$. Then v dominates G and $\sigma(g) = 3$. Hence no such g exists and $\Gamma_b(G_n) = \sigma(f) = 2n$.

On the other hand, we show that $\alpha_{\rm bn}(G_n) = 3$ for each $n \geq 2$. Let h be a broadcast derived from the characteristic function of a maximum independent set. Since $\alpha(G_n) = 3$, $\sigma(h) = 3$. Recall that such a broadcast is a maximal *bn*-independent broadcast, hence $\alpha_{bn}(G_n) \geq 3$. Consider any *bn*-independent broadcast h' on G. To maintain *bn*-independence, h' can have at most one broadcast vertex in each of X,Y, and Z. And further if a vertex is broadcasting with strength 2 either it dominates the entire graph and thus is a maximal *bn*-independent broadcast or without loss of generality, it is in X and dominates X and Y. In the latter case, there can be at most one other broadcast vertex, say z, and $h'(z) = 1$. Hence, in every case, $\sigma(h') \leq 3$ and $\lim_{n\to\infty} \Gamma_b(G)/\alpha_{\rm bn}(G) = \lim_{n\to\infty} 2n/3 = \infty$. □

5.3 Summary

We showed, by example, that $\Gamma_b(G) - \alpha_{bn}(G)$ and $\Gamma_b(G) - \alpha_{bnr}(G)$ are unbounded for graphs in general. We gave a second example, showing that $\Gamma_b(T) - \alpha_{bnr}(T)$ is also unbounded for trees.

We gave two examples showing that $\alpha_{bnr}(G) - \Gamma_b(G)$ and $\alpha_{bn}(G) - \Gamma_b(G)$ are unbounded for trees and for graphs in general.

For the ratios, we showed that $\alpha_{\rm bnr}(G)/\Gamma_b(G) \leq \alpha_{\rm bn}(G)/\Gamma_b(G) < 2$ for all graphs G. This bound is tight for $\alpha_{\rm bn}(G)/\Gamma_b(G)$. The tree T in Figure 5.2 satisfies $\alpha_{\rm bnr}(T)/\Gamma_b(T) = \frac{14}{13}$.

Problem 5.3.1. *Determine the smallest constant c such that $\alpha_{\rm bnr}(T)/\Gamma_b(T) \leq c$ for all trees T. Similarly, determine the smallest constant k such that $\alpha_{\rm bnr}(G)/\Gamma_b(G) \leq k$ for more general graphs G.*

Finally, we showed that for bipartite graphs with $n \geq 2$,

$$\Gamma_b(G)/\alpha_{bn}(G) \leq \Gamma_b(G)/\alpha_{bnr}(G) < 2.$$

And we gave an example to show that $\Gamma_b(G)/\alpha_{bn}(G)$ and $\Gamma_b(G)/\alpha_{bnr}(G)$ are both unbounded for graphs in general.

Chapter 6

Trees

We now focus on boundary independence and study $\alpha_{bn}(T)$ and $\alpha_{bnr}(T)$ for trees other than paths and spiders. To get tighter bounds and exact results, we look at subclasses of trees based on the number of vertices of degree 3 or more, the subtree induced by these vertices, and the lengths of the paths connecting these vertices. We obtain improved upper and lower bounds for $\alpha_{bn}(T)$ for trees in general and give examples of trees which meet our new bounds. Recall that for a graph G and any spanning tree T of G, $\alpha_{bn}(G) \leq \alpha_{bn}(T)$. Thus, we also find tight bounds for $\alpha_{bn}(G)$ for graphs in general. We calculate exact formulas and provide an algorithm for calculating $\alpha_{bn}(T)$ for several large classes of trees.

6.1 Maximum Boundary Independent Broadcasts, $\alpha_{bn}(T)$

If f is a bn-independent broadcast such that all edges are covered, or equivalently $UE_f = \emptyset$, then f is a maximal (but not necessarily maximum) bn-independent broadcast. The converse is not true. In fact, there exist graphs such that no α_{bn}-broadcast covers all edges. For trees, consider certain double spiders. For example, create the graph G by attaching 3 pendant edges to each leaf of a P_2. On G, a broadcast f from every leaf of strength 1 is a bn-independent broadcast with $\sigma(f) = 6$ and with the edge of P_2 uncovered. By examining all possibilities, we see that the largest cost broadcast which covers all edges of this double spider has strength 5. For an example which is not a tree consider odd cycles C_n, $n \geq 5$.

A tree with $\deg v \leq 2$ for all $v \in V(T)$ is a path and a tree with exactly one vertex

v such that $\deg v > 2$ is a generalized spider. As noted in Remark 2.3.6, if T is a path or a spider then it meets the upper bound $\alpha_{bn}(T) = n - 1$ and any maximum bn-broadcast covers all edges. To further our study on maximum bn-independent broadcasts on trees, we focus on the number of vertices which have degree greater than two. This structural focus motivates the following definitions which are illustrated in Figure 6.1. If $\deg v \geq 3$, then we refer to v as a *branch vertex.*

Definition 6.1.1. *The* branch-leaf *representation $BL(T)$ of a tree T is a graph obtained by suppressing all vertices v with* $\deg v = 2$. *Equivalently, to form $BL(T)$, a vertex of degree 2 in T is chosen and a new graph $T - \{v\}$ is created in which the two neighbours or v are now adjacent. This process is repeated until no vertices of degree 2 remain.*

Definition 6.1.2. *The* branch representation $B(T)$ *of a tree T is formed from $BL(T)$ by deleting all leaves.*

Definition 6.1.3. *The* branch set $V(B(T))$ *of a tree T is* $V(B(T)) = \{v : v \in V(T)$ *and* $\deg_T(v) \geq 3\}$ *or equivalently the set of branch vertices in T.*

Definition 6.1.4. *The* branch number $b(T)$ *of a tree T is* $b(T) = |V(B(T))|$.

Definition 6.1.5. *The* leaf set $L(T)$ *of a tree T is the set of all leaves of T.*

Definition 6.1.6. *If a leaf l of a tree T is adjacent to a branch vertex v in $BL(T)$ then it belongs to $L(v)$, the* leaf set *of v, and we refer to it as a leaf of v (even though l is not necessarily adjacent to v in T).*

Since $BL(T)$ is unique, we can talk about $L(v)$ for any branch vertex v of T where the reference to $BL(T)$ is implied. Given an α_{bn}-broadcast on T, if a branch vertex v is overdominated by one of its leaves then it is possible that the leaf also overdominates $L(v)$, all the leaves attached to this branch. In this case, no other vertex in $L(v)$ is broadcasting. To maximize the weight of this type of broadcast, the leaf which is furthest from the branch vertex is broadcasting. To describe the weight of this choice, we define the following concepts for a tree T.

Definition 6.1.7. *For a tree T with $b(T) \geq 1$, the* max leaf value *of a branch vertex v is* $\max(v) = \max\{d_T(v, x) : x \in L(v)\}$.

Definition 6.1.8. *For a tree T with $b(T) \geq 1$, the* sum of a branch vertex v *is* $\mathrm{sum}(v) = \sum_{x \in L(v)} d_T(v, x)$.

Definition 6.1.9. *For a tree T with $b(T) \geq 1$, the* loss of a branch vertex *is* $\text{loss}(v) = \text{sum}(v) - \max(v)$.

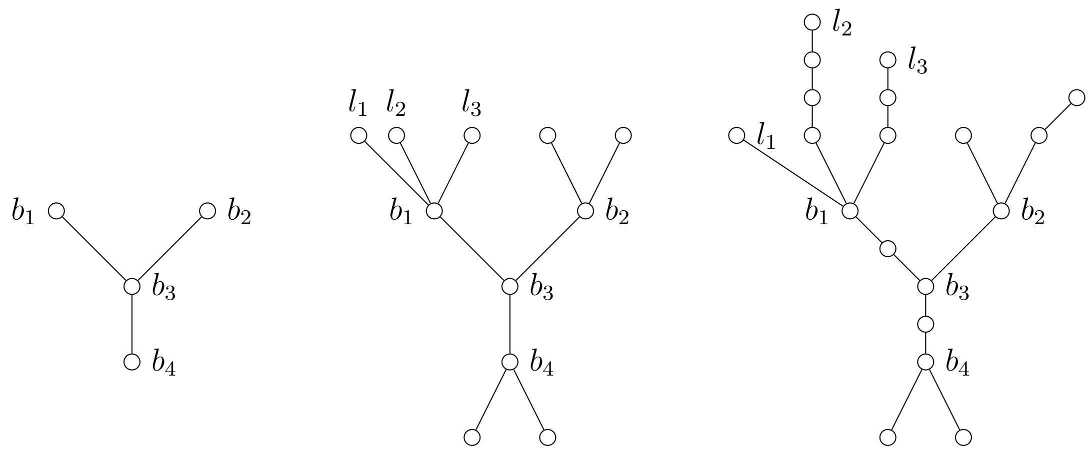


Figure 6.1: A tree T (right) , its *branch-leaf* representation $BL(T)$ (middle) and its *branch* representation $B(T)$ left. The branch set of T is $V(B(T)) = \{b_1, b_2, b_3, b_4\}$ and its branch number is $b(T) = 4$. The branch vertex b_1 has a leaf set $L(b_1) = \{l_1, l_2, l_3\}$. The $\text{sum}(b_1) = 8$, the $\max(b_1) = 4$ and the $\text{loss}(b_1) = 4$.

6.1.1 Broadcasting leaves

We now develop some results regarding the role of broadcasting leaves in $\alpha_{bn}(T)$-broadcasts.

Lemma 6.1.10. *Let f be an α_{bn}-broadcast on a tree T such that a leaf l dominates a branch vertex w. If l' is a leaf in $L(w)$ that does not hear f from l, then $l' \in V_f^+$, the $l' - w$ path Q in T contains a vertex $b \in B_f(l)$, and $f(l') = d(l', b)$.*

Proof. Since $l' \in L(w)$, each internal vertex of Q has degree 2 in T. Since f is dominating and leaves only hear leaves (Lemma 2.3.12), some leaf $t \neq l$ broadcasts to l'. Since l dominates w but does not dominate l', the $l' - w$ path Q contains a vertex $b \in B_f(l)$. (Possibly, $b = w$.) If t does not belong to Q, then the $t - w$ path is internally disjoint from Q, $d(t, w) < d(t, l') \leq f(t)$ and t overdominates w. But then $N_f(t) \cap N_f(l) - B_f(l) \neq \emptyset$, which is impossible. Hence $t \in V(Q)$ and so $t = l'$. Let a be the broadcast vertex on the $l' - b$ subpath R of Q nearest to b. Then $a \neq b$. Suppose $a \neq l'$. Let $a_1, ..., a_k$ be all the broadcast vertices on R strictly between l'

and b. Since f is bn-independent, $a_1, ..., a_k$ broadcast to exactly $2\sum_{j=1}^{k} f(a_j)$ edges. Hence R has length $\ell(R) \geq f(l') + 2\sum_{j=1}^{k} f(a_j)$. Define the broadcast f' by

$$f'(x) = \begin{cases} \ell(R) & \text{if } x = l' \\ 0 & \text{if } x \in V(R) - \{l'\} \\ f(x) & \text{otherwise.} \end{cases}$$

Then with respect to f', l' broadcasts to b but no farther along Q. Together with the fact that each internal vertex of Q has degree 2 in T, this implies that f' is bn-independent. But

$$\begin{aligned} \sigma(f') &= \ell(R) + \sum_{x \in V_f^+ - V(R)} f(x) \\ &\geq f(l') + 2\sum_{i=1}^{k} f(a_i) + \sum_{x \in V_f^+ - V(R)} f(x). \end{aligned}$$

Since $k \geq 1$:

$$\sigma(f') > f(l') + \sum_{i=1}^{k} f(a_i) + \sum_{x \in V_f^+ - V(R)} f(x) = \sigma(f).$$

Hence $\sigma(f') > \sigma(f)$ which contradicts f being an α_{bn}-broadcast. Therefore l' is the only vertex in V_f^+ on R. Moreover, since f is maximal bn-independent, l' broadcasts to b but no farther along Q. Therefore $f(l') = d(l', b)$ and the result follows. □

Although the broadcast in Figure 6.2 is not an α_{bn}-broadcast, it is maximal given the broadcast values on l and l_6, and thus shows an example of a bn-independent broadcast where l and l' must have the relationship described in Lemma 6.1.10.

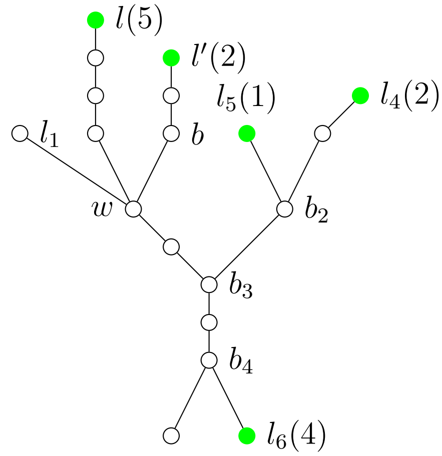


Figure 6.2: A bn-independent broadcast f on a tree T. Given that $f(l) = 5$ and $f(l_6) = 4$, f is a maximal bn-independent broadcast. The leaf l dominates w and there exists a leaf $l' \in L(w)$ such that l' is not dominated by l. There is a vertex b on the $l' - w$ path such that $b \in N_f(l) \cap N_f(l')$ as required by Lemma 6.1.10.

Lemma 6.1.11. *For any tree T with $b(T) \geq 2$ there exists an α_{bn}-broadcast f with two leaves $l_1, l' \in V_f^+$ and two distinct branch vertices $w, w' \in V(T)$ such that $l_1 \in L(w)$ and $l' \in L(w')$.*

Proof. Let T be a tree with $b(T) \geq 2$. Notice that $B(T)$ is a tree and $|V(B(T))| = b(T) \geq 2$. Thus $B(T)$ has at least two leaves or, equivalently, there are at least two branch vertices b_1, b_2 in T such that $\deg_{B(T)}(b_i) = 1$ and thus $|L_T(b_i)| \geq 2$ for $i = 1, 2$. Let f be any α_{bn}-broadcast on T. Since f is dominating and leaves only hear leaves (Lemma 2.3.12), there is at least one broadcasting leaf. We assume that there is exactly one branch vertex, say w, such that $L(w)$ contains all broadcasting leaves, else f is the required broadcast and our statement is satisfied. Let $l_1, ..., l_r$, $r \geq 1$, be the broadcasting leaves in $L(w)$. Any leaf in $L(w)$ that broadcasts to leaves in $L(T) - L(w)$ overdominates w. By bn-independence, at most one leaf in $L(w)$ can overdominate w. Hence there is a unique leaf, say l_1, in $L(w)$ that dominates $L(T) - L(w)$. Since l_1 dominates $L(T) - L(w)$ and since leaves only hear leaves (Lemma 2.3.12) the only vertices it does not dominate must lie on $l_i - w$ paths where $1 < i \leq r$. By Lemma 6.1.10, these vertices must be dominated by the respective l_i, $1 < i \leq r$. Hence the only broadcasting vertices are $l_1, ..., l_r$. Let l' be a leaf, $l' \in L(T) - L(w)$, such that $d(l', w) \geq d(l, w)$ for all $l \in L(T) - L(w)$. Since $f(l_1) \leq e(l_1)$, $B_f(l_1) \neq \emptyset$.

And either $l' \in B_f(l_1)$ or there exists $l'' \in L(w)$ such that a vertex v on the $l'' - w$ path is in $B_f(l_1)$ or both. Possibly $v = l''$. Since f is bn-independent, if $l' \in B_f(l_1)$ then $f(l') = 0$. And, by Lemma 6.1.10, if $v \in B_f(l_1)$ then $f(l'') = d(l'', v)$. Define a broadcast f_1:

$$f_1(x) = \begin{cases} d(x, w) & \text{if } x = l' \text{ or } x \in L(w) \\ 0 & \text{otherwise.} \end{cases}$$

Then either $f_1(l_1) + f_1(l') = f(l_1) + f(l')$ or $f_1(l_1) + f_1(l'') = f(l_1) + f(l'')$ or both. For all $x \in L(w)$, $x \neq l_i$ for any $1 \leq i \leq r$, $f_1(x) > f(x) = 0$. Suppose $r \geq 2$. Since l_1 overdominates w in f and f is bn-independent, $f_1(l_i) > f(l_i)$ for $2 \leq i \leq r$. Hence, $\sigma(f_1) \geq \sigma(f)$. For all leaves $x \in L(w) \cup \{l'\}$, $B_{f_1}(x) = \{w\}$. Hence f_1 is bn-independent and either violates the maximality of f or supplies the required broadcast. □

6.1.2 The role of branch vertices

In Theorem 6.1.12, below, we see the importance of the branch vertices of a tree in determining the broadcast values which may be assigned to its leaves. Recall that there is an α_{bn}-broadcast in which $f(v) = 1$ or $\deg(v) = 1$ for all $v \in V_f^+$ (Theorem 2.3.14). Hence, knowing the broadcast values on the leaves will be very helpful in determining the weight of any α_{bn}-independent broadcast. The combination of Theorem 6.1.12, Theorem 2.3.14, and Lemma 2.3.12 allows us, through a case by case approach, to determine an upper bound for $\alpha_{bn}(T)$ based on the order of T and the number and type of branch vertices of T. This result applies to graphs in general since $\alpha_{bn}(G) \leq \alpha_{bn}(T)$ for all spanning trees T of G. We also use these results to present strategies for generating α_{bn}-broadcasts and thus exact values for $\alpha_{bn}(T)$ on any tree T with $B(T) \cong K_{1,k}$ for all $k \geq 1$.

Suppose a vertex v overdominates a branch vertex w of degree k. Once it has dominated w, for the remaining broadcast of $f(v) - d(w, v)$, it covers up to $k - 1$ distinct paths for the same strength of broadcast required for a single path of this length. Initially, it seems that maximizing weight would require dominating each of these $k - 1$ paths with individual broadcasting vertices. However, as we will see, sometimes a maximum weight broadcast is only produced by overdominating such branch vertices.

By choosing α_{bn}-broadcasts which minimize the number of overdominated branch vertices and examining the cost/benefit (for the total weight) of different types of branch overdomination, Theorem 6.1.12 provides restrictions on the way in which a leaf may overdominate a branch vertex. Informally, Theorem 6.1.12 tells us that a leaf l may never overdominate a branch b by exactly 2. Either l overdominates a branch vertex b by exactly one and b has no leaves except possibly l, or l overdominates b by 3 or more and has exactly one vertex in its boundary and that vertex is not on a $b' - l'$ path for any $b' \in B(T)$ and $l' \in L(b')$. In addition to b, l may dominate an unlimited number of branch vertices by 3 or more so long as it also overdominates all of their leaves.

Theorem 6.1.12. *Any tree T with $b(T) \geq 2$ has an α_{bn}-broadcast f with the minimum number of overdominated branch vertices which satisfies the following statement:*

For any leaf l, let X be the set of all branch vertices overdominated by l. If $X \neq \emptyset$ and $v \in B_f(l)$ then v is not a leaf and v is not an internal vertex on any $x - y$ path where $x \in B(T)$ and $y \in L(x)$. Moreover,

- *(i) there exists $w \in X$ such that $f(l) = d(l, w) + 1$, and either $L(w) = \{l\}$ and $X = \{w\}$, or $L(w) = \emptyset$ and $f(l) \geq d(l, w') + 3$ for all $w' \in X - \{w\}$, or*
- *(ii) $f(l) \geq d(l, w) + 3$ for all $w \in X$ and $B_f(l) = \{v\}$.*

Proof . Let T be a tree with branch number $b(T) \geq 2$ and let f be an α_{bn}-broadcast on T for which the number of overdominated branch vertices is a minimum. Assume that there is a leaf l overdominating a branch vertex. Let $X = \{w \in B(T) : l$ overdominates $w\}$. We first show that if $w \in X$, then $f(l) \neq d(l, w) + 2$. Suppose, for a contradiction, that there exists $w \in X$ such that $f(l) = d(l, w) + 2$. Since $\deg(w) \geq 3$, w is adjacent to two vertices v_1, v_2 that do not lie on the $l - w$ path. Define the broadcast g_1 by $g_1(l) = f(l) - 2$, $g_1(v_1) = g_1(v_2) = 1$ and $g_1(u) = f(u)$ otherwise. Then $\sigma(g_1) = \sigma(f)$. Since $B_{g_1}(v_1) \cap B_{g_1}(v_2) = \{w\}$, $B_{g_1}(l) \cap B_{g_1}(v_i) = \{w\}$ for $i = 1, 2$ and $N_{g_1}(v_1) \cup N_{g_1}(v_2) \subseteq N_f(l)$, g_1 is bn-independent. See Figure 6.3. If g_1 is not maximal bn-independent, then g_1 can be extended to a maximal bn-independent broadcast, contrary to f being an α_{bn}-broadcast. Hence g_1 is an α_{bn}-broadcast. However, since w is no longer overdominated, g_1 has fewer overdominated branch vertices than f does, contrary to the choice of f. Hence:

$$\text{no vertex } w \in X \text{ satisfies } f(l) = d(l, w) + 2. \tag{6.1}$$

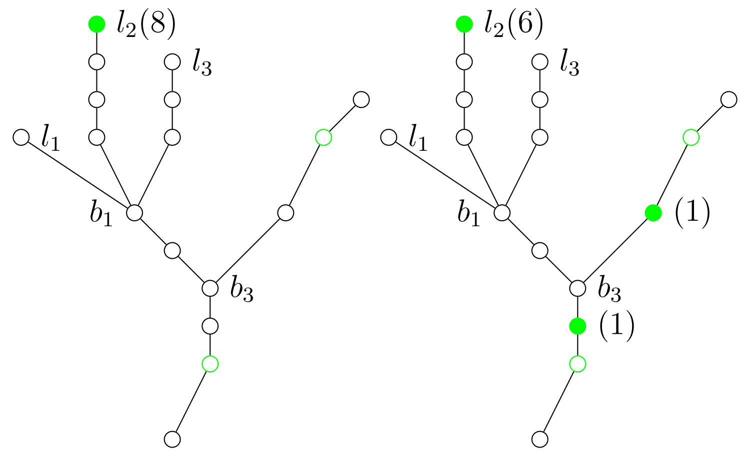


Figure 6.3: A tree T (left) in which a broadcasting leaf overdominates a branch b_3 by 2. Broadcasting vertices are shown as solid green and the boundary of the broadcast is shown by vertices outlined in green. The broadcast can be changed (right) so that there are fewer overdominated branches. The two broadcasts have the same weight. It is easy to see that the new broadcast is bn-independent. Hence any bn-broadcast which minimizes overdominated branch vertices does not have a leaf overdominating a branch vertex by exactly 2.

Let $P = \{v : v \text{ is on a } w - l' \text{ path where } w \in X \text{ and } l' \in L(w)\}$. We now show that $B_f(l) \cap P = \emptyset$. Suppose, for a contradiction, that $y \in B_f(l) \cap P$. Thus there exists $w \in X$ such that y is on a $l' - w$ path and $f(l) = d(l, y)$. (See Figure 6.4). Notice that $l \neq l'$ else $X = \emptyset$. By Lemma 6.1.10, $f(l') = d(l', y)$. Make a new broadcast g_2 with $g_2(l') = d(l', w)$, $g_2(l) = d(l, w)$ and $g_2(u) = f(u)$ otherwise. Notice that $N_{g_2}(l') \cup N_{g_2}(l) \subset N_f(l)$ and $N_{g_2}(l') \cap N_{g_2}(l) = \{w\}$. Hence g_2 is a bn-independent broadcast. Since $f(l) + f(l') = d(l, l') = g_2(l) + g_2(l')$, g_2 has the same weight as f. However, since $\deg(w) \geq 3$, g_2 is not dominating. Hence g_2 can be extended and violates the maximality of f. Hence, $B_f(l) \cap P = \emptyset$. Notice that l does not overdominate any vertex in $B(T) - X$. Hence, l does not dominate any vertex on an $x - l'$ path where $x \in B(T) - X$ and $l' \in L(x)$. Hence:

$$\text{if } l \text{ overdominates a branch vertex and } v \in B_f(l), \text{ then there is no } x \in B(T) \text{ such that } v \text{ is on an } x - l' \text{ path for any } l' \in L(x). \tag{6.2}$$

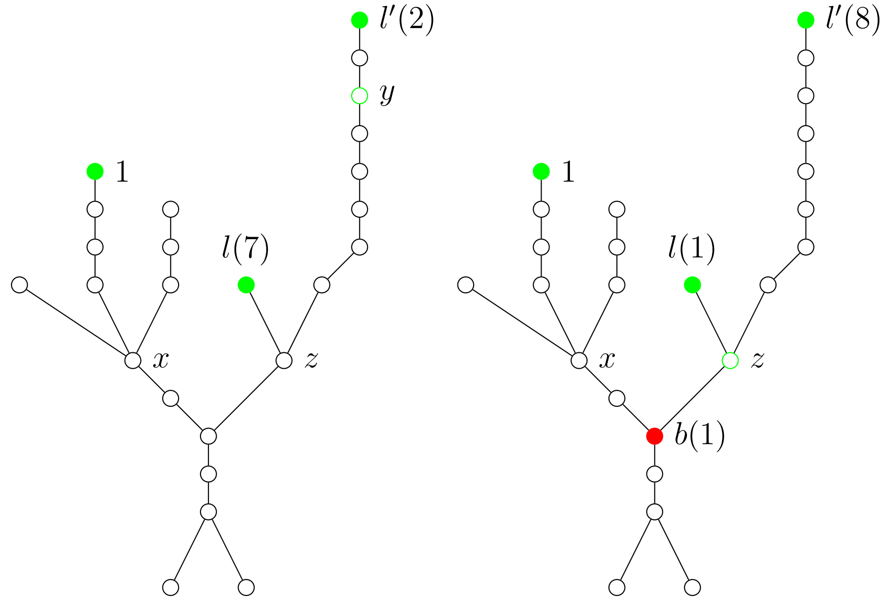


Figure 6.4: A tree T (left) in which a broadcast f has a leaf l which overdominates a branch vertex z. Notice that $y \in B_f(l)$ and y is on a $z - l'$ path where $l' \in L(z)$. Creating the broadcast g_2 (right) with $g_2(l') = d(l', z)$ and $g_2(l) = d(l, z)$ leaves at least one vertex non-dominated. Assigning a broadcast value of 1 to b creates a larger broadcast which is still bn-independent. Hence f is not maximal.

