IRREDUNDANCE IN THE QUEENS' GRAPH

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Abstract

The vertices of the queens' graph Q_n are the squares of an $n \times n$ chessboard and two squares are adjacent if a queen placed on one covers the other. Informally, a set I of queens on the board is irredundant if each queen in I covers a square (perhaps its own) which is not covered by any other queen in I. It is shown that the cardinality of any irredundant set of vertices of Q_n is at most $\lfloor 6n + 6 - 8\sqrt{n+3} \rfloor$ for $n \geq 6$. We also show that the bound is not exact since $IR(Q_8) \leq 23$.

1. Introduction

The lower (upper) domination numbers $\gamma(G), (\Gamma(G))$, independence numbers $i(G)$ $(\beta(G))$ and irredundance numbers $ir(G)$ $(IR(G))$ of a graph G are respectively the smallest (largest) cardinalities of minimal dominating, maximal independent and maximal irredundant vertex sets of G.

These six parameters are well-studied in the literature (see [3]) and satisfy the following chain of inequalities:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

In particular there has been considerable recent interest in the evaluation of these parameters for graphs defined from $n \times n$ chessboards ([2]). This is perhaps due to the fact that two of these problems, namely the determination of γ and i for the Queens' graph

Q_n (defined in the following paragraph), have defied all efforts at solution for at least a hundred years (see [2]).

The Queens' graph Q_n has the n^2 squares of the chessboard as its vertex set and two vertices are adjacent if a queen placed on one covers the other, *i.e.* if the two squares are on the same *line* (row, column or diagonal) of the board.

The survey paper ([2]) gives an excellent account of recent results on the six parameters for Q_n. Since that paper was written, Weakley ([5]) and Burger, Cockayne and Mynhardt ([1]) have established new values of $\gamma(Q_n)$.

This paper is concerned with the upper irredundance number $IR(Q_n)$. Informally, a set I of queens on the board is irredundant if each queen in I covers a square (perhaps its own) which is not covered by any other queen in I. Weakley ([5]) has shown $\Gamma(Q_n)$ (and hence $IR(Q_n)$) $\geq 2n-5$ and McCrae ([4]) has used computer techniques to generate examples which show this lower bound is not exact. In the present work we show that $IR(Q_n) \leq \lfloor 6n+6-8\sqrt{n+3} \rfloor$ and show that our bound is also not exact since $IR(Q_8) \leq 23$.

2. The Upper Bound for $IR(Q_n)$

The following notation and terminology will be required. The rows and columns are numbered in obvious matrix fashion. The *sum-diagonal* numbered k contains the squares (i, j) such that $(i+j)-(n+1)=k$. The *difference-diagonal* numbered k contains the squares (i, j) satisfying $(i-j)=k$. There are $(2n-1)$ sum-diagonals (difference-diagonals) which are numbered $0, \pm 1, \pm 2, \ldots, \pm(n-1)$.

For a vertex v of Q_n, $r(v)$, $c(v)$, $d(v)$, $s(v)$ denote respectively the row, column, difference-diagonal and sum-diagonal which contain v. A set I of vertices of a graph G is *irredundant* if for each $v \in I$

$$N[v]-\bigcup_{u \in I-\{v\}} N[u] \neq \emptyset .$$

Let I be an irredundant vertex set of Q_n, A be the set of isolated vertices of $G[I]$ where $|A|=\alpha \leq n$ (since $\beta(Q_n)=n$) and $X=\{x_1, \ldots, x_t\}=I-A$. Since I is irredundant, for each $i=1, \ldots, t$, x_i is adjacent to $y_i \in V-I$ (a *private neighbour of* x_i) which is not adjacent to any vertex of $I-\{x_i\}$. Vertices x_i and y_i are on a line ℓ_i. Let $\{y_1,\ldots,y_t\}=Y$, and $Z=V-(I\cup Y)$. The private neighbour property implies that $\ell_1,\ldots,\ell_t$ are distinct. Define $U=\{\ell_1,\ldots,\ell_t\}$.

We begin with a few simple propositions.

Proposition 1. *If x_i,x_j (or y_i,y_j) are adjacent on a line ℓ, then $\ell\notin U$.*

Proof. If $\{x_i,x_j\}\subseteq\ell$ and