If $f(l) = d(l, w) + 1$ then statement (6.2) implies that $L(w) \subseteq \{l\}$. Now suppose $w' \in X$, $w \neq w'$ and $f(l) = d(l, w') + 1$. Since l is a leaf, $d(l, w) = d(l, w') \geq 2$. Let v_1 and v_2 be neighbours of w and w' on $l - w$ and $l - w'$ paths, respectively. Possibly $v_1 = v_2$. Create a broadcast g_3 with $g_3(l) = f(l) - 2$, $g_3(w) = g_3(w') = 1$ and $g_3(u) = f(u)$ otherwise. Notice that $N_{g_3}(l) \cup N_{g_3}(w) \cup N_{g_3}(w') \subset N_f(l)$, $N_{g_3}(l) \cap N_{g_3}(w) = \{v_1\}$, $N_{g_3}(l) \cap N_{g_3}(w') = \{v_2\}$ and either $N_{g_3}(w) \cap N_{g_3}(w') = \emptyset$ or $v_1 = v_2$ and $N_{g_3}(w) \cap N_{g_3}(w') = \{v_1\}$. Hence g_3 is a bn-independent broadcast. Notice that $\sigma(f) = \sigma(g_3)$ and g_3 overdominates fewer branch vertices, violating the choice of f. Hence, l overdominates at most one branch vertex by exactly one and statement (6.1) now implies (i).

We show that if $f(l) \geq d(l, w) + 3$ for all $w \in X$ then $|B_f(l)| = 1$. Suppose $|B_f(l)| \geq 2$ and consider two distinct vertices $v_1, v_2 \in B_f(l)$. Label the $l - v_i$ paths P_i, $i = 1, 2$. Of all $x \in X \cap V(P_1) \cap V(P_2)$ choose one such that $d(x, l)$ is a maximum. Such a vertex exists because $X \neq \emptyset$. If $w \in X \cap V(P_i) - \{v_1, v_2\}$ for $i \in \{1, 2\}$ then, by (6.2), l dominates $L(w)$. Let Q_i be the $x - v_i$ subpath of P_i, $i = 1, 2$. By our choice of x, Q_1 and Q_2 are internally disjoint. Since $d(l, v_1) = d(l, v_2) = f(l)$, Q_1 and Q_2 have the same length, say k. By our choice of f, $k \geq 3$. Say Q_i is the path $x = q_{i,0}, q_{i,1}, ..., q_{i,k} = v_i$, $i = 1, 2$.

- If $k \equiv 0 \pmod 2$, define the broadcast g_4 by $g_4(l) = d(l, x) = f(l) - k$, $g_4(q_{i,j}) = 1$ if $j \equiv 1 \pmod 2$ and $g_4(u) = f(u)$ otherwise. Since $\bigcup_{i=1}^{2} \bigcup_{j \equiv 1 \pmod 2} N_{g_4}(q_{i,j}) \subseteq N_f(l)$, g_4 is bn-independent, and since k is even, there are $2(\frac{k}{2}) = k$ vertices $q_{i,j}$ in $V_{g_4}^+$, which implies that $\sigma(g_4) = \sigma(f)$. But g_4 overdominates fewer branch vertices than f does, and we have a contradiction as before.

- If $k \equiv 1 \pmod 2$, define the broadcast f' by $f'(l) = d(l, x) + 1$, $f'(q_{i,j}) = 1$ if $j \geq 2$ and $j \equiv 0 \pmod 2$, and $f'(u) = f(u)$ otherwise. Since k is odd and $k \geq 3$, there are $2(\frac{k-1}{2}) = k - 1 \geq 2$ vertices $q_{i,j}$ in $V_{f'}^+$. As for g_4, f' is bn-independent and $\sigma(f') = \sigma(f)$. If f' is not maximal independent, it can be extended to a bn-independent broadcast with weight greater than $\sigma(f)$, which is impossible. Hence f' is an α_{bn}-broadcast. Either f' overdominates fewer branch vertices and violates the choice of f, or f and f' dominate the same number of branch vertices and since $d(l, x) = f'(l) - 1$, we have already shown that (i) holds for l with respect to f'. In this latter case, we consider the broadcast f' instead of f.

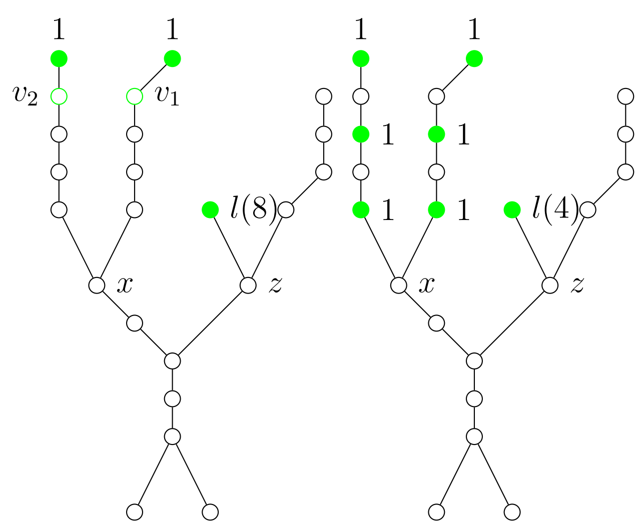


Figure 6.5: A tree T (left) in which a broadcast f has a leaf l which overdominates a branch x by more than 2 and $|B_f(l)| = \{v_1, v_2\}$ (left). The broadcast can be changed (right) without reducing its weight so that x is no longer overdominated.

Hence either $B_f(l) = \{v\}$ and (ii) holds for l and f, or there exists another α_{bn}-broadcast f' such that (i) holds for l and f'. This exhausts all possibilities, showing that the statement of our theorem is correct. $\square$

We restate Theorem 6.1.12 for broadcasts in which only leaves broadcast with strength greater than 1. Corollary 6.1.13 will be used in the proof of our upper bound, Theorem 6.1.16.

Corollary 6.1.13. *For any tree T with a branch number $b(T) \geq 2$, let F represent the set of all α_{bn}-broadcasts f on T such that $V_f^{++} - V_f^1 \subseteq L(T)$. There is a member of F, say f, with minimum number of overdominated branch vertices of all broadcasts in F which satisfies the following statement:*

For any leaf l, let X be the set of all branch vertices overdominated by l. If $X \neq \emptyset$ and $v \in B_f(l)$ then v is not a leaf and v is not an internal vertex on any $x - y$ path where $x \in B(T)$ and $y \in L(x)$. Moreover,

- (i) *there exists $w \in X$ such that $f(l) = d(l, w) + 1$, and either $L(w) = \{l\}$ and $X = \{w\}$, or $L(w) = \emptyset$ and $f(l) \geq d(l, w') + 3$ for all $w' \in X - \{w\}$, or*
- (ii) *$f(l) \geq d(l, w) + 3$ for all $w \in X$ and $B_f(l) = \{v\}$.*

Proof. Consider a tree T and an α_{bn}-broadcast f as described in the corollary statement which exists by Theorem 2.3.14. Use the techniques in Theorem 6.1.12 on f to guarantee that it satisfies the statements of Theorem 6.1.12. These techniques either reduce an existing broadcast value, increase the broadcast value on a leaf, or introduce a new broadcast of strength one. Hence although the broadcast which results, say f', may or may not be the same as f, it will satisfy Theorem 6.1.12 and it will also satisfy the conditions $f'(v) = 1$ or $\deg(v) = 1$ for all $v \in V_{f'}^{+}$. $\square$

6.1.3 Upper bound for $\alpha_{bn}(T)$

The following comments and figures provide some intuition for new definitions, for our results and for Question 6.1.17 regarding the upper bound for $\alpha_{bn}(T)$. Recall that if f is a bn-independent broadcast on G then the set of all uncovered edges together with the sets of edges covered by each $N_f(v)$ for all $v \in V_f^+$ forms a partition on $E(G)$. Thus the upper bound, $\alpha_{bn}(G) = n-1$, is achieved when all edges are covered and each $v \in V_f^+$ covers exactly $f(v)$ edges. However, if a tree has two or more branch vertices then this upper bound is not achievable. We can see this by thinking about how the broadcast covers the edges between two branch vertices. There are four possibilities:

Figure 6.6: Case analysis shows that this is an α_{bn}-broadcast. The branch vertices are covered by leaves and the edge between them is uncovered; $\alpha_{bn}(T) = n - b(T) = 6 - 2 = 4$.

Figure 6.7: A bn-independent broadcast f in which the edge between two branch vertices is covered by a branch vertex, b_1. Notice that b_1 covers $f(b_1) + 2$ edges. The broadcast is not an α_{bn}-broadcast and $\sigma(f) = 3$.

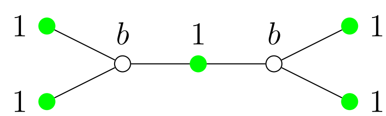


Figure 6.8: Case analysis show that this is an α_{bn}-broadcast. The edges between the branch vertices are covered by a vertex v, but $f(v) = 1$ and v covers $f(v) + 1 = 2$ edges; $\alpha_{bn}(T) = n - b(T) = 7 - 2 = 5$.

Finally, Figure 6.9 shows an α_{bn}-broadcast where the edges between branch vertices are covered by a leaf v. Since v overdominates a branch vertex it covers at least $f(v) + 1$ edges.

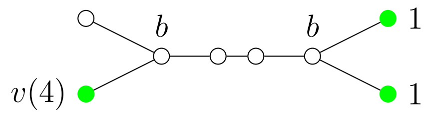


Figure 6.9: Case analysis shows that this is an α_{bn}-broadcast. The edges between branch vertices are covered by a leaf v; $\alpha_{bn}(T) = n - b(T) = 8 - 2 = 6$.

Initially, it seems that, after the first branch, every additional branch vertex reduces the value of $\alpha_{bn}(T)$. However, this is not always the case and it is necessary to consider the subgraphs shown in Figures 6.10 and 6.11 which motivate some new definitions. Let R_T be the subset of $V(T)$ consisting of all branch vertices w of $B(T)$ with $|L(w)| \leq 1$ and define $\rho(T) = |R_T|$. If T has no such branch vertices, then $\rho(T) = 0$. Equivalently, $\rho(T)$ *is the number of branch vertices in* T *with at most one leaf.*

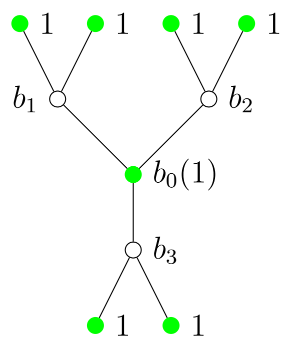


Figure 6.10: Case analysis shows that this is an α_{bn}-broadcast. A tree T with $L(b_0) = \emptyset$, $R_T = \{b_0\}$ and $\alpha_{bn}(T) = 7 = n - b(T) + \rho(T) > n - b(T)$.

In the above Figure 6.10, $B(T)$ induces a star $K_{1,3}$ and $L(b_0) = \emptyset$. If f is a *bn*-independent broadcast on T such that the central branch vertex b_0 of the star is a broadcast vertex with $f(b_0) = 1$ then $\deg(b_0)$ edges are covered by b_0 while $\deg(b_0)+1$ branch vertices are dominated. Note that $\{b_0\} = R_T$, hence $\rho(T) = 1$.

In Figure 6.11 , we see an α_{bn}-broadcast without the $K_{1,3}$ subgraph of $B(T)$ but $|L(b_0)| = 1$ and $B(T)$ induces a $K_{1,2}$ star. Again note that $\alpha_{bn}(T) > n - b(T)$.

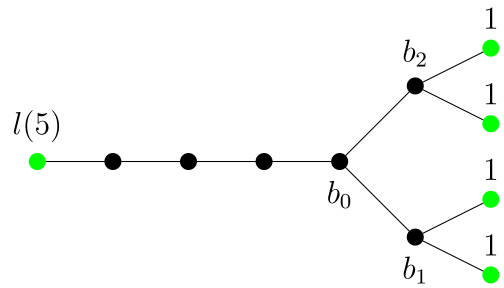


Figure 6.11: Case analysis shows that this is an α_{bn}-broadcast. A tree with $|L(b_0)| = 1$, $T[R_T] = K_1$, $\rho(T) = 1$ and $\alpha_{bn}(T) = 9 = n - b(T) + \rho(T) > n - b(T)$.

In Figure 6.11, we see an α_{bn}-broadcast on a tree T. The branch vertices b_1, b_0, b_2 induce a $K_{1,2}$ star . Notice that $L(b_0) = \{l\}$ and $f(l) = d(l, b_0) + 1$. We have $N_f(l)$ covering $f(l) + \deg(b_0) - 1$ and $\deg(b_0)$ branch vertices.

We further refine our discussion on branch vertices with the following definitions. A branch vertex of T of degree 1 in $B(T)$ is called an *end-branch vertex*. Denote the subset of end-branch vertices by $B_{end}(T)$. If $w \in B_{end}(T)$, then exactly one edge incident with w does not lie on a path from w to a leaf in $L(w)$. Hence, $|L(w)| \geq 2$ and $B_{end}(T) \cap R_T = \emptyset$. We need the following proposition:

Proposition 6.1.14. *For any tree T of order n,* $\operatorname{diam}(T) \leq n - b(T) - 1$.

Proof. Let P be a diametrical path of T. Each branch vertex of T is incident with at least one unique edge not on P. Therefore, by counting edges,

$$diam(P) + b(T) \leq n - 1. \square$$

We next prove a lemma which will serve as a base case for an inductive proof of an upper bound for $\alpha_{bn}(T)$ in terms of its order, the number of branch vertices, and $\rho(T)$.

Lemma 6.1.15. *For a tree T with $b(T) = 2$, $\alpha_{bn}(T) \leq n - b(T) + \rho(T)$.*

Proof 1. Let T be a tree with exactly two branch vertices, say b_1, b_2. In this case $\rho(T) = 0$ and $n - b(T) + \rho(T) = n - 2$. We have given a characterization of trees which meet the upper bound $\alpha_{bn}(T) = n - 1$ in Corollary 2.3.7. This characterization shows that only spiders and paths meet the bound. If $b(T) = 2$ then T is neither a spider nor a path and we have our result. □

Proof 2. Let f be an α_{bn}-broadcast on T with $f(v) = 1$ or $\deg v = 1$ or both. Such a broadcast exists by Proposition 2.3.14. Considering the domination of the branch vertices, there are two possible cases.

Case 1: Neither branch vertex is overdominated. This means that either there is an uncovered edge on the $b_1 - b_2$ path or there is a vertex v on the $b_1 - b_2$ path with $f(v) = 1$ or both. If there is an uncovered edge then $\alpha_{bn}(T) \leq |CE_f| \leq n - 2$. If v covers $f(v) + 1$ edges, then by counting covered edges we have $\alpha_{bn}(T) + 1 \leq |CE_f| \leq n - 1$. Either way $\alpha_{bn}(T) \leq n - 2 = n - b(T)$.

Case 2: A branch vertex is overdominated by a vertex v which means that v covers at least $f(v) + 1$ edges, so again $\alpha_{bn}(T) = \sigma(f) \leq n - 1 - 1 = n - b(T)$.

This exhausts all cases. Hence $\alpha_{bn}(T) \leq n - b(T) = n - b(T) + \rho(T)$. □

The upper bound for $\alpha_{bn}(T)$ nows follows in Theorem 6.1.16 by induction:

Theorem 6.1.16. *For a tree T with $b(T) \geq 1$, $\alpha_{bn}(T) \leq n - b(T) + \rho(T)$.*

Proof. We have shown in Remark 2.3.6 that the claim is true if $b(T) = 1$, and Lemma 6.1.15 shows that it is true for $b(T) = 2$. Now that we have established our base cases, we proceed with induction on $b(T)$. Assume that the claim is true for all trees with fewer than t branch vertices. Suppose it is false for some tree with t branch vertices, $t \geq 3$. Amongst all trees with t branch vertices for which it is false, let T be one of minimum order n. By Proposition 2.3.14 we may consider α_{bn}-broadcasts on T in which each $v \in V_f^+$ satisfies $f(v) = 1$ or $\deg(v) = 1$ (or both). From these broadcasts, we choose a broadcast f in which the number of overdominated branch vertices is a minimum and such that the statement of Corollary 6.1.13 applies. Regarding our counterexample, we consider the described $\alpha_{bn}(T)$-broadcast f with $\sigma(f) > n - b(T) + \rho(T)$. First we look at the degrees of the vertices in V_f^+ and show that each vertex in V_f^+ is a leaf. Then we use Corollary 6.1.13 to examine the possible ways in which the end-branch vertices are dominated. All possibilities lead to contradictions. Recall that every maximal bn-independent broadcast is dominating and since $b(T) = t \geq 3$, $B_{end}(T) \neq \emptyset$. Hence, we conclude that no such t exists and our statement is true.

If f has a broadcast vertex v which is not a leaf then either v is a branch vertex

(Case A) or $\deg(v) = 2$ (Case B).

Case A: There is a broadcast vertex b_0 such that $\deg(b_0) \geq 3$. By the choice of f, $f(b_0) = 1$. Note that b_0 is not adjacent to a leaf because no leaf hears a non-leaf in a maximal bn-broadcast. So either b_0 is adjacent to a vertex v such that $\deg(v) = 2$ or all neighbours of b_0 are branch vertices and $L(b_0) = \varnothing$. Assume the former, say $N(v) = \{b_0, b'\}$ and consider the two subtrees T_1, T_2 created by reconnecting v to each component of $T - v$ in the obvious manner, where T_1 is the subtree that contains b_0. Examples of possible T_1 and T_2 decompositions of the tree in Figure 6.12 are shown in Figures 6.13 and 6.15. For $i = 1, 2$, let $g_i = f \restriction T_i$ and note that each g_i is a bn-independent broadcast on T_i. If b_0 does not belong to R_T, then $|L_T(b_0)| \geq 2$, hence $|L_{T_1}(b_0)| \geq 3$ and b_0 does not belong to R_{T_1}. On the other hand, if b_0 does belong to R_T, then b_0 may or may not belong to R_{T_1}. Similarly, if b' does not belong to R_T (possibly b' is not even a branch vertex), then b' does not belong to R_{T_2}, and if b' does belong to R_T, then b' may or may not belong to R_{T_2}. Any other vertex of T_i that belongs to R_{T_i} also belongs to R_T. Therefore $\rho(T) \geq \rho(T_1) + \rho(T_2)$. Since $\deg_T(v) = 2$, $b(T) = b(T_1) + b(T_2)$. Since $\sigma(f) = \sigma(g_1) + \sigma(g_2)$ and, by the assumption on T, $\sigma(f) \geq n - b(T) + \rho(T) + 1$, we have

$$\begin{aligned}\sigma(g_1) + \sigma(g_2) &= \sigma(f) \geq n - b(T) + \rho(T) + 1 \\ &\geq |V(T_1)| + |V(T_2)| - 1 - b(T_1) - b(T_2) + \rho(T_1) + \rho(T_2) + 1 \\ &= |V(T_1)| - b(T_1) + \rho(T_1) + |V(T_2)| - b(T_2) + \rho(T_2). \qquad (6.3)\end{aligned}$$

But since $b(T_i) \leq b(T)$ and $|V(T_i)| < |V(T)|$, the choice of T implies that $\alpha_{bn}(T_i) \leq n - b(T_i) + \rho(T_i)$, for $i = 1, 2$. Hence by the pigeon hole principle, equation (6.3) implies that:

$$\sigma(g_i) = \alpha_{bn}(T_i) = |V(T_i)| - b(T_i) + \rho(T_i),\ i = 1, 2. \qquad (6.4)$$

Define the broadcast g_1' on T_1 by $g_1'(v) = 2,\ g_1'(b_0) = 0$ and $g_1'(x) = g_1(x)$ otherwise. Since $B_{g_1'}(v) = B_{g_1}(b_0) - \{v\}$, g_1' is a bn-independent broadcast. But $\sigma(g_1') > \sigma(g_1) = \alpha_{bn}(T_1)$, a contradiction. Examples of the g_1'-broadcasts are shown in Figures 6.14 and 6.16.

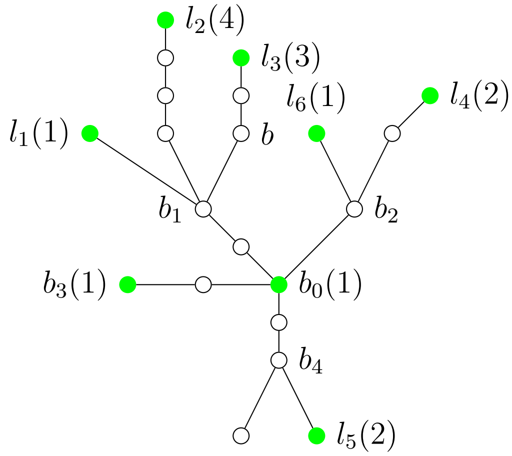


Figure 6.12: A maximal bn-broadcast with a broadcasting branch vertex.

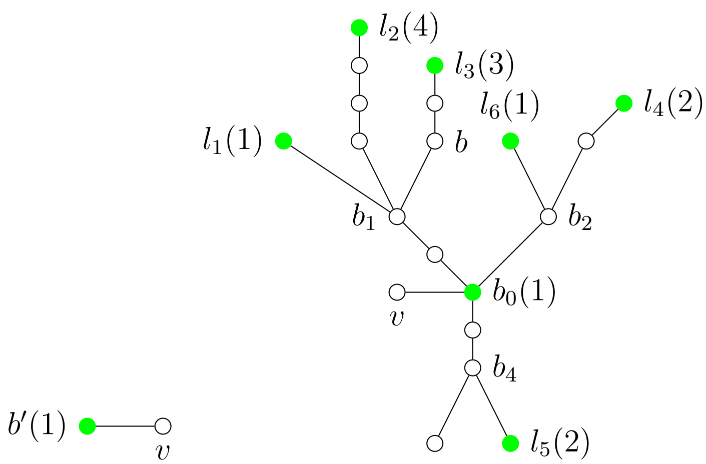


Figure 6.13: The broadcasting branch vertex has a neighbour v with $\deg(v) = 2$. The tree from Figure 6.12 is decomposed into two trees T_1 and T_2 as described in the proof.

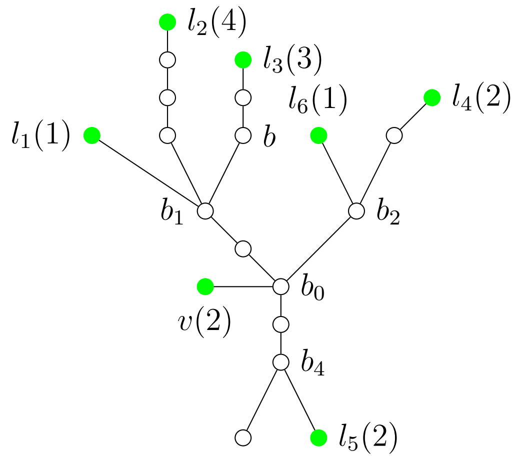


Figure 6.14: The broadcast on T_2 has been increased and is still bn-independent. Hence the original broadcast on T_2 had weight less than $\alpha_{bn}(T_2)$ which contradicts Equation 6.4.

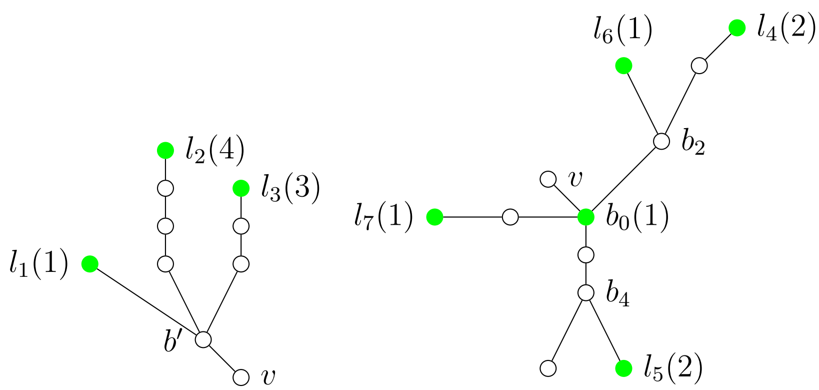


Figure 6.15: The tree from Figure 6.12 has a vertex v where $\deg(v) = 2$ and v is not on a $b_0 - x$ path for any $x \in L(b_0)$. Here T is decomposed into two trees T_1 and T_2 (left) as described in the proof.

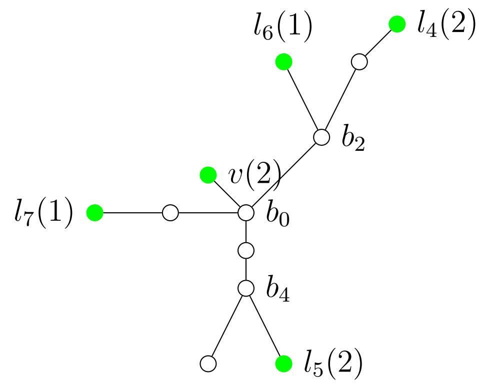


Figure 6.16: The tree T_2 from Figure 6.15 has a broadcast f which exceeds $\alpha(T_2)$, a contradiction showing that broadcasting branch vertices are not adjacent to vertices of degree 2.

We conclude that all neighbours of b_0 are branch vertices, thus $L(b_0) = \varnothing$ and $b_0 \in R_T$. Let $b_1, ..., b_k$, $k \geq 3$, be the neighbours of b_0 in T and let $T_1, ..., T_k$ be the subtrees of T obtained by reconnecting b_0 to each component of $T - b_0$ in the obvious manner. See Figure 6.17. For $i \in \{1, ..., k\}$, let $g_i = f \upharpoonright T_i$ and note that each g_i is a bn-independent broadcast on T_i. Hence $\sigma(g_i) \leq \alpha_{bn}(T_i)$. By the induction hypothesis, $\alpha_{bn}(T_i) \leq |V(T_i)| - b(T_i) + \rho(T_i)$ and:

$$\sum_{i=1}^{k} \sigma(g_i) \leq \sum_{i=1}^{k} \alpha_{bn}(T_i) \leq \sum_{i=1}^{k} [|V(T_i)| - b(T_i) + \rho(T_i)].$$

Hence: (6.5)

$$\sum_{i=1}^{k} \sigma(g_i) \leq \sum_{i=1}^{k} |V(T_i)| - \sum_{i=1}^{k} b(T_i) + \sum_{i=1}^{k} \rho(T_i). \tag{6.6}$$

Similar to the case of T_1 and T_2 above, $R_{T_i} \subseteq (R_T - \{b_0\}) \cap V(T_i)$, but $b_0 \notin R_{T_i}$ for each i. Hence $\rho(T) \geq \sum_{i=1}^{k} \rho(T_i) + 1$. By construction, $b(T) = \sum_{i=1}^{k} b(T_i) + 1$. By the assumption on T we now have:

$$\sum_{i=1}^{k} \sigma(g_i) - k + 1 = \sigma(f) > n - b(T) + \rho(T)$$

$$\geq \sum_{i=1}^{k} |V(T_i)| - k + 1 - \sum_{i=1}^{k} b(T_i) - 1 + \sum_{i=1}^{k} \rho(T_i) + 1.$$

Hence

$$\sum_{i=1}^{k} \sigma(g_i) > \sum_{i=1}^{k} |V(T_i)| - \sum_{i=1}^{k} b(T_i) + \sum_{i=1}^{k} \rho(T_i), \tag{6.7}$$

contradicting (6.6). We conclude that no branch vertex of our counterexample is a broadcast vertex.

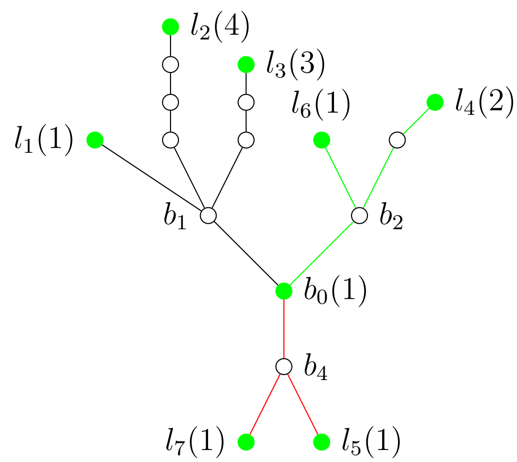


Figure 6.17: A maximal bn-broadcast with a broadcasting branch vertex b_0 and $N_f(b_0) \subset B(T)$. The trees T_1,T_2 and T_3, as described in the proof, are induced by the black, green and red edges, respectively. Notice that $\alpha_{bn}(T_1)+\alpha_{bn}(T_2)+\alpha_{bn}(T_3)-2 = 9+4+3-2 = 14 = \sigma(f) = n - b(T) + \rho(T) = 17 - 4 + 1$. Hence, f is not a counterexample to our upper bound.

Case B: There is a broadcast vertex v with $\deg(v) = 2$. By the choice of f, $f(v) = 1$. Say $N(v) = \{b_1, b_2\}$ and for $i = 1, 2$, let T_i be the subtree of T obtained by joining v to b_i in $T - v$. As before, let $g_i = f \restriction T_i$. Since $\deg(v) = 2$, $b(T) = b(T_1) + b(T_2)$ and $\rho(T) \geq \rho(T_1) + \rho(T_2)$. Hence, by the induction hypothesis,

$$\begin{aligned}\sigma(f) = \sigma(g_1) + \sigma(g_2) - 1 &\leq |V(T_1)| + |V(T_2)| - b(T_1) - b(T_2) + \rho(T_1) + \rho(T_2) - 1 \\ &\leq |V(T)| - b(T) + \rho(T),\end{aligned}$$

contradicting the choice of f.

Therefore in our counterexample,

$$\textbf{only leaves of } T \textbf{ are broadcast vertices,} \tag{6.8}$$

that is, $V_f^+ \subseteq L(T)$. To complete our proof, we show that no branch vertex is overdominated by exactly one (Case 1). We then consider the way the end-branch vertices are dominated. By Corollary 6.1.13 and Case 1, there are two remaining possible ways to dominate an end-branch vertex b_0. Either b_0 is dominated but not overdominated (Case 2) or it is overdominated by 3 or more (Case 3). Once these three cases are shown to be impossible, we see that there is no way to dominate an end-branch vertex. However, all maximal bn-independent broadcasts are dominating. Hence, no counterexample to our claim exists.

Case 1: Suppose that a branch vertex b_0 is overdominated by exactly 1. By statement (6.8), $V_f^+ \subseteq L(T)$. Hence, there is a leaf l such that $f(l) = d(b_0, l) + 1$ and by Corollary 6.1.13 (i), $L(b_0) = \{l\}$. Thus $b_0 \in R_T$. Let $b_1, ..., b_{k-1}$, $k \geq 3$, be the neighbours of b_0 that do not lie on the $b_0 - l$ path and note that $\{b_1, ..., b_{k-1}\} \subseteq B_f(l)$. Further note that the neighbours of b_0 may or may not be branch vertices. See Figure 6.18. For $i \in \{1, ..., k-1\}$, let T_i be the subtree of T obtained by joining b_0 to b_i in $T - b_0$. Let T_k be the tree induced by $N_f(l)$.

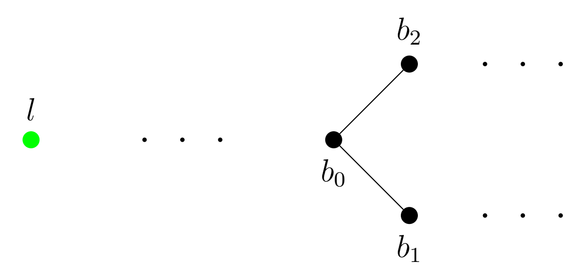


Figure 6.18: A tree with branch vertex b_0 which is overdominated by a leaf l by exactly 1. We note that $L(b_0) = \{l\}$, hence $b_0 \in R_T$ and b_0 has $t = k - 1 \geq 2$ neighbours not on a $b_0 - l$ path. None of the vertices shown except l are leaves hence the three sets of ellipsis represent the missing portions of the tree. Here $t = 2$. For the purpose of a contradiction, three new graphs will be formed from T. These subtrees are shown in Figure 6.19.

Define the broadcast g_i by $g_i = (f \restriction T_i - \{(b_0, 0)\}) \cup \{(b_0, 1)\}$ for $i \in \{1, ..., k-1\}$, and $g_k = f \restriction T_k$. Since $B_{g_i}(b_0) \subseteq B_f(l)$ for $i \in \{1, ..., k-1\}$ and $B_{g_k}(l) = B_f(l)$, each

g_i is a bn-independent broadcast on T_i.

Consider T_k and g_k. Define the broadcast h on T_k by $h(l) = g_k(l) - 1 = f(l) - 1$ and $h(b_i) = 1$ for $i \in \{1, ..., k-1\}$. Then h is bn-independent and $\alpha_{bn}(T_k) \geq \sigma(h) = \sigma(g_k) + k - 2$, hence by the induction hypothesis:

$$\sigma(g_k) \leq \alpha_{bn}(T_k) - (k-2) \leq |V(T_k)| + b(T_k) - \rho(T_k) - k + 2. \tag{6.9}$$

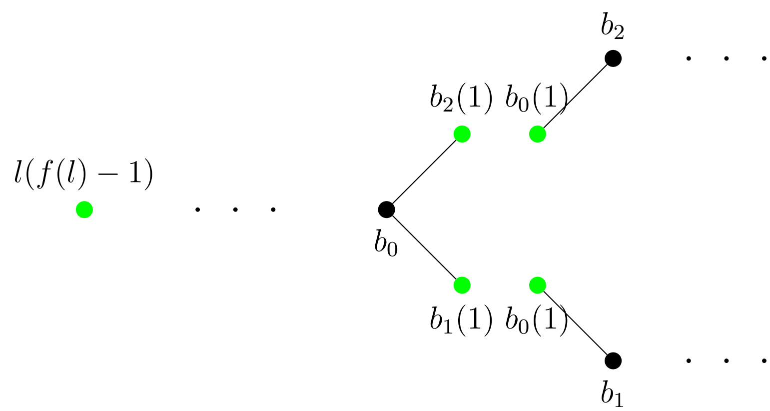


Figure 6.19: The subtrees of T from Figure 6.18: T_1 (top right), T_2 (bottom right) and T_3 (left). A new broadcast g with $g(b_0) = 1$ and $g(x) = f(x)$ otherwise is shown on T_1 and T_2. A new broadcast h is shown on T_3 with $h(l) = f(l) - 1$ and $h(b_i) = 1$ for $i = 1, ...k-1$. These subtrees are used to show that for our counterexample no leaf overdominates a branch by exactly one.

Note that $b(T) = \sum_{i=1}^{k} b(T_i)$. Since $\deg(b_0) \geq 3$ and each b_i, $i \in \{1, ..., k-1\}$, is a leaf in T_k, $b_0 \notin R_{T_k}$. Since $b_0 \in L(T_i)$ for $i \in \{1, ..., k-1\}$, $b_0 \notin R_{T_i}$ for $i \in \{1, ..., k-1\}$. And since $L_T(b_i) \subseteq L_{T_i}(b_i)$ for all $i = 1, ..., k-1$, each b_i that does not belong to R_T also does not belong to R_{T_i} for $i \in \{1, ..., k-1\}$. Hence, $\rho(T) \geq \sum_{i=1}^{k} \rho(T_i) + 1$. By the construction of the T_i we therefore have

$$|V(T)| = \sum_{i=1}^{k} |V(T_i)| - 2(k-1), \quad b(T) = \sum_{i=1}^{k} b(T_i) \;\text{ and }\; \rho(T) \geq \sum_{i=1}^{k} \rho(T_i) + 1.$$

Since for all $i \neq k$, $f(b_0) = 0$ while $g_i(b_0) = 1$ and $f(x) = g(x)$ otherwise,

$$\sigma(f) = \sum_{i=1}^{k} \sigma(g_i) - (k-1).$$

Therefore

$$\sum_{i=1}^{k} \sigma(g_i) - (k-1) = \sigma(f) > n - b(T) + \rho(T)$$
$$\geq \sum_{i=1}^{k} |V(T_i)| - 2(k-1) - \sum_{i=1}^{k} b(T_i) + \sum_{i=1}^{k} \rho(T_i) + 1$$

and so

$$\sum_{i=1}^{k} \sigma(g_i) > \sum_{i=1}^{k} |V(T_i)| - \sum_{i=1}^{k} b(T_i) + \sum_{i=1}^{k} \rho(T_i) - k + 2. \tag{6.10}$$

By the induction hypothesis $\sigma(g_i) \leq \alpha_{bn}(T_i) \leq |V(T_i)| + b(T_i) - \rho(T_i)$ for $i \in \{1, ..., k-1\}$, and by (6.9), $\sigma(g_k) \leq \alpha_{bn}(T_k) - k + 2$. Hence

$$\sum_{i=1}^{k} \sigma(g_i) \leq \sum_{i=1}^{k} |V(T_i)| - \sum_{i=1}^{k} b(T_i) + \sum_{i=1}^{k} \rho(T_i) - k + 2,$$

which contradicts (6.10). Hence no branch vertex is overdominated by exactly 1.

We now focus on the possible ways to dominate an end-branch vertex.

Case 2: Suppose there exists a vertex $b_0 \in B_{end}(T)$ that is dominated, but not overdominated, by a leaf l. Then $f(l) = d(b_0, l)$ and $b_0 \in B_f(l)$. By Lemma 6.1.10, $f(l') = d(b_0, l')$ for each $l' \in L(b_0)$. It is possible that the only leaves dominating b_0 are in $L(b_0)$. Let v_1 be the neighbour of b_0 that does not lie on a $b_0 - l'$ path for any $l' \in L(b_0)$. Since b_0 is not overdominated no vertex in $L(b_0)$ dominates v_1. See Figure 6.20. Let T_0 be the subtree of $T - b_0v_1$ that contains b_0 and let T_1 be the subtree obtained by joining b_0 to v_1 in the subtree of $T - b_0v_1$ that contains v_1. Note T_0 is a path or a generalized spider and by Corollary 2.3.7, $\alpha_{bn}(T_0) = |V(T_0)| - 1 \leq |V(T_0)| - b(T_0) + \rho(T_0)$. Also, $b(T_1) = b(T) - 1$ and, since b_0 is an endbranch vertex, $\rho(T_1) \leq \rho(T)$.

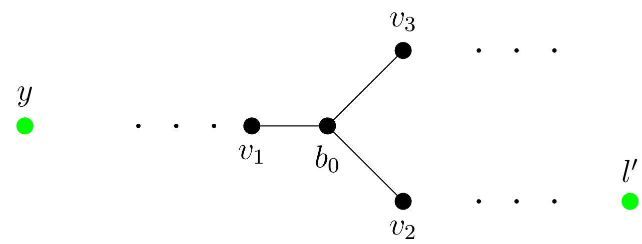


Figure 6.20: A tree with end-branch vertex b_0 which is dominated but not overdominated by a leaf. Here v_2 and v_3 are either leaves or have degree 2. All leaves in $L(b_0)$ dominate b_0. There is a leaf $y \notin L(b_0)$ such that either $f(y) = d(y, b_0)$ (Case 2.1) or $f(y) = d(y, v_1)$ (Case 2.2).

We consider two subcases, depending on whether some vertex in T_1 dominates b_0 or not.

Subcase 2.1: No vertex in T_1 dominates b_0. Let $f_1 = f \upharpoonright T_1$ and define g_1 by $g_1(b_0) = 1$ and $g_1(x) = f_1(x)$ otherwise. Let $g_0 = f \upharpoonright T_0$. We have

$$\begin{aligned}\sigma(g_0) + \sigma(g_1) - 1 = \sigma(f) &> n - b(T) + \rho(T)\\ &\geq |V(T_0)| + |V(T_1)| - 1 - (b(T_1) + 1) + \rho(T_1)\end{aligned}$$

and so

$$\sigma(g_0) + \sigma(g_1) > |V(T_0)| + |V(T_1)| - b(T_1) + \rho(T_1) - 1. \tag{6.11}$$

But by the induction hypothesis,

$$\sigma(g_1) \leq \alpha_{bn}(T_1) \leq |V(T_1)| - b(T_1) + \rho(T_1)$$

while $\sigma(g_0) = \alpha_{bn}(T_0) = |V(T_0)| - 1$, which gives

$$\sigma(g_0) + \sigma(g_1) \leq |V(T_0)| + |V(T_1)| - b(T_1) + \rho(T_1) - 1,$$

which contradicts (6.11).

Subcase 2.2: Some vertex y in T_1 dominates b_0. Since $V_f^+ \subseteq L(T)$ (Statement 6.8) y is a leaf. Since (by assumption) v_1 does not lie on a $b_0 - l$ path for any $l \in L(b_0)$, the $v_1 - y$ path contains a branch vertex; let c be the branch vertex on this path nearest to v_1. By our choice of f, y does not overdominate b_0. Hence $b_0 \in B_f(y)$ and $f(y) = d(b_0, y) = d(v_1, y) + 1$. As shown in Case 1, no leaf overdominates a branch vertex by exactly 1, hence v_1 is not a branch vertex and therefore $c \neq v_1$. By Corollary 6.1.13 , $f(y) \geq d(c, y) + 3$ and $B_f(y)$ consists of a single non-leaf vertex. But $b_0 \in B_f(y)$, hence $B_f(y) = \{b_0\}$. This, however, implies that y dominates all of T_1, otherwise $B_f(y)$ would contain another vertex. For $i = 0, 1$, define $g_i = f \restriction T_i$. We now have that $\sigma(g_1) = f(y) = e_{T_1}(y) \leq \text{diam}(T_1) \leq |V(T_1)| - b(T_1) - 1$ (by Proposition 6.1.14), hence

$$\sigma(g_0) + \sigma(g_1) \leq |V(T_0)| - 1 + |V(T_1)| - b(T_1) - 1 = |V(T_0)| + |V(T_1)| - b(T_1) - 2.$$

However, since $\rho(T_0) = 0$, $\rho(T_1) \leq \rho(T)$ and $b(T) = b(T_1) + 1$,

$$\begin{aligned} \sigma(g_0) + \sigma(g_1) = \sigma(f) &> n - b(T) + \rho(T) \\ &\geq |V(T_0)| + |V(T_1)| - 1 - (b(T_1) + 1) + \rho(T_1) \\ &= |V(T_0)| + |V(T_1)| - b(T_1) + \rho(T_1) - 2. \end{aligned}$$

This contradiction concludes the proof of Subcase 2.2 and thus the proof of Case 2. Hence no $v \in B_{end}(T)$ is dominated without being overdominated.

Each end-branch vertex that is dominated by a leaf l is overdominated by more than 1 by l. By Corollary 6.1.13, only one case remains to be considered.

Case 3: There exists a vertex $b_0 \in B_{end}(T)$ that is overdominated by a leaf l and $f(l) > d(l, b) + 2$. Let $L(b_0) = \{l_1, ..., l_k\}$. Since b_0 is an end-branch vertex, $k \geq 2$. There are two subcases: $l \in L(b_0)$ or $l \notin L(b_0)$. In either case, by Corollary 6.1.13, $B_f(l) = \{v\}$ for some vertex v. Let v' be the vertex on the $v - l$ path such that $d(v, v') = 2$. By Corollary 6.1.13, $\deg(v') = 2$. Form two subgraphs of T by reconnecting v' to each component of $T - \{v'\}$ in the obvious way. Let T_1 be the component which contains l and let T_2 be the subgraph which does not contain l. Let $f_i = f \restriction T_i$ for $i = 1, 2$. Notice that the $v' - v$ path is not f_2-dominated. Extend f_2 by creating a broadcast f_2' on T_2 with $f_2'(v') = 2$, $f_2'(x) = f_2(x)$ otherwise. Since f is a bn-independent broadcast and $N_{f_2'}(v') \subset N_f(l)$, f_2' is bn-independent and $\sigma(f_2) + 2 = \sigma(f_2') \leq \alpha_{bn}(T_2)$.

Suppose $l \in L(b_0)$. Without loss of generality, say $l = l_1$. Since $B_f(l_1) = \{v\}$, l_1 dominates T_1. (And since $f_1(l) = e_{T_1}(l) + 2$, f_1 is not a valid broadcast on T_1). Define a new broadcast f_1' on T_1 with $f_1'(y) = d(y, b_0)$ for all $y \in L(b_0)$, $f_1'(v') = d(v', b_0)$ and $f_1'(x) = 0$ otherwise. For all $1 \le i, j \le k$, $i \ne j$, $N_{f_1'}(l_i) \cap N_{f_1'}(l_j) = B_{f_1'}(l_i) \cap B_{f_1'}(l_j) = \{b_0\}$ and $N_{f_1'}(l_i) \cap N_{f_1'}(v') = B_{f_1'}(l_i) \cap B_{f_1'}(v') = \{b_0\}$. Hence, f_1' is a bn-independent broadcast (It is possible that f_1' is not maximal). Since $d(v, v') = 2$ and $\sigma(f_1) = f_1(l_1) = d(l_1, v') + 2 = f_1'(l_1) + f_1'(v') + 2$, $\sigma(f_1') \ge \sigma(f_1) - 2 + k - 1$. And since $k \ge 2$, $\sigma(f_1) \le \sigma(f_1') + 3 - k \le \sigma(f_1') + 1 \le \alpha_{bn}(T_1) + 1$. Further, both graphs have fewer vertices than T and $b(T_1), b(T_2) < b(T)$. Hence, by the induction hypothesis, $\sigma(f_1) \le V(T_1) - b(T_1) + \rho(T_1) + 1$ and $\sigma(f_2) + 2 = \sigma(f_2') \le V(T_2) - b(T_2) + \rho(T_2)$. Hence,

$$\sigma(f_1) + \sigma(f_2) \le V(T_1) + V(T_2) - b(T_1) - b(T_2) + \rho(T_1) + \rho(T_2) - 1.$$

Since $L(T) \subset L(T_1) \cup L(T_2)$, $\rho(T) \ge \rho(T_1) + \rho(T_2)$. And, since $\deg(v) = 2$, $B(T_1) + B(T_2) = B(T)$. Hence,

$$\sigma(f) = \sigma(f_1) + \sigma(f_2) \le V(T) + 1 - b(T) + \rho(T) - 1$$

contradicting our choice of f.

Suppose that $l \in L(b)$ where $b \ne b_0$. Since l overdominates b_0, Corollary 6.1.13 implies that l overdominates $L(b_0)$. In this case, make a new broadcast h_1 on T_1 with $h_1(l) = d(l, b)$, $h_1(l_1) = d(l_1, b)$, $h_1(v') = d(v', b)$ and $h(x) = 0$ otherwise. Note that

$$N_{h_2}(l) \cap N_{h_2}(l_1) = B_{h_2}(l) \cap B_{h_2}(l_1) = \{b\},$$

$$N_{h_2}(l) \cap N_{h_2}(v') = B_{h_2}(l) \cap B_{h_2}(v') = \{b\}$$

and

$$N_{h_2}(v') \cap N_{h_2}(l_1) = B_{h_2}(v') \cap B_{h_2}(l_1) = \{b\}.$$

Hence h_1 is a bn-independent broadcast and $\sigma(h_1) \le \alpha_{bn}(T_1)$. Since $b_0 \ne b$, $h_1(l_1) \ge 2$. Also, $h_1(l) + h_1(v') + 2 = f_1(l)$. Hence, $\sigma(h_1) \ge \sigma(f_1)$ and by the induction hypothesis,

$$\sigma(f_1) + \sigma(f_2) \le \sigma(h_1) + \sigma(f_2) \le V(T_1) + V(T_2) - b(T_1) - b(T_2) + \rho(T_1) + \rho(T_2) - 2.$$

As before, $\rho(T) \ge \rho(T_1) + \rho(T_2)$ and $B(T_1) + B(T_2) = B(T)$. Hence,

$$\sigma(f) = \sigma(f_1) + \sigma(f_2) \le V(T) + 1 - b(T) + \rho(T) - 2$$

contradicting our choice of f.

This final contradiction shows that the end-branch vertices of our counterexample are not dominated. But all α_{bn}-broadcasts are dominating. Hence no counterexample exists and the statement holds. □

There exist trees for which strict inequality holds in the bound in Theorem 6.1.16. Consider the tree T in Figure 6.21, for example. The vertices in R_T are indicated in red; $\rho(T) = 4$, $b(T) = 10$ and $|V(T)| = 22$, so the bound is $\alpha_{bn}(T) \leq 16$. However, if we broadcast with a weight of 1 from each blue or blue circled vertex we get a *bn*-independent broadcast with a weight of 15. By considering all possible *bn*-independent broadcasts where leaves only hear leaves, it can be shown that indeed $\alpha_{bn}(T) = 15$. The vertices in red make up R_T and the circled vertices form a maximum independent set of $G[R_T]$, the subgraph of T induced by R_T. This raises the following question.

Question 6.1.17. *Is it true that for any tree T of order n and $b(T) \geq 1$, $\alpha_{bn}(T) \leq n - b(T) + \alpha(G[R_T])$?*

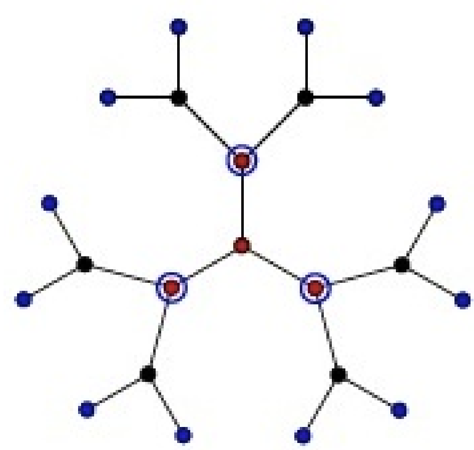


Figure 6.21: A tree T with $\alpha_{bn}(T) = 15 < n - b(T) + \rho(T) = 16$. While T falls below the bound of Theorem 6.1.16 it meets our conjectured bound $\alpha_{bn}(T) = n - b(T) + \alpha(G[R_T]) = 22 - 10 + 3 = 15$.

6.1.4 Lower bound for $\alpha_{bn}(T)$

A broadcast of strength $\text{diam}(G)$, from a peripheral vertex, is a *bn*-independent broadcast. Thus $\text{diam}(G)$ forms a lower bound for $\alpha_{bn}(G)$ for all graphs. For trees, equality holds for paths but for trees with branch vertices the inequality is strict. Let P be a diametrical path of such a tree. Then P has a branch vertex b. Let u be a neighbour of b not on P and let v and w be leaves of P. Define f by $f(u) = 1$, $f(v) = d(v, b)$, $f(w) = d(w, b)$ and $f(x) = 0$ otherwise. Then f is bn-independent and $\sigma(f) = d(v, w) + 1 = \text{diam}(G) + 1$. Hence $\alpha_{bn}(T) \geq \sigma(f) > \text{diam}(G)$. We prove a better lower bound for $\alpha_{bn}(T)$ in Theorem 6.1.18. For $i \in \{0, ..., b(T)\}$, define subsets of $B(T)$ by

$$\begin{aligned} B_i(T) &= \{b \in B(T) : |L(b)| = i\}, \text{ and} \\ B_{\geq i}(T) &= \{b \in B(T) : |L(b)| \geq i\}. \end{aligned}$$

Then $B_0 \cup B_1 \cup B_{\geq 2}$ is a partition of $B(T)$. We also partition the set $W(T)$ of *vertices of* T *with degree* 2 into two subsets, $W_e(T)$ for the *external vertices of degree* 2, and $W_i(T)$ for the *internal vertices of degree* 2, as follows:

$$\begin{aligned} W_e(T) &= \{u \in V(T) : \deg(u) = 2 \text{ and } u \text{ lies on an } l - b \text{ path} \\ &\qquad \text{for some } b \in B(T) \text{ and some } l \in L(b)\}, \text{ and} \\ W_i(T) &= W(T) - W_e(T). \end{aligned}$$

Note that $B_0(T) \cup B_1(T) = R_T$. For any branch vertex b, let T_b be the subtree of T induced by all the $b-l$ paths from b to leaves $l \in L(b)$. Then $T_b = K_1$ if $b \in B_0(T)$, T_b is a path of length $d(l, b)$ if $b \in B_1(T)$ and $L(b) = \{l\}$, T_b is a path of length $d(l_1, b) + d(l_2, b)$ if $L(b) = \{l_1, l_2\}$, and T_b is the generalized spider $S(d(l_1, b), , ..., d(l_k, b))$ if $L(b) = \{l_1, ..., l_k\}$, $k \geq 3$. Define $G_{int}(T)$, the *interior subgraph of* T, to be the subgraph of T induced by $B_0(T) \cup B_1(T) \cup W_i(T)$. Alternatively, $G_{int}(T)$ is the subgraph of T obtained by deleting the vertices of T_b for each $b \in B_{\geq 2}(T)$, and the $b - l$ path (except for b) if $b \in B_1(T)$ and $L(b) = \{l\}$.

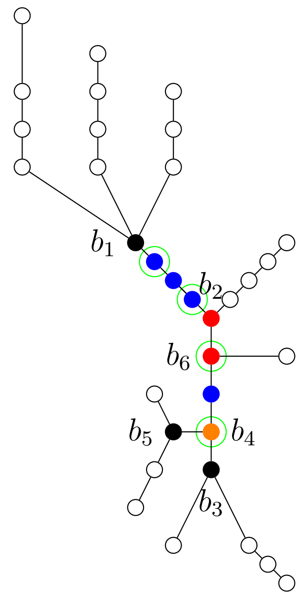


Figure 6.22: A tree T with the vertices of $W_i(T)$ coloured blue, $B_0(T)$ coloured orange, $B_1(T)$ coloured red, $B_{\geq 2}(T)$ coloured black, and $W_e(T)$ and the leaves uncoloured. The subgraph $G_{int}(T)$ is induced by the blue, red and orange vertices. A maximal independent set of $G_{int}(T)$ is given by the vertices which are circled in green.

Theorem 6.1.18. *If T is a tree of order n such that $b(T) \geq 1$, then*

$$\alpha_{bn}(T) \geq n - b(T) - |W_i(T)| + \alpha(G_{int}(T)).$$

Proof. It is sufficient to define a bn-independent broadcast f on T such that $\sigma(f) = n - b(T) - |W_i(T)| + \alpha(G_{int}(T))$. To do this, let X be any maximum independent set of $G_{int}(T)$ and let $Y = B_1(T) - X$. The definition of f below is illustrated in Figure 6.23.

(i) For each vertex $b \in B_{\geq 2}(T) \cup Y$ and each leaf $l \in L(b)$, let $f(l) = d(b, l)$. Then $\bigcup_{l \in L(b)} N_f(l) = V(T_b)$ and $\sum_{l \in L(b)} f(l) = |V(T_b)| - 1$.

(ii) For each vertex $b \in X \cap B_1(T)$ and the leaf $l \in L(b)$, let $f(l) = d(b, l) + 1$. Then $N_f(l) = V(T_b) \cup N(b)$ and $f(l) = |V(T_b)|$.

(iii) For each vertex $v \in X \cap (B_0(T) \cup W_i(T))$, let $f(v) = 1$.

(iv) Otherwise let $f(v) = 0$.

Then

$$\begin{aligned}\sigma(f) &= \sum_{b\in B_{\geq 2}(T)\cup Y} (|V(T_b)|-1) + \sum_{b\in X\cap B_1(T)} |V(T_b)| + |X\cap(B_0(T)\cup W_i(T))| \\ &= \sum_{b\in B_{\geq 1}(T)} (|V(T_b)|-1) + |X\cap B_1(T)| + |X\cap(B_0(T)\cup W_i(T))| \\ &= \sum_{b\in B_{\geq 1}(T)} (|V(T_b)|-1) + |X|.\end{aligned}$$

Since the expression $\sum_{b\in B_{\geq 1}(T)}(|V(T_b)|-1)$ counts all vertices in all subtrees T_b except the branch vertex itself, and $T_b = K_1$ if $b \in B_0(T)$,

$$\sum_{b\in B_{\geq 1}(T)} (|V(T_b)|-1) = n - b(T) - |W_i(T)|,$$

from which we obtain

$$\begin{aligned}\sigma(f) &= n - b(T) - |W_i(T)| + |X| \\ &= n - b(T) - |W_i(T)| + \alpha(G_{int}(T)).\end{aligned}$$

If f is not bn-independent then there are two vertices $u, v \in V_f^+$ such that, for some edge $xy \in E(T)$, $\{u, v\} \subseteq H_f(x) \cap H_f(y)$. By the construction of f, u satisfies (i),(ii) or (iii).

Case 1: Suppose u satisfies (i). Then $u \in L(b_i)$ and $f(u) = d(u, b_i)$. Hence $f(v) \geq d(v, b_i) + 1$. We show that this is not possible. Suppose $v \in L(b_j)$. If $b_j = b_i$ then $f(v) = d(v, b_j) = d(v, b_i) < d(v, b_i) + 1$. If $b_j \neq b_i$ then by construction of f, $f(v) = d(v, b_j) + 1$, hence $f(v) \leq d(v, b_j) + d(b_i, b_j) = d(b_i, v) < d(v, b_i) + 1$. Finally, suppose $v \in X \cap (B_0(T) \cup W_i(T))$. By construction of f, $v \neq b_i$ and $f(v) = 1$, hence $f(v) \leq d(v, b_i) < d(v, b_i) + 1$, a contradiction.

Case 2: Suppose u satisfies (ii). Then $u \in L(b_i)$, $b_i \in X$ and $f(u) = d(u, b_i) + 1$. Hence, $f(v) \geq d(v, b_i)$. Case 1 has already shown that v does not satisfy (i). Hence either $v \in L(b_j)$, $b_j \in X$ and $f(v) = d(v, b_j) + 1$, or $f(v) = 1$ and $v \in X \cap (B_0(T) \cup W_i(T))$. As $f(v) \geq d(v, b_i)$, either $b_j \in N(b_i)$, or $v \in N(b_i)$, which contradicts the fact that X is an independent set.

Case 3: The only remaining possibility is that $u, v \in X \cap (B_0(T) \cup W_i(T))$. In this case $f(u) = f(v) = 1$, hence $d(u, v) = 1$ which contradicts the fact that $u, v \in X$, an independent set.

This exhausts all possible cases, hence f is bn-independent and we have a lower bound for $\alpha_{bn}(T)$. $\square$

A tree with the broadcast described in the proof of Theorem 6.1.18 is shown in Figure 6.23.

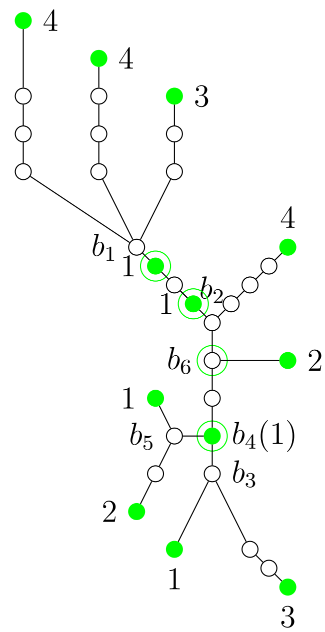


Figure 6.23: A tree T with a bn-independent broadcast f as described in the proof of Theorem 6.1.18. Note that $n - b(T) - |W_i(T)| + \alpha(G_{int}(T)) = 33 - 6 - 4 + 4 = \sigma(f) = 27 \leq \alpha_{bn}(T) \leq n - |b(T)| + \rho(T) = 33 - 6 + 3 = 30$. The vertices b_4 and b_6 form a maximal independent set on $V(G[R_T]) = \{b_4, b_6, b_2\}$. Hence $\sigma(f) \leq n - b(T) + \alpha(G[R_T]) = 33 - 6 + 2 = 29$ and our conjectured upper bound also holds.

We have therefore proved the main result of this dissertation, as stated below.

Theorem 6.1.19. *For any tree T,*

$$n - b(T) - W_i(T) + \alpha(G_{int}(T)) \leq \alpha_{bn}(T) \leq n - b(T) + \rho(T).$$

Proof. Apply Theorem 6.1.16 and Theorem 6.1.18. $\square$

Corollary 6.1.20. *If T is a tree of order n such that $b(T) \geq 1$ and $W_i(T) = \varnothing$, then*

$$n - b(T) + \alpha(G[R_T]) \leq \alpha_{bn}(T) \leq n - b(T) + \rho(T).$$

Proof. If $W_i(T) = \varnothing$ then $G_{int}(T)$ is the subgraph of T induced by $G[R_T]$ and the result follows from Theorems 6.1.19. $\square$

6.1.5 Exact formula for large classes of trees

If our conjectured upper bound holds then Corollary 6.1.20 gives us an exact formula for trees T with $b(T) \geq 1$ and $W_i(T) = \varnothing$, namely, $\alpha_{bn}(T) = n - b(T) + \alpha(G[R_T])$. Next, we use Corollary 6.1.20 to determine $\alpha_{bn}(T)$ exactly for subclasses of the above mentioned of trees:

Corollary 6.1.21. *If T is a tree of order n such that $b(T) \geq 1$, $W_i(T) = \varnothing$, and $R_T = \varnothing$ or $G[R_T] = \overline{K_s}$ for some integer $s \geq 1$, then $\alpha_{bn}(T) = n - b(T) + \rho(T)$.*

Proof. If $R_T = \varnothing$ or $G[R_T] = \overline{K_s}$ for some integer s, then $\alpha(G[R_T]) = |R_T| = \rho(T)$ and the result follows from Corollary 6.1.20. $\square$

We now determine a formula for $\alpha_{bn}(T)$ where T is a tree with exactly two branch vertices. We use this result to show that there exist trees such that α_{bn} lies strictly between the upper bound in Theorem 6.1.16 and the lower bound in Theorem 6.1.18. Of course, none of these trees have $W_i(T) = \varnothing$, and all support a positive answer to Question 6.1.17. Recall that loss(b) is defined on page 89.

Theorem 6.1.22. *Let T be a tree with order n and branch number $b(T) = 2$. Let b_1, b_2 be the two branch vertices. Then $\alpha_{bn}(T) = n - 1 - \min\{\lceil \frac{d(b_1,b_2)}{2} \rceil, \text{loss}(b_1), \text{loss}(b_2)\}$.*

Proof. Let f be an α_{bn}-broadcast on T in which each non-leaf vertex is either dominated by a leaf, or it is dominated by a broadcast of strength 1. The broadcast f exists by Theorem 2.3.14. Of all such broadcasts, let f have the minimum number of overdominated branch vertices. By Lemma 2.3.12, each leaf is dominated by a leaf. Further from all such broadcasts choose a broadcast f such that Corollary 6.1.13 applies.

Label the path joining the two branch vertices P: $b_1 = v_0, v_1, ... v_k = b_2$. Notice that $d(b_1, b_2) = k$. In an α_{bn}-broadcast all vertices are dominated. If $k = 1$ then there are no internal vertices on P and by Corollary 6.1.21, $\alpha_{bn}(T) = n - b(T) + |R_T| = n - 2 = n - 1 - \min\{\lceil \frac{d(b_1,b_2)}{2} \rceil, \text{loss}(b_1), \text{loss}(b_2)\}$. If $k \neq 1$ then we examine the way in which the interior vertices of P are dominated. Let $P_{int} : v_1, ..., v_{k-1}$ be the path induced by the internal vertices of P. There are two possibilities for dominating P_{int}.

Case 1: Suppose that no vertex in P_{int} is dominated by a leaf in f. Since f has maximum weight, by Lemma 6.1.10, for every leaf $x \in L(b_1)$ we have $f(x) = d(x, b_1)$. Similarly for every leaf $y \in L(b_2)$, we have $f(y) = d(y, b_2)$. For $i \in \{1, 2\}$, let T_i be the subtree induced by all vertices on $b_i - l$ paths for all $l \in L(b_i)$. Notice that $f(T_i) = V(T_i) - 1$ for $i = 1, 2$. Since T_1 and T_2 are already dominated, the vertices on P_{int} are dominated by the vertices on P_{int}. Hence a maximum broadcast coincides with a maximum dominating set of P_{int} which has size $\alpha(P_{int}) = \lceil \frac{k-1}{2} \rceil$. Hence the weight of f is $\sigma(f) =$V(T$_1$)$+ V(T_2) - 2 + \lceil \frac{k-1}{2} \rceil = n - 2 - \lfloor \frac{k-1}{2} \rfloor = n - 1 - \lceil \frac{d(b_1,b_2)}{2} \rceil$.
Case 2: A leaf x dominates some or all of the vertices on P_{int} and thus overdominates a branch. Assume without loss of generality that $x \in L(b_1)$. Since $|L(b_1)| \geq 2$, statement (i) of Corollary 6.1.13 does not apply to x. Thus statement (ii) of Corollary 6.1.13 applies and $B_f(x) = \{v\}$ where v is a vertex on P_{int}. Hence, x overdominates $L(b_1) - \{x\}$.

Let v_j be the vertex on the path P such that $B_f(x) = \{v_j\}$. First, we show that $j \neq k - 1$. If $y \in L(b_2)$ then, since $|L(b_2)| \geq 2$, by Corollary 6.1.13, y does not overdominate b_2 by exactly 1. Hence, if $j = k - 1$ then $\{v_{k-1}\} = PB_f(x) = B_f(x)$ and f is not maximal. If $j < k - 1$ then by our choice of f and because f is dominating, v_{j+1} is either dominated by $t \in \{v_{j+1}, v_{j+2}\}$ or by a leaf $l' \in L(b_2)$. In the first case, the broadcast g with $g(x) = d(x, t) + 1$, $g(t) = 0$ and $g(x) = f(x)$ otherwise has greater weight than f, and $N_g(x) \subset N_f(t) \cup N_f(x)$. Hence, g is bn-independent and violates the maximality of f. In the second case, Corollary 6.1.13 (ii) applies to l', hence l' overdominates $L(b_2) - \{l'\}$ and $f(l') = d(l', v_j)$, $f(x) = d(x, v_j)$ and $f(u) = 0$ otherwise. Make a new broadcast g_1 with $g_1(y) = d(y, b_2)$ for all $y \in L(b_2)$ and $g_1(x) = d(x, b_2)$ and $g_1(x) = 0$ otherwise. For all $\{w, w'\} \in V_g^+$ such that $w \neq w'$, $N_{g_1}(w) \cap N_{g_1}(w') = \{b_2\} = B_{g_1}(w) \cap B_{g_1}(w')$, hence g_1 is bn-independent. Since $g_1(x) + g_1(l) = f(x) + f(y)$ and $|L(b_2)| \geq 2$, $\sigma(g_1) > \sigma(f)$ and g_1 violates the maximality of f. Hence if f overdominates b_1 then $f(x) = d(x, b_2)$. If $d(x, b_1) < \max(b_1)$ then let x' be a leaf such that $d(x', b_1) = \max(b_1)$. Create a new broadcast g_1' with $g_1'(x) = 0$, $g_1'(x') = d(x', b_2)$ and $g_1'(u) = g_1(u)$ otherwise. Since $d(x, b_1) < d(x', b_1)$, $\sigma(g_1') > \sigma(g_1)$. Notice that $N_{g_1'}(x') = N_{g_1}(x)$ and all other boundaries are unchanged. Hence g_1' is bn-independent and contradicts the maximality of g_1. Hence we assume that $d(x, b_1) = \max(b_1)$. If $\text{loss}(b_1) > \lceil \frac{d(b_1,b_2)}{2} \rceil$ then the broadcast from Case 1 has greater weight than f. Hence we only choose this broadcast if $\text{loss}(b_1) \leq \lceil \frac{d(b_1,b_2)}{2} \rceil$. And finally, by symmetry, we have $\alpha_{bn}(T) = n - 1 - \min\{\text{loss}(b_1), \text{loss}(b_2)\}$ where $\min\{\text{loss}(b_1), \text{loss}(b_2)\} \leq \lceil \frac{d(b_1,b_2)}{2} \rceil$. These are the only cases, thus our result holds. □

Theorem 6.1.22 allows us to determine a special class of trees T for which $\alpha_{bn}(T)$ lies strictly in between our upper and lower bounds.

Corollary 6.1.23. *Let T be a tree with branch number $b(T) = 2$. Let b_1, b_2 be the two branch vertices. Let $2 \leq \text{loss}(b_1) \leq \text{loss}(b_2) < \lceil \frac{d(b_1,b_2)}{2} \rceil$. Then*

$$n - b(T) - |W_i(T)| + \alpha(G_{int}(T)) < \alpha_{bn}(T) < n - b(T) - \rho(T).$$

Proof. For a tree T with branch number $b(T) = 2$, $\rho(T) = 0$. Hence the upper bound becomes $n-2$. Also $G_{int}(T)$ and $W_i(T)$ are both equivalent to P_{int}, as described above in the proof of Theorem 6.1.22. Hence $n - b(T) - |W_i(T)| + \alpha(G_{int}(T)) = n - 2 - (k - 1) + \lceil \frac{k-1}{2} \rceil = n - 2 - \lfloor \frac{k-1}{2} \rfloor = n - 1 - \lceil \frac{d(b_1,b_2)}{2} \rceil$. By Theorem 6.1.22 and our choice of T, $n - 1 - \lceil \frac{d(b_1,b_2)}{2} \rceil < n - 1 - \text{loss}(b_1) = \alpha_{bn}(T) < n - 3 < n - b(T) - \rho(T) = n - 2$. $\square$

Figure 6.24, shows an example of a tree for which $\alpha_{bn}(T)$ lies strictly between the bounds of Theorem 6.1.22 and Theorem 6.1.18.

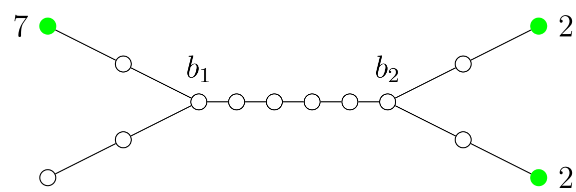


Figure 6.24: A tree T with $b(T) = 2$, $|R_T| = 0$, $|W_i(T)| = 4$ and $\alpha(G_{int}(T)) = 2$. Hence $10 = n - b(T) - |W_i(T)| + \alpha(G_{int}(T)) < \sigma(f) = 11 = \alpha_{bn}(T) \leq n - b(T) + \rho(T) = 12$.

Remark 6.1.24. *For the trees of Corollary 6.1.23, $b_1, b_2 \in B_{\geq 2}(T)$, hence $R_T = \emptyset$ and $\alpha_{bn}(T) \leq n - b(T) + \alpha(G[R_T]) = n - b(T)$. So for these trees, the answer to Question 6.1.17 is yes and the inequality is strict.*

For another example of a subclass of trees which meet our lower, upper and conjectured upper bounds, we consider a subclass of the caterpillars studied by Ahmane et al in [2]. Recall that a caterpillar is a tree with a diametrical path $P : v_1, v_2, ...v_m$ such that every vertex is either on the path or adjacent to a vertex v_i where $2 \leq i \leq m-1$. A trunk is a vertex of degree 2 on the spine of the caterpillar and for the all spine vertices v_i, where $2 \leq i \leq m-1$, $\lambda_i = \deg(v_i) - 2$. Ahmane et al calculate $\alpha_h(T)$

for all caterpillars T with no two adjacent branch vertices. Corollary 2.3.28 adapted from Ahmane et al. [2] gives one of their results and states: For T, a caterpillar with $\lambda_i \geq 3$ or $\lambda_i = 0$ for all $2 \leq i \leq |\operatorname{diam}(T)| - 1$ and with $\tau(T)$ trunks, no two adjacent,

$$\alpha_h(T) = \sum_{i=2}^{|\operatorname{diam}(T)|-1} \lambda_i + \tau(T) + 2.$$

For such a caterpillar with the additional condition that $\lambda_2, \lambda_{m-1} \geq 3$, a broadcast f which meets this bound and has $V_f^+ = V_f^1$ where all leaves and all trunk vertices are broadcasting with a strength of 1 and none of the branch vertices are broadcasting. Hence $\alpha_h(T) = n - b(T)$. Recall that an h-independent broadcast f with $V_f^+ = V_f^1$ is also bn-independent. Also, recall that $\alpha_{bn}(G) \leq \alpha_h(G)$. Hence, for these caterpillars, $\alpha_{bn}(T) = n - b(T)$. Since, for all $2 \leq i \leq \operatorname{diam}(T) - 1$, either $\lambda_i \geq 3$ or $\deg(v_i) = 2$, every branch vertex has at least 3 leaves. Hence $\rho(T) = 0$ and $R_T = \emptyset$. Also $V(G_{int}[T]) = W_{int}(T) = \{v : v \text{ is a trunk of } T\}$ and since no two trunks are adjacent $|W_{int}(T)| = \alpha(G_{int}[T])$. Hence $\alpha_{bn}(T) = n - b(T) = n - b(T) + \rho(T) = n - b(T) - |W_i(T)| + \alpha(G_{int}[T]) = n - b(T) + \alpha(R_T)$. In this specific case, it is also true that $\alpha_h(T) = \alpha_{bn}(T) = \alpha(T)$.

This result can be generalized to give an exact formula for $\alpha_{bn}(T)$ for another category of trees:

Corollary 6.1.25. *Let T be a tree such that $\rho(T) = 0$ and $|W_i(T)| = \alpha(G_{int}(T))$. Then*

$$\alpha_{bn}(T) = n - b(T).$$

Proof. From Theorems 6.1.16 and 6.1.18, $n - b(T) - |W_i(T)| + \alpha(G_{int}(T)) \leq \alpha_{bn}(T) \leq n - b(T) + \rho(T)$. And since both bounds equal $n - b(T)$ the result follows. □

Any tree in which $\rho(T) = 0$ and $G[W_i(T)] = \overline{K_s}$ for $s \geq 0$ will satisfy the conditions of Corollary 6.1.25. Notice, the trees in Corollary 6.1.25 also meet our conjectured upper bound.

We give an example of how Theorem 6.1.16 can be used to write an algorithm to output an α_{bn}-broadcast for all trees for which $B(T)$ is isomorphic to a particular tree. Algorithm 1 outputs an α_{bn}-broadcast for a trees with $B(T) \cong K_{1,k}$, $k \geq 2$. Note that sometimes the algorithm outputs the broadcast described in Theorem 6.1.18 and thus meets the lower bound. However, this is not always the case and for many trees, Algorithm 1 shows that $\alpha_{bn}(T)$ lies strictly between the two bounds.

For a tree with $B(T) \cong K_{1,k}$, $k \geq 2$, let b_0 be the central branch vertex of $B(T)$ and $b_1, ...b_k$ be the end-branch vertices of $B(T)$. Let T_i be the subtree induced by all vertices on a $x - b_i$ path where $x \in L(b_i)$. Sometimes it is advantageous for $l \in L(b_j)$, $b_j \neq b_0$ to overdominate b_0 and all but one of the other end-branches and the associated T_i's. For example, suppose there is an end-branch vertex b_1 with relatively large loss(b_1) and large $d(b_1, b_0)$. If a leaf in $L(b_1)$ dominates with strength $d(l, b_1) + d(b_1, b_0)$, to take advantage of the large $d(b_0, b_1)$, then it must dominate all other leaves of $L(b_1)$. Hence, the broadcast loses out on the large loss(b_1). In this case, the α_{bn}-broadcast might use a leaf from another branch vertex b_j to dominate $T - T_1$ with a strength of $\max(b_j)+d(b_j, b_0)+d(b_0, b_1)$ and let all leaves of b_1 dominate b_1. This will give a broadcast of weight $\sigma(f) = V(T_1) - 1 + \max\{d(x, b_1) : x \in L(T) - L(T_1)\}$. See Figure 6.25 for an example.

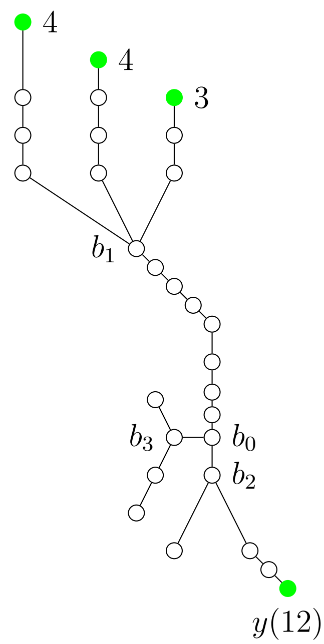


Figure 6.25: A tree T and broadcast f with $\sigma(f) = V(T_1) - 1 + \max\{d(x, b_1) : x \in L(T) - L(T_1)\}$. Here b_1 has relatively large loss$(b_1) = 7$ and large $d(b_1, b_0) = 9$. Hence, to get the maximum weight bn-independent broadcast $y \in L(b_2)$ overdominates $T - T_1$.

An Algorithm for producing an α_{bn}-broadcast on a tree with $B(T) \cong K_{1,k}$. The following data specifies the tree and is required for the algorithm:

1 Let k be the number of non-central branch vertices.

2 Let b_0 be the central branch.

3 Label the k non-central branch vertices $b_1, b_2...b_k$ such that $d(b_1, b_0) \geq d(b_2, b_0) \geq ... \geq d(b_k, b_0)$.

4 Input $d(b_0, b_0), d(b_1, b_0), d(b_2, b_0), ..., d(b_k, b_0)$.

5 Label the vertices of each $b_i - b_0$ path $b_i = v_{i,0}, ..., v_{i,d(b_0,b_i)} = b_0$.

6 For $0 \leq i \leq k$, label the vertices of $L(b_i) : x_{i1}, ..., x_{i|L(b_i)|}$ such that $d(x_{i1}, b_i) \geq ... \geq d(x_{i|L(b_i)|}, b_i)$ and let $l_{i,j} = d(x_{i,j}, b_i)$.

7 Let $l = \max\{|L(b_i)| : i = 0, ..., k\}$.

8 If $|L(b_i)| < l$ then let $l_{i,j} = 0$ for all j such that $|L(b_i)| < j \leq l$ for all $i = 0, 1, ..., k$.

The input data in matrix form is

$$\begin{bmatrix} d(b_0, b_0) \\ \vdots \\ d(b_0, b_k) \end{bmatrix} \quad \begin{bmatrix} l_{0,1} & \dots & l_{0,l} \\ \vdots & & \vdots \\ l_{k,1} & \dots & l_{k,l} \end{bmatrix}$$

where the entries of the leaf matrix are increasing in value along the rows except in the case of $l_{i,j} = 0$. And the entries of the branch matrix increase moving down the column.

Begin Algorithm 1

Input Data: The $d(b_0, b_i)$ and $l_{i,j}$ matrices, k, l and $|L(b_0)|$

Procedure: An α_{bn}-broadcast on a tree with $B(T) \cong K_{1,k}$

BROADCAST="b_0 is overdominated by more than 2"

CHOICE=0 % It is possible for b_0 to be broadcasting or overdominated by 1 only if CHOICE=0 or 1.

COUNTER =$|L(b_0)|$ % If $|L(b_0)| > 1$ then b_0 will not broadcasting or be overdominated by exactly 1.

EQUAL=$\emptyset$ % If $\max(b_i) + d(b_i, b_0) = \text{sum}(b_i) + \lceil \frac{d(b_i,b_0)}{2} \rceil$ then it is possible that x_{i1} could overdominate b_0 by exactly 1.

% The following section determines x_i which is the maximal weight for covering the $L(b_i) - b_i$ paths and the $b_i - b_0$ path given that b_0 is neither broadcasting nor overdominated. The COUNTER is used to determine whether or not the conditions for "overdominating b_0 by 1 " are met. If $|L(b_0)| > 1$ then b_0 is not overdominated by exactly 1. Also, every time the broadcast to cover the $L(b_i) - b_i - b_0$ paths requires dominating from l_{i1} with a strength of $l_{i1} + d(b_i, b_0)$ the counter is incremented by 1.

If $|L(b_0)| > 0$

  Then let $x_0 = \sum_{j=1}^{l} l_{0j}$ **and** $label(x_0) = 1$

  Else $label(x_0) = 2$ % b_0 has no leaves.

For $1 \leq i \leq k$

  If $l_{i,1} + d(b_i, b_0) > \sum_{j=1}^{l} l_{ij} + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$

  Then $x_i = l_{i,1} + d(b_1, b_0)$ and $label(x_i) = 1.$

  Else $x_i = \sum_{j=1}^{l} l_{ij} + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ and $label(x_i) = 0.$

  If CHOICE ≤ 1

    Then

      If $Label(x_i) = 1$

      Then Increase COUNTER by 1

        If COUNTER > 1

        Then CHOICE=2 % b_0 will not be overdominated by 1.

      Else

        If $d(b_0, b_i) = 0 \pmod 2$

        Then Increase CHOICE by 1

        If CHOICE=1

          Then If $l_{i,1} + d(b_i, b_0) = \sum_{j=1}^{l} l_{ij} + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$

          Then EQUAL $= x_{i1}$

          Else CHOICE=2

% In our base broadcast, the vertices on the $b_0 - b_i$ path cover themselves and b_0's neighbour on the path is broadcasting. Hence, it will not be advantageous to overdominate b_0 by any vertex other that x_{i1}.

Increment i

ENDFor

% Next we determine whether or not b_0 will be overdominated by at least 3. Suppose a leaf x_{ti} overdominates b_0 by 3 or more. To maximize the broadcast, $i = 1$. Corollary 6.1.13 and Lemma 6.1.11 imply that exactly one branch, say b_n is not overdominated by x_{t1}. What we can gain with this choice of broadcast is the ability to capture both loss(b_n) and $d(b_n, b_0)$ in our broadcast weight. What we lose from the weight is all x_i for $0 \le i \le k$, $i \ne n, t$ and loss(b_t). Recall that our labeling choice guarantees that $d(b_1, b_0) \ge d(b_2, b_0) \ge d(b_i, b_0)$ for all $0 \le i \le k$, $i \ne 1, 2$. Hence, if f_4 is greater than f_1, f_2, and f_3 then either x_{11} overdominates b_0 and $B(x_{11}) = \{b_2\}$ or x_{t1}, $t \ne 1$, overdominates b_0 and $B(l_{t1}) = \{b_1\}$. Below x_{t1} is the leaf that overdominates b_0 and $B(y) = b_n$.

Let $y = \max\{l_{j,1} + d(b_0, b_j)\}$ for $0 \le j \le k$, $j \ne 1$
Let $t =$ the smallest j such that $y = l_{j,1} + d(b_0, b_j)$.
If $\min\{\lceil \frac{d(b_1,b_0)}{2} \rceil, \sum_{j=2}^{l} l_{1,j}\} > \sum_{i \ne 1, i=0}^{k} (x_i) - l_{t,1} - d(b_t, b_0)$
Then $b_n = b_1$.
Else

  If $\min\{\lceil \frac{d(b_2,b_0)}{2} \rceil, \sum_{j=2}^{l} l_{2,j}\} > \sum_{i \ne 2, i=0}^{k} (x_i) - l_{1,1} - d(b_1, b_0)$
  then $b_n = b_2$, $y = l_{1,1} + d(b_0, b_1)$ and $t = 1$.
  else set BROADCAST= ATMOST 1.

If BROADCAST="b_0 is overdominated by more than 2",
Then output $f_4(x)$ with the assigned values for n and t and EXIT.

$$f_4(x) = \begin{cases} d(x, b_n) & : \text{ if } x = x_{n,i} \text{ for } i = 1..., |L(b_n)| \\ d(x, b_n) & : \text{ if } x = x_{t,1} \\ 0 & : \text{otherwise.} \end{cases}$$

Else % b_0 will be dominated by at most 1. Next we make the base case for ATMOST 1.

$$f_1(x) = \begin{cases} d(x, b_0) & : \text{ if } label(x_i) = 1 \text{ and } x = x_{i,1} \\ d(x, b_j) & : \text{ if } label(x_i) = 0 \text{ and } x = x_{i,j}, \text{ for } 1 \leq j \leq |L(b_j)|, \\ 1 & : \text{ if } label(x_i) = 0 \text{ and } x = v_{i,j} \text{ and } j \equiv 1 \pmod 2 \\ & \text{ for } 1 \leq j \leq |L(b_j)|, \\ 0 & : \text{otherwise.} \end{cases}$$

If CHOICE=2

Then output f_1. % $f_1(b_0) = 0$ and b_0 is not overdominated.

Else %(CHOICE≤ 1)

If CHOICE=1

Then If COUNTER $= 0$

Then let label(EQUAL) $= 1$

Else Do nothing. % COUNTER $= 0$ and there will be exactly one x_i with $label(x_i) = 1$.

then Define f_3

$$f_3(x) = \begin{cases} f_1(x) + 1 & : \text{If } label(x_i) = 1 \text{ and } x = x_{i1} \\ f_1(x) & : \text{otherwise.} \end{cases}$$

Output $f_3(x)$ and Print "b_0 is overdominated by 1." and EXIT.

Else (CHOICE =0) % In this case, COUNTER$= 0$, $L(b_0) = \emptyset$ and for all $1 \leq i \leq k$, $x_i = \sum_{j=1}^{l} l_{ij} + \lfloor \frac{d(b_i, b_0)}{2} \rfloor$ where $d(b_0, b_i) = 1 \pmod 2$. A bn-broadcast does not overdominate b_0 by 1 and f_1 does not dominate b_0. Hence, b_0 will dominate itself.

Define f_2

$$f_2(x) = \begin{cases} f_1(x) & : \text{ if } x \neq b_0 \\ 1 & : \text{ if } x = b_0. \end{cases}$$

Output $f_2(x)$ and Print "b_0 is broadcasting." and EXIT.

END Algorithm 1

The following lemma, Lemma 6.1.26, gives a relationship between an α_{bn}-broadcast f on a tree T and the broadcast $f \restriction T_i$ where T_i is a certain subtree of T. The lemma is used in Theorem 6.1.27 to approach the generation of an α_{bn}-broadcast on a tree T such that $B(T) \cong K_{1,k}$ on a case by case basis and to show the validity of Algorithm 1.

Lemma 6.1.26. *Let f be an α_{bn}-broadcast on a tree T. For any $v \in V_f^+$, let $T_i, ..., T_k$ be the k components of $T - N_f(v)$ and let v_i be the vertex in T_i such that v_i has a neighbour in $N_f(v)$. The broadcast $f_i = f \restriction T_i$ is a maximum weight bn-independent broadcast on T_i given the restriction that $f(v_i) \leq 1$ and v_i is not overdominated by more than 1. Conversely, if we require $f(v) = c > 0$ and f is a broadcast such that $f_i = f \restriction T_i$ is a maximum weight bn-independent broadcast on T_i under the restriction that $f(v_i) \leq 1$ and v_i is not overdominated by more than 1 then f is a maximum weight bn-independent broadcast on T under the restriction that $f(v) = c$.*

Proof. If $T - N_f(v) = \emptyset$ then the result is trivially true. Let T_i be a component of $T - N_f(v)$. Since f is bn-independent and v_i has a neighbour in $N_f(v)$, $f(v_i) \leq 1$. It is clear that T_i inherits bn-independence from T. Hence $f_i = f \restriction T_i$ is a bn-independent broadcast on T_i. Suppose that g was a bn-independent broadcast on T_i such that $g(v_i) \leq 1$ and v_i is not overdominated by more than 1 and $\sigma(g) > \sigma(f_i)$. Define a new broadcast on T with $g'(x) = g(x)$ if $x \in T_i$ and $g'(x) = f(x)$ otherwise. Since $g'(v_i) \leq 1$ and g does not overdominate v_i by more than 1, $|N_{g'}(T - T_i) \cap N_{g'}(T_i)| \leq 1$. Hence, g' is bn-independent. Notice that $\sigma(g') > \sigma(f)$ which contradicts the maximality of f. The result follows.

To see the converse suppose that there exists a broadcast f as described but f is not maximal bn-independent for the given restrictions. Hence, there is a broadcast g with $g(v) = c$ and $\sigma(g) > \sigma(f)$. Let $g'_i = g \restriction T_i$ and $f'_i = f \restriction T_i$ for $i = 1, ..., k$. Notice that $\sigma(g) = g(v) + \sum_{i=1}^{k} \sigma(g'_i) > \sigma(f) = f(v) + \sum_{i=1}^{k} \sigma(f'_i)$. By design, $g(v) = f(v)$, hence, by the pigeonhole principle, there exists j such that $1 \leq j \leq k$ and $\sigma(g'_j) > \sigma(f'_j)$, which contradicts the choice of f. $\square$

Theorem 6.1.27. *Let T be a tree such that $B(T) \cong K_{1,k}$ with $k \geq 3$. Algorithm 1 produces an α_{bn}-broadcast f on T.*

Proof. There are three steps to prove that the algorithm is correct. First, we limit the structure of our broadcasts so that any maximal bn-independent broadcast on any tree can be placed into one of four different categories. Second, we show that each of the four broadcasts from the algorithm, f_1, f_2, f_3, and f_4, correspond one to one to each of these four possible categories. Third, we show that Algorithm 1 outputs the correct category of α_{bn}-broadcast given the structure of the input tree.

Let T be a tree as described in the theorem statement. Of all α_{bn}-broadcasts on T such that $f(v) = 1$ or $\deg(v) = 1$ or both, for each $v \in V_f^+$, let f be one with the fewest overdominated branch vertices. Such a broadcast exists by Theorem 2.3.14. Further, ensure that f is a broadcast such that Corollary 6.1.13 applies. Let b_0 be the central vertex of the star $B(T)$ and label the remaining branch vertices $b_1, ..., b_k$. Lemma 6.1.11 and Corollary 6.1.13 imply that, for $0 \leq i \leq k$, any leaf $l \in L(b_i)$ that overdominates b_j must also overdominate all leaves in $L(b_j)$ and there must be at least one branch which is not overdominated by l.

For $1 \leq i \leq k$, label the vertices of each $b_i - b_0$ path $b_i = v_{i,0}, ..., v_{i,d(b_0,b_i)} = b_0$. For $0 \leq i \leq k$, label the vertices of $L(b_i) : x_{i1}, ..., x_{i|L(b_i)|}$ such that $d(x_{i1}, b_i) \geq ... \geq d(x_{i|L(b_i)|}, b_i)$ and let $l_{i,j} = d(x_{i,j}, b_i)$. The values $\max(b_i)$, $\text{sum}(b_i)$ and $\text{loss}(b_i)$ are as in definitions 6.1.7, 6.1.8 and 6.1.9. Next we define x_i, $label(x_i)$, $x_{i,j}$ and $l_{i,j}$. Notice these are the same variables as in Algorithm 1.

Let $x_0 = \sum_{j=1}^{|L(b_0)|} l_0 j$.

If $L(b_0) \neq \emptyset$ then let $label(x_0) = 1$ otherwise let $label(x_0) = 2$.

For $i = 1, ..., k$, if

$$\max(b_i) + d(b_i, b_0) > \text{sum}(b_i) + \lfloor \frac{d(b_i, b_0)}{2} \rfloor$$

then let

$$x_i = \max(b_i) + d(b_i, b_0)$$

and if

$$\max(b_i) + d(b_i, b_0) \leq \text{sum}(b_i) + \lfloor \frac{d(b_i, b_0)}{2} \rfloor$$

then let

$$x_i = \text{sum}(b_i) + \lfloor \frac{d(b_i, b_0)}{2} \rfloor.$$

For $i = 1, ..., k$:

If $x_i = \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ then let $label(x_i) = 0$.

If $x_i = \max(b_i) + d(b_i, b_0)$ then let $label(x_i) = 1$.

There are four possible cases for f. One case occurs when b_0 is broadcasting. If b_0 is not broadcasting then since maximal bn-broadcasts are dominating, by our choice of f, Corollary 6.1.12 gives us three other types of broadcasts: b_0 is dominated but not overdominated, b_0 is overdominated by 1 or b_0 is overdominated by more than 2.

In three of our cases, the broadcast overdominates b_0 by at most 1. Algorithm 1 first checks the conditions which determine whether the broadcast overdominates b_0 by at least 3. If the conditions are not met then it will assign BROADCAST=ATMOST 1 and it proceeds to choose between f_1, f_2 or f_3. Otherwise, the conditions are met and it assigns BROADCAST="b_0 is overdominated by more than 2" and outputs f_4. This latter broadcast is covered in Case 4 where we will also show that the algorithm chooses correctly.

Assuming that our α_{bn}-broadcast overdominates b_0 by at most 1, we choose between Cases 1, 2 and 3 corresponding to f_1, f_2 and f_3 respectively:

Case 1: Suppose that $f(b_0) = 0$ and b_0 is dominated but not overdominated. Let T_i be the component of $T - b_0$ which contains b_i for $i = 1, ..., k$ and let T_0 be the component(s) which contains no branches (i.e. the paths from b_0 to $L(b_0)$ minus b_0). Let T_i' be the subtree made by adjoining b_0 to T_i in the obvious manner. Let $f_i' = f \upharpoonright T_i'$. By Lemma 6.1.10 and the choice of f, to meet the condition of maximality, f is a broadcast such that $\sigma(f_i') = x_i$ for all $0 \leq i \leq k$. For an example see Figure 6.26. Hence, $\sigma(f) = \sum_{i=0}^{k} x_i$. This is the "base function for ATMOST 1 broadcasts" from Algorithm 1. Algorithm 1 outputs this function if the tree fails the conditions for all other cases. In this case, $f = f_1$ which is defined, for $0 \leq i \leq k$, as follows:

$$f_1(x) = \begin{cases} d(x, b_i) & \text{if } label(x_i) = 0 \text{ and } x \in L(b_i) \\ 1 & \text{if } label(x_i) = 0 \text{ and } x = v_{i,j} \text{ and } j \equiv 1 \pmod 2 \text{ for } 1 \leq j < d(b_0, b_i) \\ d(x, b_i) + d(b_i, b_0) & \text{if } label(x_i) = 1 \text{ and } x = x_{i,1}. \\ 0 & \text{otherwise.} \end{cases}$$

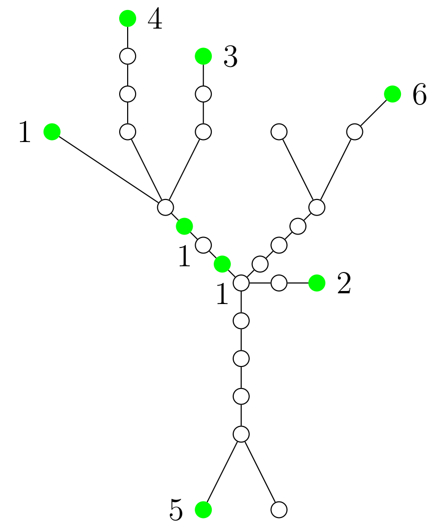


Figure 6.26: A tree with an α_{bn}-broadcast f_1 with $\sigma(f_1) = \sum_{i=0}^{k} x_i$.

Case 2: If b_0 is broadcasting then by the choice of f, $f(b_0) = 1$. Define a broadcast g, for $0 \leq i \leq k$, as follows:

$$g(x) = \begin{cases} 1 & \text{if } x = b_0 \\ d(x, b_i) & \text{if } label(x_i) = 0 \text{ and } x \in L(b_i) \\ 1 & \text{if } label(x_i) = 0 \text{ and } x = v_{i,j} \text{ where } j \equiv 1 \pmod 2, \\ & \text{for } 1 \leq j \leq d(b_0, b_i) - 2 \\ d(x, b_i) + d(b_i, b_0) - 1 & \text{if } label(x_i) = 1 \text{ and } x = x_{i,1} \\ 0 & \text{otherwise.} \end{cases}$$

If $\max(b_i) + d(b_i, b_0) > \text{sum}(b_i) + \lfloor \frac{d(b_i, b_0)}{2} \rfloor$, then

$$\max(b_i) + d(b_i, b_0) - 1 \geq \text{sum}(b_i) + \sum_{j=1}^{d(b_0, b_i)-2} g(v_{ij}).$$

Similarly, if $\max(b_i) + d(b_i, b_0) \leq \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ then,

$$\max(b_i) + d(b_i, b_0) - 1 \leq \text{sum}(b_i) + \sum_{j=1}^{d(b_0,b_i)-2} g(v_{ij}).$$

Hence, $g' = g \restriction T_i$ gives a maximum weight broadcast on T_i with the restriction that $g(b_0) = 1$. Thus, by Lemma 6.1.26, g is a maximum weight broadcast on T given that $g(b_0) = 1$.

Given that b_0 is overdominated by at most 1, there are two conditions under which no broadcast g with $g(b_0) = 1$ is maximum weight. First suppose there is a vertex v such that $v \in L(b_0)$ or, for some i, $v \in L(b_i)$ and $g(v) = d(v, b_i) + d(b_i, b_0) - 1$. Define a new broadcast g' with $g'(v) = g(v) + 2$, $g'(b_0) = 0$ and $g'(x) = g(x)$ otherwise. Since $N_{g'}(v) = N_g(v) \cup N_g(b)$, g' is bn-independent. Notice that $\sigma(g') > \sigma(g)$, which contradicts the maximality of g.

Hence b_0 will only be broadcasting in one of our α_{bn}-broadcasts if $L(b_0) = \emptyset$ and $x_i = \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ for all $1 \leq i \leq k$. Figure 6.27 shows a tree in which any broadcast f with $f(b_0) = 1$ is not maximum.

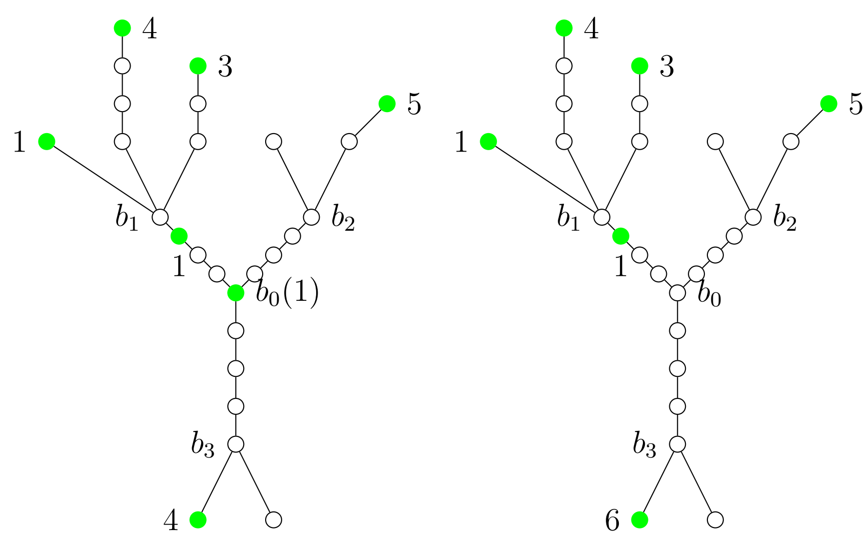


Figure 6.27: If $L(b_0) \neq \emptyset$ or if $x_i = \max b_i + d(b_i, b_0)$ then for any bn-independent broadcast with $f(b_0) = 1$ (left) we can define a new bn-independent broadcast g (right) with $g(b_0) = 0$ and $\sigma(g) > \sigma(f)$. Here $L(b_0) = \emptyset$, $x_1 = \text{sum}(b_1) + \lfloor \frac{d(b_0,b_i)}{2} \rfloor$, $x_2 = \max b_2 + d(b_2, b_0) = 6$ and $x_3 = \max b_3 + d(b_3, b_0) = 5$. The f broadcast on the left is not maximal. On the left $f(x_3) = 4$ and on the right $g(x_3) = 6$.

The second condition in which g is not maximal occurs when $d(b_i, b_0) \equiv 0 \pmod 2$ for at least two branch vertices, b_n, b_m. Since g is bn-independent, $g(v_{k,d(b_k,b_0)-1}) = 0$ for all $1 \le i \le k$. Define a new broadcast g' with $g'(v_{n,d(b_n,b_0)-1}) = g'(v_{m,d(b_m,b_0)-1}) = 1$ and $g'(b) = 0$. Here g' is bn-independent with greater weight than g and thus contradicts the maximality of g. Even if $d(b_i, b_0) \equiv 0 \pmod 2$ for a single branch vertex, say b_n, then define a new broadcast g' with $g'(v_{n,d(b_n,b_0)-1}) = 1$ and $g'(b) = 0$. The broadcast g' and g have the same weight. If this occurs, Algorithm 1 will output g' which is actually f_1 from Case 1. An example is shown in Figure 6.28.

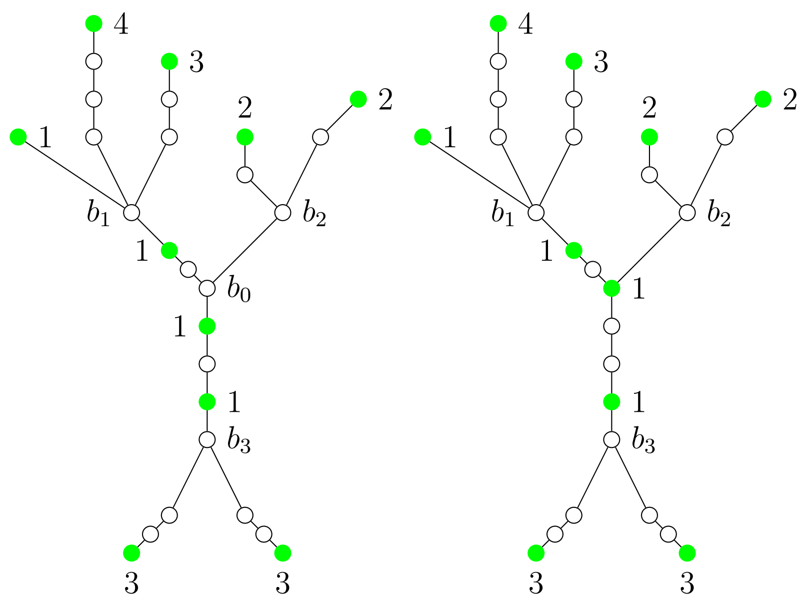


Figure 6.28: A tree with two different α_{bn}-broadcasts: f_1 (left) and f_2 (right). In this situation, $\sigma(f_1) = \sigma(f_2)$ and Algorithm 1 will choose f_1.

Hence to achieve an α_{bn}-broadcast, we only need to broadcast from b_0 when the following conditions are met:

i) $L(b_0) = \emptyset$,

ii) $x_i = \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ for all $1 \le i \le k$, and

iii) $d(b_i, b_0) \equiv 1 \pmod 2$ for all $1 \le i \le k$.

If T meets these conditions, it is easy to see that f_1 will not dominate b_0 and hence cannot be an α_{bn}-broadcast. In fact, for this structure, $\sigma(g) = \sigma(f_1) + 1$. And given the constraints on our broadcast, by the above observations and by Lemma 6.1.26, g provides the only possible way to dominate b_0 with an α_{bn}-broadcast. See Figure 6.29. We define f_2, a simplified version of g. If T meets the conditions (i), (ii), and (iii) then $f = f_2$. In Algorithm 1, if COUNTER=0 then conditions (i) and (ii) are met, and if CHOICE=YES then condition (iii) is met. If Algorithm 1 satisfies ATMOST 1=YES, COUNTER=0 and CHOICE=YES then f_2, as defined below, is the output. For $1 \leq i \leq k$,

$$f_2(x) = \begin{cases} d(x, b_i) & \text{if } x \in L(b_i) \\ 1 & \text{if } x = v_{i,j} \text{ where } j \equiv 1 \pmod 2, 1 \leq j < d(b_0, b_i) \\ 1 & \text{if } x = b_0 \\ 0 & \text{otherwise.} \end{cases}$$

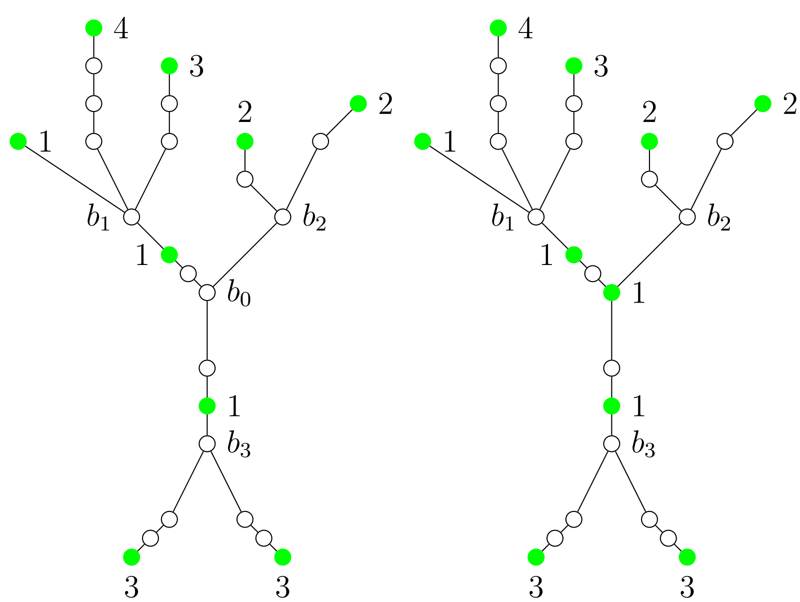


Figure 6.29: A tree which meets the conditions for an α_{bn}-broadcast with $f(b_0) = 1$. The broadcast f_1 as described in Case 1 is pictured on the left and on the right the broadcast f_2. Notice that $\sigma(f_2) = \sigma(f_1) + 1$.

We now consider the possibility that b_0 is overdominated while $f(b_0) = 0$. By our choice of f, b_0 must be overdominated by a leaf. Hence, Theorem 6.1.12 gives us two further cases. Either b_0 is overdominated by 1 or it is overdominated by at least 3.

Since we are looking at cases where b_0 is overdominated by at most 1, we examine this case first.

Case 3: Suppose b_0 is overdominated by 1. Let x_{ij} be the vertex such that $f(x_{ij}) = d(x_{ij}, b_0) + 1$. Note that x_{ij} overdominates b_i. Hence, by Corollary 6.1.13 and Lemma 6.1.11 , x_{ij} overdominates $L(b_i) - \{x_{ij}\}$ and since f is maximal $j = 1$. We claim that either $L(b_0) \cup \{x_{t1} : x_t = \max(b_t) + d(b_t, b_0)\} = \emptyset$ and $x_i = \max(b_i) + d(b_i, b_0) = \text{sum}(x_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$, or $L(b_0) \cup \{x_{t1} : x_t = \max(b_it) + d(b_t, b_0)\} = \{x_{i1}\}$. Suppose, for a contradiction, that there is a vertex $x_{y1} \in L(b_0) \cup \{x_{t1} : x_t = \max(b_t) + d(b_t, b_0)\}$ with $y \neq i$. Note that $label(x_i)$ could be 1 or 0. Regardless of $label(x_i)$, define a new broadcast g with $g(x_{i1}) = f(x_{i1}) - 1$, $g(u) = 0$ for all $u \in L(b_y) - \{x_{y1}\}$, $g(v) = 0$ for all vertices v on the $b_y - b_0$ path, $g(x_{y1}) = d(x_{y1}, b_0)$ and $g(x) = f(x)$ otherwise. Note that $B_g(x_{i1}) \cap B_g(x_{y1}) = \{b_0\}$ and all other boundaries are unchanged. Hence g is bn-independent. Note, it is possible that $f(x_{y1}) = d(x_{y1}, b_y)$, however, since $label(x_y) = 1$, $d(x_{y1}, b_0) - 1 \geq \text{sum}(x_y) + \lfloor \frac{d(b_y,b_0)}{2} \rfloor$. Further since f overdominates b_0 by 1, $f(x_{y1}) \leq d(x_{y1}, b_0) - 1$. Hence, regardless of the value which f assigns to x_{y1}, $\sigma(g) = \sigma(f)$ and g overdominates fewer branches and thus contradicts our choice of f.

Hence for all b_j, $j \neq i$, $x_j = \text{sum}(b_j) + \lfloor \frac{d(b_j,b_0)}{2} \rfloor$ and we further claim that $d(b_j, b_0) \equiv 1 \pmod 2$. Suppose, for a contradiction, that there exists a b_n, $n \neq i$, such that $x_n = \text{sum}(b_n) + \lfloor \frac{d(b_n,b_0)}{2} \rfloor$ where $d(b_n, b_0) \equiv 0 \pmod 2$. Since f is bn-independent, $f(v_{n,d(b_n,b_0)-1}) = 0$. Define a new broadcast g' with $g'(l_{i1}) = f(l_{i1}) - 1$ and $g'(v_{n,d(b_0,b_n)-1}) = 1$ and $g'(x) = f(x)$ otherwise. Again g' is a bn-independent broadcast, $\sigma(g') = \sigma(f)$ and g' overdominates fewer branch vertices.

We now examine the conditions under which $label(x_i) = 0$ or equivalently $x_i = \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$. If $x_i = \text{sum}(x_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ then, by definition, $\text{sum}(x_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor \geq \max(b_i) + d(b_i, b_0)$. However, $f(x_i) = \max(b_i) + d(b_i, b_0) + 1$ and f is maximal. Hence, $\max(b_i) + d(b_i, b_0) + 1 \geq \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$. Hence $\text{sum}(x_i) + \lfloor \frac{d(b_t,b_0)}{2} \rfloor = \max(b_i) + d(b_j, b_0)$ or $\text{sum}(x_i) + \lfloor \frac{d(b_t,b_0)}{2} \rfloor = \max(b_i) + d(b_j, b_0) + 1$. In the latter case, either f is not an α_{bn}-broadcast or there is another α_{bn}-broadcast which overdominates fewer branch vertices contradicting the choice of f. Hence $\text{sum}(x_i) + \lfloor \frac{d(b_t,b_0)}{2} \rfloor = \max(b_i) + d(b_j, b_0)$. Finally, if $x_i = \text{sum}(b_i) + \lfloor \frac{d(b_i,b_0)}{2} \rfloor$ where $d(b_i, b_0) \equiv 1 \pmod 2$ then T meets the conditions for f_2, $\sigma(f_2) = \sigma(f)$ and f_2 overdominates fewer branch vertices contradicting our choice of f. Hence, we only need a broadcast which overdominates b_0 by exactly 1 if T meets the following conditions:

i) $L(b_0) \cup \{l_{j1} : x_j = \max(b_j) + d(b_j, b_0)\} = \{x_{i1}\}$

and

ii) $x_j = \text{sum}(b_j) + \lfloor\frac{d(b_j,b_0)}{2}\rfloor$ with $d(b_j, b_0) \equiv 1 \pmod 2$ for all $j = 1...k$, $j \neq i$,

or

iii) $L(b_0) \cup \{l_j : x_j = \max(b_j) + d(b_j, b_0)\} = \emptyset$ and there exists exactly one x_i such that $x_i = \text{sum}(b_i) + \lfloor\frac{d(b_i,b_0)}{2}\rfloor = \max(b_i) + d(b_i, b_0)$,and $d(b_i, b_0) \equiv 0 \pmod 2$ and $d(b_j, b_0) \equiv 1 \pmod 2$ for all $j \neq i$, $j = 1...k$.

In this case, $f = f_3$, which is defined as

$$f_3(x) = \begin{cases} d(x_{i1}, b_0) + 1 & \text{if } x = x_{i1} \\ 1 & \text{if } x = v_{t,j} \text{ where } j \equiv 1 \pmod 2 \\ & \text{and } t \neq i, \text{ for } 1 \leq t \leq k \\ d(x, b_t) & \text{if } x \in L(b_t) \text{ and } t \neq i, \text{ for t=1,...k} \\ 0 & \text{otherwise.} \end{cases}$$

In Algorithm 1, if CHOICE=0 and COUNTER=1 then conditions (i) and (ii) are met, and if CHOICE=1 and COUNTER=1 then conditions (ii) and (iii) are met. In both cases, Algorithm 1 outputs f_3. An example of an α_{bn}- broadcast on a tree which meets conditions (i) and (ii) is shown in Figure 6.30. For a tree with this structure, $\sigma(f_3) = \sigma(f_1) + 1 \geq \sigma(f_2) + 1$.

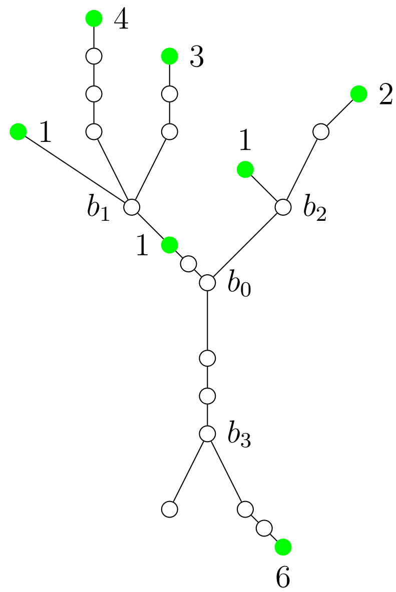


Figure 6.30: A tree which meets the conditions for an α_{bn}-broadcast with $f(b_0) = 0$ and b_0 overdominated by 1. On the left, the broadcast f_1 as described in Case 1 and, on the right, the broadcast f_3. Notice that $\sigma(f_3) = \sigma(f_1) + 1$.

An example of an α_{bn}- broadcast on a tree which meets conditions (ii) and (iii) is shown in Figure 6.31. For a tree with this structure, $\sigma(f_3) = \sigma(f_1) + 1 \geq \sigma(f_2) + 1$.

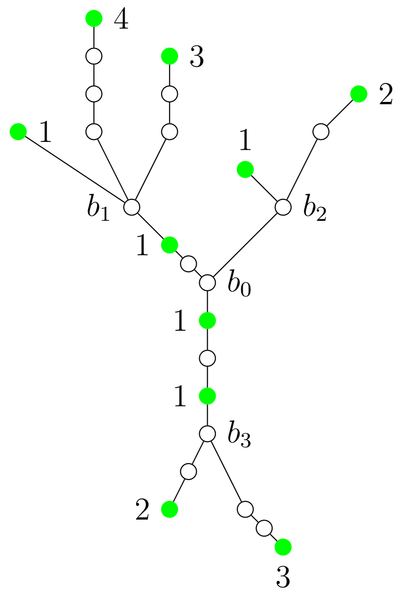


Figure 6.31: A tree which meets the conditions for an α_{bn}-broadcast with $f(b_0) = 0$ and b_0 overdominated by 1. On the left, the broadcast f_1 as described in Case 1 and, on the right, the broadcast f_3. Notice that $\sigma(f_3) = \sigma(f_1) + 1$.

Case 4: By Corollary 6.1.13, the only remaining case is that b_0 is overdominated by at least 3. Let x_{nj} be the leaf such that in our α_{bn}-broadcast f, $f(x_{nj}) > d(x_{n1}, b_0)+2$. As before, since f has maximum weight, $j = 1$. By Corollary 6.1.13, $B_f(x_{n1}) = \{v\}$, v is not a leaf and v is not an internal vertex of any $x - b$ path where $x \in L(b)$ for any branch vertex b. Hence, there is exactly one branch vertex, say b_m, which is not overdominated by x_{n1} and x_{n1} overdominates $L(T) - L(b_m)$ and v is on the $b_0 - b_m$ path. If $v = v_{m,t} \neq b_m$ then define a new broadcast g with $g(x_{n1}) = d(x_{n1}, b_m)$ and $g(x) = d(x, b_m)$ for all $x \in L(b_m)$ and $g(x) = 0$ otherwise. By the choice of f, $\sigma(f) = \sigma(g) - \lceil \frac{d(b_m, v_{m,t})}{2} \rceil < \sigma(g)$ and g contradicts the maximality of f. Hence, $v = b_m$. In this case, the maximum weight broadcast is

$$f_4(x) = \begin{cases} d(x, b_0) + d(b_0, b_m) & \text{if } x = x_{n1} \\ d(b_m, x) & \text{if } x \in L(b_m) \\ 0 & \text{otherwise.} \end{cases}$$

If f_4 is a maximum weight bn-independent broadcast with the minimum number of overdominated branches, it must have greater weight than f_1, f_2 and f_3. Recall that $\sigma(f_1) = \sum_{i=0 \text{ to } k} x_i$ and, when they exist, $\sigma(f_2)$ (or $\sigma(f_3)$)$=\sigma(f_1)+1$. In Algorithm 1, to simplify matters, if $\sigma(f_4) \geq \sigma(f_1)+1$ then the output is f_4. This means that f_4 might be chosen when its weight is the same as f_2 or f_3 even if these broadcasts are possible. In this case, the algorithm outputs a maximum weight broadcast but it does not have the least number of overdominated branch vertices of all α_{bn}- broadcasts. If $\sigma(f_4) \geq \sigma(f_1)+1$ then

$$\max(b_n) + d(b_n, b_0) + \min\{\lceil \frac{d(b_m, b_0)}{2} \rceil, \text{loss}(b_m)\} > \sum_{i=0, i \neq m}^{k} x_i.$$

Hence, to achieve an α_{bn}-broadcast, we overdominate b_0 by more than two from a leaf $l \in L(b_n)$ for some $0 \leq n \leq k$ when both of the following conditions are met:

(i) $\lceil \frac{d(b_m, b_0)}{2} \rceil > \sum_{i=0 \text{ to } k, i \neq m} x_i - \max(b_n) - d(b_n, b_0)$

(ii) $\text{loss}(b_m) > \sum_{i=0 \text{ to } k, i \neq m} x_i - \max(b_n) - d(b_n, b_0)$.

When (i) is met, for any $j \neq m, n$, $1 \leq j \leq k$,

$$\begin{aligned} \lceil \frac{d(b_m, b_0)}{2} \rceil &> \sum_{i=1 \text{ to } k, i \neq m} x_i + \sum_{i=1 \text{ to } |L(b_0)|} l_{0,i} - \max(b_n) - d(b_n, b_0) \\ &> \sum_{i=1 \text{ to } k, i \neq m,n} x_i + \sum_{i=1 \text{ to } |L(b_0)|} l_{0,i} \\ &> \lfloor \frac{d(b_j, b_0)}{2} \rfloor + 1 \geq \lceil \frac{d(b_j, b_0)}{2} \rceil. \end{aligned}$$

Hence $d(b_m, b_0) > d(b_j, b_0)$ for all $j \neq n, m$. In Algorithm 1, the inputs representing the branch vertices are indexed by decreasing magnitude. Hence, either $m = 1$ or $n = 1$ and $m = 2$. Algorithm 1 checks conditions (i) and (ii) first using $m = 1$ and if both conditions are not satisfied then it tries again with $n = 1$ and $m = 2$. If the conditions are still not satisfied then the algorithm sets BROADCAST= ATMOST 1 and outputs f_1, f_2 or f_3 as described in cases 1-3.

If conditions (i) and (ii) are satisfied then Algorithm 1 will output f_4. For this case, suppose that there was a $j \neq m, n$, $0 \leq j \leq k$, such that $x \in L(b_j)$ and $d(x, b_0) > d(l, b_0)$. Define a broadcast f_4' with $f_4'(l) = 0$, $f_4'(x) = d(x, b_0)$ and $f_4'(u) =$

$f_4(u)$ otherwise. Since f_4' has the same structure as f_4 it is bn-independent but $\sigma(f_4') > \sigma(f_4)$. Hence, l is the vertex such that $d(l, b_0) = \max\{d(x, b_0) : 0 \leq i \leq k$ and $i \neq m$ and $x \in L(b_i)\}$. If $m = 1$ in Algorithm 1, then the statement $d(l_{i,j}, b_0) = \max\{d(l_{i,1}, b_0) : 0 \leq i \leq k$ and $i \neq 1\}$ chooses the appropriate $n = j$. In this case the algorithm outputs f_4 which is the same as our f_4. If $n = 1$ and $m = 2$ then suppose that there exists an x_j with $j \notin \{1, 2\}$ such that $l_{j,1} + d(b_j, b_0) > l_{1,1} + d(b_1, b_0)$. In this case, $x_j \geq l_{j,1} + d(b_j, b_0) > l_{1,1} + d(b_1, b_0) > \lceil \frac{d(b_1,b_0)}{2} \rceil > \lceil \frac{d(b_2,b_0)}{2} \rceil$ which contradicts (i). Hence, we can be confident in assigning $l = l_{1,1}$.

If T has the correct structure to meet the conditions of case 2 and case 4 then $\sigma(f_4) = \sigma(f_2)$. Similarly, it is possible that $\sigma(f_4) = \sigma(f_3)$. In both of these cases, Algorithm 1 will output f_4. Hence, although Algorithm 1 generates an α_{bn}-broadcast, it may not have the minimum number of overdominated branches of all such broadcasts. See Figure 6.32 for an example.

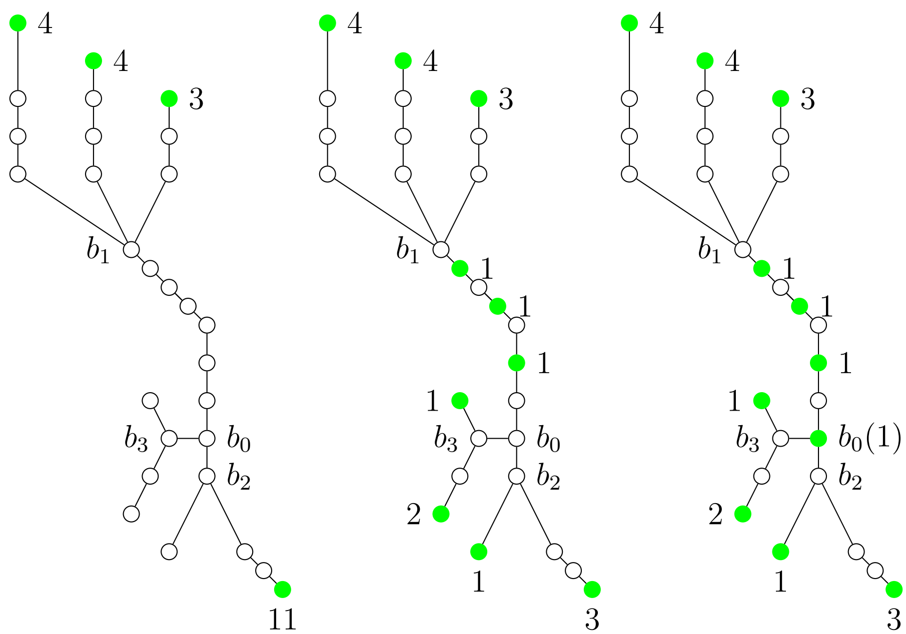


Figure 6.32: A tree which meets the algorithms conditions for an α_{bn}-broadcast with $f = f_4$. The broadcast f_4 as described in case 4 is pictured on the left. In the middle f_1 and on the right the broadcast f_2. Notice that $\sigma(f_4) = \sigma(f_2) = \sigma(f_1) + 1$. Also, $\lceil \frac{d(b_1,b_0)}{2} \rceil = 4$, loss$(b_1) = 7$ and $\sum_{i=1 \text{ to } k, i \neq m} x_i + \sum_{i=1 \text{ to } |L(b_0)|} l_{0,i} - d(b_n, b_0) = 1+2+4-4 = 3$, hence $\lceil \frac{d(b_m,b_0)}{2} \rceil$, loss$(b_1) > \sum_{i=1 \text{ to } k, i \neq m} x_i + \sum_{i=1 \text{ to } |L(b_0)|} l_{0,i} - d(b_n, b_0)$.

This exhausts all cases. We have shown that each case occurs under distinct conditions. For all trees meeting the hypothesis statement, Algorithm 1 checks these conditions and outputs the corresponding broadcast. □

Figure 6.33 shows an example of a tree for which $\alpha_{bn}(T)$ lies between the bounds of Theorem 6.1.16 and Theorem 6.1.18. The broadcast shown is the one assigned by Algorithm 1.

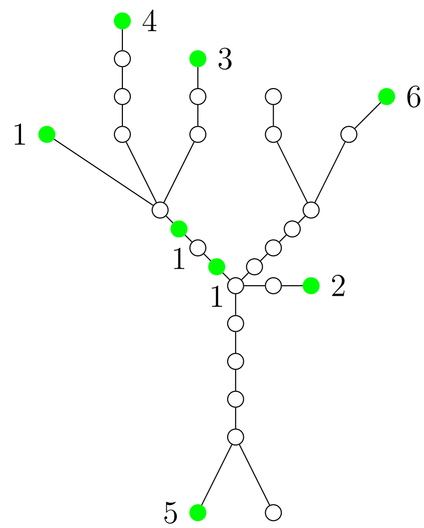


Figure 6.33: A tree with an α_{bn}-broadcast f with $n - b(T) - |W_{int}| + \alpha(G_{int}(T)) = 29 - 4 - 9 + 6 = 22 < \sigma(f) = 23 < n - b(T) + \rho(T) = 29 - 4 + 1 = 26$.

For any tree with $B(T) \cong K(1,k)$, $R_T \subseteq \{b_0\}$. Hence $\alpha(T[R_T]) \leq 1$. If $\alpha(T[R_T]) = 1$ then the upper bound of of Theorem 6.1.16 becomes $\alpha_{bn}(T) \leq n - b(T) + 1$. If $|W_i(T)| = \alpha(G_{int}(T)) + 1$ then the two bounds of Theorem 6.1.16 and Theorem 6.1.18 are equal and we have another class of graphs which meet the upper bound of Theorem 6.1.16.

Note that for trees with $B(T) \cong K(1,k)$, $\alpha(T[R_T]) = \rho(T) = 1$, or $\alpha(T[R_T]) = \rho(T) = 0$ hence the answer to Question 6.1.17, in this case, is yes.

6.2 Maximum Irredundant Boundary Independent Broadcasts on Trees

We make some preliminary observations about the structure of bnr-broadcasts on trees using an approach similar to that for bn-independence.

6.2.1 Non-dominating α_{bnr}-broadcasts

Recall that if a tree T has an α_{bnr}-broadcast f which is dominating then, since f is irredundant, $\alpha_{bnr}(T) \leq \Gamma_b(T)$. However, not all trees have such a broadcast and, as seen in Chapter 5, there exist trees such that $\alpha_{bnr}(T) > \Gamma_b(T)$.

We take some first steps towards characterizing trees for which $\alpha_{bnr}(T) > \Gamma_b(T)$. We investigate the existence of trees for which all α_{bnr}-broadcasts are non-dominating and examine what conditions accompany a non-dominated vertex. We start with observations about the structure of any bnr-broadcast on any tree.

Lemma 6.2.1. *Any tree T has an α_{bnr}-broadcast f in which there are no non-dominated leaves.*

Proof. Suppose T is a tree for which there is no α_{bnr}-broadcast which dominates all the leaves of T. Of all α_{bnr}-broadcasts on T, let f be one with the smallest number of non-dominated leaves. Let u be a non-dominated leaf. Since f is maximal bnr-independent, the function $(f - \{(u,0)\}) \cup \{(u,1)\}$ is not irredundant. Hence there are distinct vertices w and w' such that w is adjacent to u, $w' \in V_f^+$ and $\{w\} = PB_f(w')$. Since $w' \notin PB_f(w')$ and f is bnr-independent, $f(w') > 1$. Define a new broadcast g with $g(w') = f(w') - 1$, $g(u) = 1$ and $g(x) = f(x)$ otherwise. Let w_1 be the vertex adjacent to w on the $w' - w$ path. Since f is bnr-independent, $H_f(w_1) = \{w'\}$ and thus $w_1 \in PB_g(w')$. Also note that $PB_g(u) = \{u, w\}$. Hence g is bnr-independent. Since $\sigma(g) = \sigma(f)$, we may assume that g is an α_{bnr}-broadcast. Since $PB_f(w') = \{w\}$, f overdominates every leaf in $N_f(w')$. Hence the number of leaves non-dominated by g is less that the number non-dominated by f and g contradicts our choice of f, and the result follows. $\square$

Lemma 6.2.2. *Any tree T with $b(T) \geq 2$ satisfies at least one of the following statements:*

(i) There is an α_{bnr}-broadcast f on T with at least two leaves $l_1, l_2 \in V_f^+$ and two branch vertices $b_1, b_2 \in V(T)$ such that $l_1 \in L(b_1)$, $l_2 \in L(b_2)$ and $b_1 \neq b_2$, or

(ii) *$B(T)$ is a path and every branch vertex has degree 3. Let b_1, b_2 denote the two end-branch vertices. For $i = 1, 2$, $L(b_i) = \{u_i, v_i\}$ where $d(u_i, b_i) = 1$ and $d(v_i, b_i) > 1$. For all branch vertices b_i, $i \neq 1, 2$, $L(b_i) = \{v_i\}$ and $d(v_i, b_i) = 1$. (Or, equivalently, T is a caterpillar such that every branch vertex has at most one neighbour which is a leaf.) And, there is, without loss of generality, an α_{bnr} broadcast f on T with $V_f^+ = \{v_1\}$ and $f(v_1) = d(v_1, v_2)$, which implies that $\alpha_{bnr}(T) = \mathrm{diam}(T)$.*

Proof. Given a tree T with $b(T) \geq 2$ let f be an α_{bnr}-broadcast in which there are no non-dominated leaves. Such a broadcast exists by Lemma 6.2.1. By Theorem 2.3.13 and the definition of the broadcast g in the proof of Lemma 6.2.1, we may assume that leaves only hear leaves, hence there is at least one leaf l_1 such that $l_1 \in V_f^+$. Let b_1 be the branch vertex such that $l_1 \in L(b_1)$.

Suppose that f does not satisfy statement (i). Then $L(b_1)$ contains all leaves l_i such that $f(l_i) > 0$. Since f is *bnr*-independent only one leaf in $L(b_1)$ can overdominate b_1. Without loss of generality let l_1 be the leaf which overdominates b_1. Since leaves only hear leaves and all leaves are dominated, l_1 dominates all leaves l such that $l \notin L(b_1)$.

Suppose that l_1 does not overdominate $L(b_1)$. Let l_2 be a leaf such that $l_2 \in L(b_1)$ and l_2 is not overdominated by l_1. Let v be the vertex on the $b_1 - l_2$ path such that $d(v, l_1) = f(l_1)$. It is possible but not necessary that $v = l_2$. Choose a leaf l such that $l \in L(b)$ where $b \neq b_1$. Define a broadcast g with $g(l_2) = d(l_2, b_1) - 1$, $g(l_1) = d(l_1, b_1)$, $g(l) = d(l, b_1) - 1$ and $g(x) = f(x)$ otherwise. Since b_1 is a branch vertex, l_2, l_1 and l all have nonempty private boundaries. Since $b \neq b_1$, $g(l) \geq 1$ and $\sigma(g) \geq \sigma(f)$. Since $f(x) = 0$ for all x not on a $l_i - b_1$ path where $l_i \in L(b_1)$, g is a *bnr*-independent broadcast. Either g is an α_{bnr}-broadcast which satisfies (i), or it can be extended and contradicts the maximality of f.

Hence if T does not have a broadcast f which satisfies statement (i) then in any α_{bnr}-broadcast f on T, there exists a leaf l_1 which overdominates $L(b_1)$ and dominates all other leaves. Hence, $\sigma(f) = f(l_1)$ and $f(l_1) = d(l_1, l)$ where l is a leaf $l \notin L(b_1)$. For maximality of the broadcast, l and l_1 are peripheral vertices and $\alpha_{bnr}(T) = \mathrm{diam}(T)$. An example of a tree T which does not satisfy statement (i) is shown in Figure 6.34.

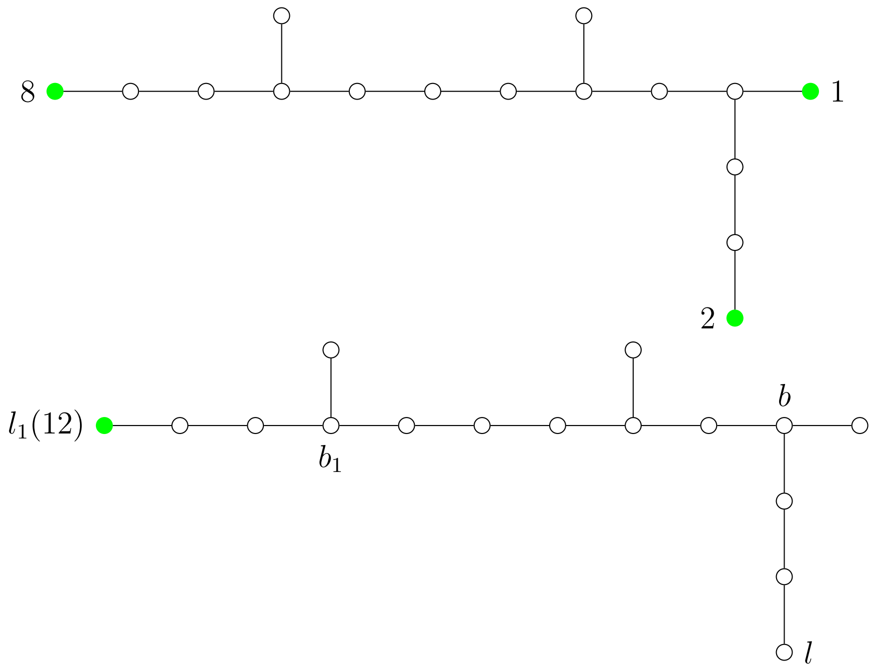


Figure 6.34: The top figure shows an example of a broadcast on the tree T which, while it satisfies the structure of (i), is not an $\alpha_{bnr}(T)$-broadcast. By considering all such possibilities we see that T does not satisfy (i). Notice, bottom figure, that it appears that T satisfies (ii) and $\alpha_{bnr}(T) = 12 = \Gamma_b(T)$. By symmetry and exhaustion, this can be shown to be true.

Suppose T does not have a broadcast which satisfies statement (i). Let f be any α_{bnr}-broadcast on T. From the above arguments, $\sigma(f) = f(l_1)$, $l_1 \in L(b_1)$, $d(l, l_1) = f(l_1)$ and $l \in L(b)$ where $b \neq b_1$. We proceed to show that T meets the remaining conditions in statement (ii). First we show that $\deg_{B(T)}(b_1) = \deg_{B(T)}(b) = 1$ or, equivalently, b_1 and b are end-branch vertices. Suppose that $\deg_{B(T)}(b_1) \geq 2$. Then let $b_3 \neq b$ be a branch vertex with $\deg_{B(T)}(b_3) = 1$ such that there is a $b_1 - b_3$ path which is internally disjoint from the $b_1 - b$ path. Choose a leaf vertex $l_3 \in L(b_3)$. Define a broadcast

$$g(x) = \begin{cases} d(x, b_1) & \text{if } x = l_3 \\ d(x, b_1) - 1 & \text{if } x \in \{l_1, l\} \\ 0 & \text{otherwise.} \end{cases}$$

Since all three branch vertices are distinct, g is bnr-independent and $d(l_3, b_1) \geq 2$. Hence $\sigma(g) \geq \sigma(f)$ and either g violates the maximality of f or it satisfies statement (i) and violates the choice of T. Hence b_1 is an end-branch vertex. By symmetry, the broadcast f' defined by $f'(l) = f(l_1)$ and $f'(x) = 0$ if $x \neq l$ has the same properties as f, and as for b_1 we obtain that $\deg_{B(T)}(b) = 1$.

Suppose that $B(T)$ is not a path. Let b_3 be a branch vertex such that $\deg_{B(T)}(b_3) \geq 3$ and b_3 lies on a $b - b_1$ path in $B(T)$. Let b_2 be a branch vertex adjacent to b_3 in $B(T)$ which does not lie on same path as b, b_1 and b_3 in $B(T)$. Let l_2 be a leaf in $L(b_2)$. Define a broadcast:

$$g_1(x) = \begin{cases} d(x, b_3) & \text{if } x = l_1 \\ d(x, b_3) - 1 & \text{if } x \in \{l_2, l\} \\ 0 & \text{otherwise.} \end{cases}$$

Since b, b_2 and b_3 are all distinct, g_1 is bnr-independent and $d(l_2, b_3) \geq 2$. Also $d(l_1, b) + d(b, l) = f(l_1) - 1$, hence $\sigma(g_1) \geq \sigma(f)$. Hence either g_1 satisfies condition (i) of our statement, violating the choice of T, or it contradicts the maximality of f. Thus, we conclude that $B(T)$ is a path.

Suppose there is a branch vertex b_3 such that $\deg_{B(T)}(b_3) = 2$ and $\{u, v\} \subseteq L(b_3)$. Define a broadcast:

$$g_2(x) = \begin{cases} 1 & \text{if } x \in \{u, v\} \text{ and } d(x, b_3) = 1 \\ d(x, b_3) - 1 & \text{if } x \in \{l, l_1\}, \text{ or } x \in \{u, v\} \text{ and } d(x, b_3) \geq 2. \\ 0 & \text{otherwise.} \end{cases}$$

Notice that g_2 is bnr-independent and $\sigma(g_2) \geq \sigma(f)$. Hence either g_2 contradicts the maximality of f or it satisfies condition (i) of our statement and contradicts the choice of T. This shows that $|L(b_3)| \leq 1$. Since b_3 is a branch vertex and $\deg_{B(T)}(b_3) = 2$, $|L(b_3)| = 1$. Therefore, $\deg(x) = 3$ for all x such that $\deg_{B(T)}(x) = 2$.

For the end-branch b, suppose that $\{u, v\} \subseteq L(b)$ and either $d(u, b), d(v, b) > 1$ or

$d(u,b) = d(v,b) = 1$. Define a broadcast:

$$g_3(x) = \begin{cases} d(x,b) & \text{if } x \in L(b) \text{ and } d(x,b) = 1 \\ d(x,b) - 1 & \text{if } x \in L(b) \text{ and } d(x,b) > 1 \\ d(x,b) - 1 & \text{if } x = l_1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $b \neq b_1$, g_3 is a *bnr*-independent broadcast. If $L(b) = \{u, v\}$ and, without loss of generality, $g_3(u) = 1$ and $g_3(v) > 1$ then $\sigma(g_3) < \sigma(f)$. Since $\deg_{B(T)}(b) = 1$, $|L(b)| \geq 2$. Hence, in all other possibilities for $L(b)$, $\sigma(g_3) \geq \sigma(f)$ and either g_3 contradicts the maximality of f, or it satisfies statement (i) and contradicts the choice of T. We conclude that $d(u,b) = 1$ and $d(v,b) > 1$. By symmetry, $L(b_1) = \{u_1, v_1\}$, $d(u_1, b_1) = 1$, and $d(v_1, b_1) > 1$. Finally, assuming that $B(T)$ is a path, suppose that T has a branch vertex $b' \neq b, b_1$ such that there exists $w \in L(b')$ with $d(w, b') > 1$. Create a new broadcast:

$$g_4(x) = \begin{cases} d(x,b') - 1 & \text{if } x = v, v_1 \\ d(x,b') & \text{if } x = w \\ 0 & \text{otherwise.} \end{cases}$$

Since, $d(w,b') \geq 2$ and $d(v,b') + d(b', v_1) = \text{diam}(T)$, $\sigma(g_4) \geq \sigma(f)$. Hence, either $\sigma(g_4)$ meets condition (i) or $\sigma(g_4)$ is not maximal and f is not a maximum weight *bnr*-broadcast.

We have shown that if T does not satisfies statement (ii) then it satisfies statement (i). $\square$

Remark 6.2.3. *There are trees which satisfy conditions* (i) *and* (ii) *from Lemma 6.2.2. An example is shown in Figure 6.35.*

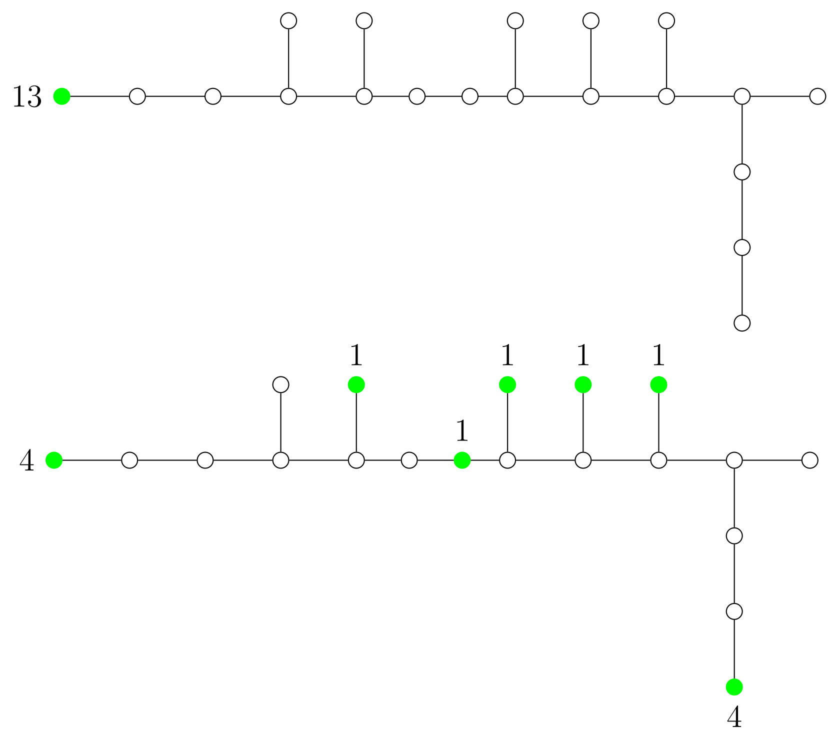


Figure 6.35: A tree T which satisfies (i) and (ii) from Lemma 6.2.2; $\alpha_{bnr}(T) = 13 = \Gamma_b(T) = \text{diam}(T)$.

Remark 6.2.4. *There are trees which satisfy condition* (i) *and have the structure described in condition* (ii) *but do not satisfy condition* (ii) *from Lemma 6.2.2. An example is shown in Figure 6.36.*

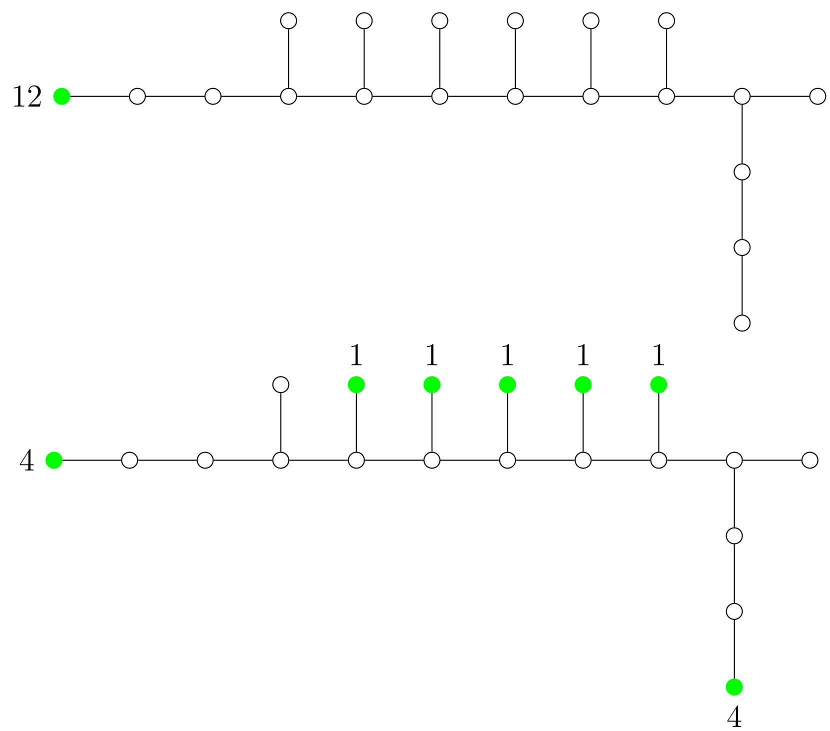


Figure 6.36: A Tree T which satisfies (i) and the structure of (ii) from Lemma 6.2.2; but $\alpha_{bnr}(T) = 13 \neq \text{diam}(T) = 12$. Hence T does not fully satisfy condition (ii).

Conjecture 6.2.5. *Let T be a tree that satisfies the structure described in condition (ii) from Lemma 6.2.2. If $W_{int}(T) \neq \emptyset$, then $\alpha_{bnr}(T) = \text{diam}(T)$, and if $W_{int}(T) = \emptyset$ then $\alpha_{bnr}(T) = \text{diam}(T) + 1$ and T satisfies condition (i).*

Theorem 6.2.6. *Any tree T with branch number $b(T) \geq 2$, has an α_{bnr}-broadcast with the minimum number of overdominated branch vertices which satisfies the following statement:*

If a leaf, say l, overdominates a branch vertex b in f where $d(b, l) \geq d(b', l)$ for all $b' \in B(T) \cup N_f(l)$ then:

i) $f(l) = \sigma(f)$ *and T meets condition (ii) from Theorem 6.2.2*

ii) $f(l) = d(l, b) + 1$, *b is adjacent to at most one leaf, and, for all $l_i \in L(b) - \{l\}$,* $d(l_i, b) \leq 2$;

iii) $f(l) = d(l, b) + 2$, $\deg b = 3$, $d(l, b) > 1$ *and* $|L(b)| \leq 2$; *or* *iv)* $f(l) > d(l,b)+2$, $L(b) \subset N_f(l)$, $|B_f(l)| \le 2$, *and either* $B_f(l) \cap L(b) = \emptyset$ *or* b *is an end-branch vertex,* $B_f(l) \cap L(b) = \{l'\}$, $L(b) = \{l', l''\}$ *and* $d(b, l'') = 1$.

Proof. Let T be a tree with branch number $b(T) \ge 2$ and f an α_{bnr}-broadcast on T for which the number of overdominated branch vertices is a minimum. By Lemma 6.2.2, f satisfies at least one of the two conditions in Lemma 6.2.2.

Case 1: If f does not satisfies condition (i) of Lemma 6.2.2 then it satisfies our condition (i) (which is condition(ii) of Lemma 6.2.2) .

Hence, for the remaining cases, we assume that f satisfies condition (i) of Lemma 6.2.2.

Case 2: Suppose that there is a leaf l overdominating a branch vertex b such that $f(l) = d(l,b)+1$. If $L(b) - \{l\} = \emptyset$ then there is nothing to prove. Let l' be a leaf $l' \in L(b) - \{l\}$. Suppose $d(l', b) > 2$. Leaves only hear leaves, b is overdominated and f is bnr-independent. Thus l' must be broadcasting and since f is maximal, $f(l') = d(l', b) - 2$. In this case, define a broadcast g with $g(l) = f(l) - 1$, $g(l') = f(l') + 1$ and $g(x) = f(x)$ otherwise. Since $N_g(l) \cup N_g(l') \subseteq N_f(l) \cup N_f(l')$, $PB_g(l) = \{b\}$ and $PB_g(l')$ contains the vertex adjacent to b on the $b - l'$ path, g is bnr-independent. Since $\sigma(f) = \sigma(g)$, either g contradicts the maximality of f or it contradicts the minimality of the number of overdominated vertices in f.

Suppose there are two leaves, l_1, l_2, adjacent to b, possibly $l_1 = l$. In both cases, since f is bnr-independent and l overdominates b by one, $f(l_2) = 0$. Define a broadcast g as follows: if l is not adjacent to b, then let $g(l) = f(l) - 2$, $g(l_1) = g(l_2) = 1$, and $g'(x) = f(x)$ otherwise. If l is adjacent to b, then let $g(l) = g(l_2) = 1$ and $g'(x) = f(x)$ otherwise. If l is not adjacent to b then $g(l) + g(l_1) + g(l_2) = f(l)$ and if l is adjacent to b, then $g(l_1) + g(l_2) = f(l)$. Thus, in both cases, $\sigma(f) = \sigma(g)$. Notice that $N_g(l) \cup N_g(l_1) \cup N_g(l_2) \subseteq N_f(l)$ and $PB_g(l_i) = \{l_i\}$ for $i = 1, 2$. If l is not adjacent to b, then l is the only broadcast covering the internal vertices of the $l - b$ path and $g(l) = d(l, b) - 1$, hence $PB_g(l) \neq \emptyset$. Therefore, in either case, g is bnr-independent and either g contradicts the maximality of f or it contradicts the minimality of the number of overdominated vertices in f. Thus f satisfies condition (ii) of our theorem statement.

Case 3: Suppose that there is a leaf l overdominating a branch vertex b such that $f(l) = d(l, b) + 2$. Since it is a branch vertex, $\deg(b) \geq 3$. First we note that l is not adjacent to b. Suppose it was. Let v_1 and v_2 be vertices adjacent to b different from l. Make a new broadcast g with $g(l) = g(v_1) = g(v_2) = 1$ and $g(x) = f(x)$ otherwise. Notice that $N_g(b) \cup N_g(v_1) \cup N_g(v_2) \subseteq N_f(l)$ and v_1, v_2 and l are all in their own private g-boundaries. Hence g is bnr-independent and $\sigma(f) = \sigma(g)$ while g overdominates fewer branch vertices. Hence either g violates the maximality of f or it violates the choice of f. We conclude that l is not adjacent to b.

If $\deg(b) > 3$ then let v_1, v_2, v_3 be vertices adjacent to b but not on the $l - b$ path. Make a new bnr-independent broadcast g with $g(l) = f(l) - 3$ and $g(v_1) = g(v_2) = g(v_3) = 1$. Since $N_g(l) \cup N_g(v_1) \cup N_g(v_2) \cup N_g(v_3) \subseteq N_f(l)$, and $PB_g(v_1) = \{v_1\}$, $PB_g(v_2) = \{v_2\}$, $PB_g(v_3) = \{v_3\}$ and $PB_g(l)$ contains the vertex adjacent to b on the $b - l$ path, g is bnr-independent. Since $\sigma(g) = \sigma(f)$ and b is no longer overdominated, g contradicts our hypothesis. Thus $\deg(b) = 3$. Since $b(T) \geq 2$, $\deg_{B(T)}(b) > 0$. Hence $|L(b)| \leq 2$. Thus condition (iii) is satisfied.

Case 4: Suppose that for any of our selected α_{bnr}-broadcasts f on T there exists at least one leaf l and one branch vertex b where $d(b, l) \geq d(b', l)$ for all $b' \in B(T) \cap N_f(l)$ and $f(l) > d(l, b) + 2$. Let O be the set of all such pairs of leaves and branches in f. Of all such broadcasts, choose one such that $\sum_{(b,l) \in O} (f(l) - d(b, l))$ is a minimum. Suppose there are three or more vertices $v_1, v_2, v_3 \in B_f(l)$. For $1 \leq i \leq 3$, let v_i' be the vertex adjacent to v_i on the $v_i - b$ path. Define a broadcast g with $g(v_1') = g(v_2') = g(v_3') = 1$, $g(l) = f(l) - 3$ and $g(x) = f(x)$ otherwise. Notice that $N_g(v_1) \cup N_g(v_1) \cup N_g(v_1) \cup N_g(l) \subseteq N_f(l)$, $v_i' \in PB_g(v_i')$, and each $b - v_i$ path contains a vertex u_i such that $H(u_i) = \{l\}$.(Possibly $u_i = b$) Hence g is a bnr-independent broadcast with the same weight as f and g contradicts the choice of f. Hence $|B_f(l)| \leq 2$. Now suppose that $l' \in L(b) - N_f(l)$. Leaves only hear leaves, b is overdominated and f is bnr-independent. Thus l' must be broadcasting. Since f is maximal and bnr-independent, $f(l') = 1$ and $f(l') + f(l) = d(l, l')$ or $f(l') + f(l) = d(l, l') - 1$. Since b is a branch vertex, there is a vertex v adjacent to b which is not on a $b - l'$ or a $b - l$ path. If $f(l') + f(l) = d(l, l') - 1$, then create a new broadcast g with $g(l') = d(l', b) - 1$, $g(l) = d(l, b) - 1$, $g(v) = 1$ and $g(x) = f(x)$ otherwise. Again g is bnr-independent , $\sigma(f) = \sigma(g)$ and g no longer overdominates b. If $f(l') = 1$ and $f(l') + f(l) = d(l, l')$ then there must be a vertex $u \in PB_f(l)$. Let u' be the vertex adjacent to u on the $u - b$ path. Create a new broadcast g with $g(l') = d(l', b) - 1$,

$g(l) = d(l, b) - 1$, $g(u') = 2$ and $g(x) = f(x)$ otherwise. Again g is *bnr*-independent , $\sigma(f) = \sigma(g)$ and g no longer overdominates b.Hence, we have contradicted our choice of T. We conclude that $L(b) \subset N_f(l)$. And either $L(b) - \{l\}$ is overdominated by l or there exists $l' \in B_f(l) \cap L(b)$.

Suppose that there exists $l' \in B_f(l) \cap L(b)$. If $\deg(b) > 3$ then there exist vertices u, v adjacent to b, neither of which is on a $b-l'$ or a $b-l$ path. Create a new broadcast g with $g(l') = d(l', b)-1$, $g(l) = d(l, b)-1$, $g(v) = g(u) = 1$ and $g(x) = f(x)$ otherwise. Again g is *bnr*-independent and $\sigma(f) = \sigma(g)$. Hence g contradicts the choice of T and we conclude that, since b is a branch, $\deg(b) = 3$. Now suppose that there is a vertex u such that u is not on a $b-l'$ or a $b-l$ path and $d(u, b) = 2$. Create a new broadcast g with $g(l') = d(l', b)$, $g(l) = d(l, b) - 1$, $g(u) = 1$ and $g(x) = f(x)$ otherwise. Again g is *bnr*-independent and $\sigma(f) = \sigma(g)$. Hence g contradicts the choice of f and we conclude no such vertex u exists. Since $\deg(b) = 3$, $L(b) = \{l', l''\}$ where without loss of generality, $d(l'', b) = 1$. Since $\deg(b) = 3$, $\deg_{\mathrm{B(T)}}(b) = 1$ or, equivalently, b is an end-branch vertex.

We have covered all possible ways for b to be overdominated and thus, the proof is complete. □

Theorem 6.2.7. *If a tree T has an α_{bnr}-broadcast f which does not dominate a vertex u, then let $\{u_1, u_2, ..., u_r\} = N(u)$. The following statements hold:*

1. *If $u \in B(T)$ then $L(u) = \emptyset$, and*

2. *There are at least two vertices $u_i \in N(u)$ and two vertices $v_i \in L(T)$ for which $\{u_i\} = PB_f(v_i)$.*

Proof. Using Theorem 2.3.14, we can consider a *bnr*-independent broadcast f in which either $\deg(v) = 1$, $f(v) = 1$ or both. Since the arguments in Lemma 6.2.1 either reduce the size of a broadcast or introduce a broadcast of strength 1, we can assume that f dominates all leaves. Of all such broadcasts, let f be the one with the fewest non-dominated vertices.

Suppose that $l \in L(u)$. Since u is not dominated and thus not overdominated and leaves only hear leaves, $f(l) > 0$. Let $B_f(l) = v'$. Notice that v' lies on the $l - u$ path. Define a broadcast g with $g(x) = 0$ if x is and internal vertex on the $l - u$ path, $g(l) = d(l, u)$, and $g(x) = f(x)$ otherwise. Since u is non-dominated by f, g is *bnr*-independent and by our choice of f, $\sigma(g) \geq \sigma(f)$ and u is now dominated. This contradicts the maximality of f or our choice of T.

Suppose that u has exactly one neighbour u_1 such that $\{u_1\} = PB_f(v_1)$ for some $v_1 \in V_f^+$. Since $v_1 \notin PB_f(v_1)$, $f(v_1) > 1$ and by our choice of f, $v_1 \in L(T)$. Define a broadcast g with $g(u) = 1$, $g(v_1) = f(v_1) - 1$ and $g(x) = f(x)$ otherwise. Since $\{u_1\} = N(u) \cap PB_f(v_1)$, $PB_g(v) - N(u) \neq \emptyset$ for all $v \neq u$. Note also that $u \in PB_g(u)$. Hence g is a *bnr*-broadcast with $\sigma(g) = \sigma(f)$. Since $\{u_1\} = PB_f(v_1)$ and u is now dominated, g dominates more vertices than f. This contradicts our choice of f.Suppose that u has no neighbours u_1 such that $\{u_1\} = PB_f(v_1)$ for some $v_1 \in V_f^+$. Define a new broadcast with $g(u) = 1$ and $g(x) = f(x)$ otherwise. Since $PB_f(v) - N(u) =\mathrm{PB}_g(v) - N(u) \neq \emptyset$ for all $v \neq u$ and $PB_f(u) = \{u\}$, g is *bnr*-independent. Since $\sigma(g) > \sigma(f)$, g contradicts the maximality of f. Hence we conclude that u has at least two neighbours such that $\{u_i\} = PB_f(v_i)$ for some $v_i \in L(T)$. $\square$

Notice that our example T in Figure 5.2 with $\alpha_{bnr}(T) > \Gamma_b(T)$ and the trees H_k constructed from T all satisfy the conclusion of Theorem 6.2.7. We leave it as an open problem to obtain a more accurate description or perhaps even a characterization of trees for which $\alpha_{bnr}(T) > \Gamma_b(T)$.

6.3 Maximum Hearing Independent Broadcasts on Trees

Recall from Chapter 2 the approach used by Bessy et al. [3] to achieve their bound on hearing independence. Starting with a $\alpha_h(G)$ broadcast f they create a new broadcast g with $g(v) = \lfloor \frac{f(v)}{2} \rfloor$ for all $v \in V_f^+$. The h-independence of f guarantees that $N_g(v) \cap N_g(w) \subseteq B_g(v) \cap B_g(w)$ for all $v, w \in V_g^+$. Or equivalently, g is *bn*-independent. If fact, we can work with the broadcast g where $g(v) = \lceil \frac{f(v)}{2} \rceil$ if $f(v) \geq 2$ and $g(x) = f(x)$ otherwise and still be guaranteed *bn*-independence. This observation led to a connection between bounds for maximum h-independence and *bn*-independence broadcasts. Now using our new bound $\alpha_{bn}(T) \leq n - b(T) + \rho(T)$, we note a new bound for maximum hearing independent broadcasts. For any graph G of order n:

$$\alpha_h(G) \leq \alpha_h(T_n) < 2 \min\{\alpha_{bn}(T_n) : T_n \text{ is a spanning tree of } G\}$$

and thus,

$$\alpha_h(G) < 2\min\{(n - b(T_n) + \rho(T_n)) : T_n \text{ is a spanning tree of } G\}.$$

6.4 Summary

We studied $\alpha_{bn}(T)$-broadcasts for trees with two or more branch vertices. We observed that in these broadcasts there are always a pair of broadcasting leaves such that the path between them contains at least two branch vertices. Also, when a leaf overdominates a branch vertex it never overdominates it by exactly two and there are restrictions on its boundary set. We used these results, along with observations from Chapter 2, to show, by examining a limited number of cases, that $\alpha_{bn} \leq n - b(T) + \rho(T)$ where $b(T)$ is the number of branch vertices and $\rho(T)$ is the number of branch vertices with at most one leaf. Many trees meet this bound and others do not. We conjectured that $\alpha_{bn} \leq n - b(T) + \alpha([R_T])$ where $[R_T]$ is the subgraph induced by the vertices which are counted by $\rho(T)$. We described a bn-independent broadcast for trees based on the number and type of branches and the tree induced by all vertices on paths joining the branches, G_{int}. This gave us a lower bound, $\alpha_{bn}(T) \geq n - b(T) - |W_i(T)| + \alpha(G_{int}(T))$. The upper and lower bounds allowed us to determine $\alpha_{bn}(T)$ exactly for many trees. If T is a tree of order n such that $b(T) \geq 1$, $W_i(T) = \varnothing$, and $R_T = \varnothing$ or $G[R_T] = \overline{K_s}$ for some integer $s \geq 1$, then $\alpha_{bn}(T) = n - b(T) + \rho(T)$.

We also described trees for which $\alpha_{bn}(T)$ lies strictly between the two bounds. Let T be a tree with order n and branch number $b(T) = 2$. Let b_1, b_2 be the two branch vertices. Then $\alpha_{bn}(T) = n - 1 - \min\{\lceil \frac{d(b_1,b_2)}{2} \rceil, \text{loss}(b_1), \text{loss}(b_2)\}$. If $2 \leq \text{loss}(b_1) \leq \text{loss}(b_2) < \lceil \frac{d(b_1,b_2)}{2} \rceil$. Then

$$n - b(T) - |W_i(T)| + \alpha(G_{int}(T)) < \alpha_{bn}(T) < n - b(T) - \rho(T).$$

For another formula, let T be a tree such that $\rho(T) = 0$ and $|W_i(T)| = \alpha(G_{int}(T))$. Then

$$\alpha_{bn}(T) = n - b(T).$$

We gave an example of how Theorem 6.1.16 can be used to write an algorithm to output an α_{bn}-broadcast for trees with well-defined structures. Algorithm 1 outputs an α_{bn}-broadcast for a tree with $B(T) \cong K_{1,k}$, $k \geq 2$.

We made some initial observations to limit the structure of α_{bnr}-broadcasts. And, finally, recalling that $\alpha_{bn}(G)$ can be used to bound $\alpha_h(G)$ we note that

$$\alpha_h(G) < 2\min\{(n - b(T_n) + \rho(T_n)) : T_n \text{ is a spanning tree of } G\}.$$

Chapter 7

Conclusion

We studied definitions for independent broadcasts from three different categories: boundary independence, hearing independence and set independence. Each of these three categories is further divided into three definitions of independence requiring, respectively, no additional conditions, irredundance, and minimal domination. To our knowledge, hearing independence and hearing independence with minimal domination are the only definitions which have been studied prior to this dissertation The weight of the minimum maximal and maximum independent broadcast for a graph G are the parameters that we studied and we illustrated that all nine definitions are distinct in this regard. We discovered that the boundary independence definition is the goldilocks of broadcast independence. Specifically, without forcing minimal domination, it most closely fits an inequality chain similar to that for independence, namely,

$$\gamma_b(G) \leq i_{bn}(G) \leq \alpha_{bn}(G) < 2\Gamma_b(G).$$

Also, bn-independent broadcasts, which are easier to study, have a nice relationship with h-independent broadcasts allowing us to share results between the two definitions. After some preliminary investigation and results on paths and grids, we discovered that the maximum independence parameter is more interesting than the minimum maximal parameter and our attention turned to investigating and comparing $\alpha_h(G)$, $\alpha_{bn}(G)$, $\alpha_{bnr}(G)$, and $\Gamma_b(G)$. Our most significant results were determining the weight of a maximum bn-independent broadcast for many classes of trees through formulas and algorithms as well as finding an upper and a lower bound for $\alpha_{bn}(G)$ for graphs in general.

These results for $\alpha_{bn}(G)$ are based on six main observations. Our first observation allowed us to get upper bounds on bn-independent broadcasts on graphs by working with trees. Given a bn-independent broadcast f on a graph G, f is a bn-independent broadcast on any connected spanning subgraph of G. Hence, $\alpha_{bn}(G) \leq \min\{\alpha_{bn}(T) : T \text{ is a spanning tree of } G\}$. Our second observation was that, for bn-independent broadcasts, the edge sets covered by each broadcasting vertex are disjoint. This second observation led us to our first upper bound for graphs G of order n: $\alpha_{bn}(G) \leq n - 1$. We characterized the graphs meeting this bound: spiders (and paths). Specifically, no tree with two or more branch vertices meets this bound. Our third and fourth observations are useful for creating an α_{bn}-broadcast on trees that adheres to a limited structure. We showed that in an α_{bn}-broadcast on a tree no leaf hears a nonleaf and also that there always exists an α_{bn}-broadcast in which no nonleaf broadcasts with a value greater than one. Using an α_{bn}-broadcast on a tree T which meets these structural conditions allowed us to draw conclusions about broadcasts on trees which do not meet our initial upper bound, i.e. those with two or more branch vertices. We observed restrictions on the way in which broadcasting leaves can overdominate branch vertices. This led to our fifth observation based on the way in which the number and type of branch vertices in a tree impact the weight of a maximum bn-independent broadcast, namely $\alpha_{bn}(T) \leq n - b(T) + \rho(T)$.

For our sixth observation, we followed our basic ideas about branch vertices and the existence of bn-broadcasts with a particular structure to produce a reasonably large weight bn-independent broadcast on any tree based on the structure of the tree. This gave us a lower bound met by large classes of trees, and the result

$$n - b(T) - W_i(T) + \alpha(G_{int}(T)) \leq \alpha_{bn}(T) \leq n - b(T) + \rho(T).$$

Although many trees meet these upper and lower bounds, some meeting both, there are trees which fall in between. We conjectured an upper bound which will meet a greater number of trees, leaving an open question:

Question 6.1.17 *Is it true that for any tree T of order n and $b(T) \geq 1$,*

$$\alpha_{bn}(T) \leq n - b(T) + \alpha(G[R_T])?$$

We now summarize our work in greater detail. Our process started in Chapter 1 with definitions and basic observations. In Chapter 2, we presented and used existing results on broadcasts and irredundance to get initial bounds for our new parameters. Using the fact that efficient broadcasts exist for all graphs G, we showed that $i_{bnd}(G) = i_{hd}(G) = i_{sd}(G) = \gamma_b(G) \leq \min\{\text{rad}(G), \gamma(G)\}$ and $\gamma_b(G) \leq i_{bn}(G), i_h(G) \leq \text{rad}(G)$ with equality for radial graphs. Using the Ball graph of a very efficient broadcast f on G, we were able to show that $i_{bn}(G) \leq \gamma_g(G) + \lceil \frac{V_f^+}{3} \rceil$. Since irredundant broadcasts are not necessarily dominating and dominating broadcasts are irredundant, we observed that $i_{sr}(G), i_{hr}(G), i_{sr}(G) \leq \gamma_b(G)$ and that the inequality is strict. We noted that all the inequalities requiring irredundance are bounded below by $ir_b(G)$.

For the maximums, we noted that parameters which are irredundant are bounded above by $IR_b(G)$ and that parameters that are minimal dominating are bounded above by $\Gamma_b(G)$. For a *bn*-, *bnr*- or *bnd*-independent broadcast f we used the fact that any edge is covered by at most one broadcasting vertex to show that
$\sigma(f) \leq m - \sum_{v_f^+} \deg(v) + |V_f^+|$. Observing that removal of edges maintains *bn*-independence, we showed that for a graph G of order n,

$$\alpha_{bn}(G) \leq \min\{\alpha_{bn}(T) : T \textit{ is a spanning tree of } G\} \leq n - 1.$$

We characterized the graphs which meet this bound as spiders (and paths). We then focused on the structure of maximal *bn* and *bnr*-independent broadcasts on trees and determined that in any $\alpha_{bn}(T)$-broadcasts no leaf hears a nonleaf and, although this in not the case for *bnr*-independence, there always is an $\alpha_{bnr}(T)$-broadcasts such that no leaf hears a nonleaf. For all trees, we showed that there exists an $\alpha_{bn(bnr)}(T)$-broadcast such that $f(v) = 1$ or $\deg(v) = 1$ or both. For $\alpha_{bn}(T)$-broadcasts, if the number of vertices broadcasting with a strength of 1 is maximized, then $PB_f(v) = \emptyset$ for all $v \in V_f^+$. We found specific examples showing that both $\alpha_{bn}(G)$ and $\alpha_{bnr}(G)$ are incomparable with $\Gamma_b(G)$.

In Chapter 3 we studied and determined values for all our parameters on paths. For s-independence, the broadcasting vertices form an independent set and for maximality, every vertex in the set broadcasts with a strength equal to its eccentricity. Exact formulas for $i_s(P_n)$ and $\alpha_s(P_n)$ are given in Theorem 3.1.10 and Theorem 3.2.3.

By examining the edges covered and uncovered by an independent broadcast, and the existence of independent broadcasts f with $V_f^+ = V_f^1$, we determined exact values for all other independence parameters for paths in terms of their size. For the minimums, for $n \geq 1$,

$$i_{sr,hr,bnr,bnd,hd,sd}(P_n) = \lceil \frac{n}{3} \rceil$$

and, for all $n \neq 3$,

$$i_{bn,h}(P_n) = \lceil \frac{2n}{5} \rceil.$$

For the maximums, $\alpha_h(P_n) = 2(n-1)$, Erwin's bound. All other maximum independent broadcast types are met by a broadcast of strength diam(G) from a leaf. Or equivalently, for $n > 1$,

$$\alpha_{sr,hr,bnr,bnd,hd,sd,bn}(P_n) = n - 1.$$

In Chapter 4, we presented some existing results on h-independence and developed new results for all other independence parameters on grids. For the minimum independence parameters, since grid graphs are radial, we showed, for all $2 \leq m \leq n$:

$$\gamma_b(G_{m,n}) = i_{bn}(G_{m,n}) = i_{bnd}(G_{m,n}) = i_h(G_{m,n}) = i_{hd}(G_{m,n}) = i_{sd}(G_{m,n}) = \text{rad}(G_{m,n})$$

where:

$$\text{rad}(G_{m,n}) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \text{if } m = 2 \text{ or } 3 \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor & \text{otherwise.} \end{cases}$$

For the minimums which require irredundance, we obtained:

$$i_{sr}(G_{m,n}) \leq i_{hr}(G_{m,n}) \leq i_{bnr}(G_{m,n}) \leq \text{rad}(G_{m,n}).$$

If our conjecture that there exists a dominating $i_{sr}(G_{m,n})$-broadcast holds, then all minimum irredundant independence parameters will equal rad$(G_{m,n})$.

For the maximums, we presented Bouchemakh and Zemir 's results for h-independence:

$$\alpha_h(G_{m,n}) = \max\{2(\text{diam}(G_{m,n}) - 1), \alpha(G_{m,n})\}$$

for all $1 \leq m \leq n, (m,n) \neq (5,5)$ and

$$\alpha_h(G_{5,5}) = 15.$$

For boundary independence, for $G_{m,n}$ where $2 \leq m \leq n$, we showed that:

$$\alpha_{bnd}(G_{m,n}) = \alpha_{bn}(G_{m,n}) = \alpha_{bnr}(G_{m,n}) = \alpha(G) = \lceil \frac{mn}{2} \rceil.$$

Our boundary independence results generalized to 2-connected bipartite graphs. Given a 2-connected bipartite graph G:

$$\alpha_{bnr}(G) = \alpha_{bnd}(G) = \alpha_{bn}(G) = \alpha(G) \geq \lceil \frac{|V(G)|}{2} \rceil.$$

In Chapter 5 we compared $\alpha_h(G)$, $\alpha_{bn}(G)$, $\alpha_{bnr}(G)$, and $\Gamma_b(G)$. We showed, by example, that $\Gamma_b(G) - \alpha_{bn}(G)$ and $\Gamma_b(G) - \alpha_{bnr}(G)$ are unbounded for graphs in general. We gave a second example, showing that $\Gamma_b(T) - \alpha_{bnr}(T)$ is also unbounded for trees. We gave two examples showing that $\alpha_{bnr}(G) - \Gamma_b(G)$ and $\alpha_{bn} - \Gamma_b(G)$ are unbounded for trees and for graphs in general.

For the ratios, we showed that $\alpha_{bnr}(G)/\Gamma_b(G) \leq \alpha(G)/\Gamma_b(G) < 2$ for all graphs G. This bound is tight for $\alpha_{bn}(G)/\Gamma_b(G)$. We gave an example with $\alpha_{bnr}(T)/\Gamma_b(T) = \frac{14}{13}$, inspiring the the following open problem:

Problem 5.3.1 *Determine the smallest constant c such that $\alpha_{bnr}(T)/\Gamma_b(T) \leq c$ for all trees T. Similarly, determine the smallest constant k such that $\alpha_{bnr}(G)/\Gamma_b(G) \leq k$ for more general graphs G.*

Finally, we showed that for bipartite graphs with $n \geq 2$,

$$\Gamma_b(G)/\alpha_{bn}(G) \leq \Gamma_b(G)/\alpha_{bnr}(G) < 2.$$

We gave an example to show that $\Gamma_b(G)/\alpha_{bn}(G)$ and $\Gamma_b(G)/\alpha_{bnr}(G)$ are both unbounded for graphs in general.

In Chapter 6 we examined *bn*- and *bnr*-independent broadcasts on trees. We furthered our observations on the structure of *bn*-independent broadcasts on trees and as a result we were able to improve the upper bound on $\alpha_{bn}(T)$. We presented a strategy for creating an α_{bn}-broadcast on trees with exactly two branch vertices and an algorithm for creating an α_{bn}-broadcast on trees for which the subgraph induced by $G_{int} \cup B(T)$ is a star.

As stated above, *bn*-independence has an inequality chain closest to that for independence. Namely,

$$\gamma_b(G) \le i_{bn}(G) \le i(G) \le \alpha(G) \le \alpha_{bn}(G) \diamond \Gamma_b(G).$$

Also note that:

$$\alpha_{bn}(G) \le \min\{n - b(T) + \rho(T) : T \text{ is a spanning tree of } G\},$$

$$\frac{1}{2}\alpha_h(G) < \alpha_{bn}(G) < 2\Gamma_b(G),$$

and

$$i_{bn}(G) \le \mathrm{rad}(G) \le \mathrm{diam}(G) \le \alpha_{bn}(G).$$

The other new independence parameter of significant interest is *bnr*-independence. Its inequality chain is not as nice as that for *bn*-independence and it is much trickier to work with. Since a minimal dominating broadcast is irredundant but an irredundant broadcast is not necessarily dominating we obtained

$$ir_b(G) \le i_{bnr}(G) \le \gamma_b(G) \le \gamma(G) \le i(G) \le \alpha(G) \le \alpha_{bnr}(G) \diamond \Gamma_b(G) \le IR_b(G).$$

We presented some preliminary results on understanding the structure of a *bnr*-broadcast on trees, leaving an interesting open problem regarding $\alpha_{bnr}(T_n)$.

Conjecture 6.2.5 *Let T be a tree that satisfies the structure described in condition (ii) from Lemma 6.2.2. If $W_{int}(T) \neq \emptyset$, then $\alpha_{bnr}(T) = \mathrm{diam}(T)$, and if $W_{int}(T) = \emptyset$ then $\alpha_{bnr}(T) = \mathrm{diam}(T) + 1$ and T satisfies condition (i).*

And, finally, recalling that α_{bn} can be used to bound α_h we noted that

$$\alpha_h(G) < 2\min\{(n - b(T_n) + \rho(T_n)) : T_n \text{ is a spanning tree of } G\}.$$

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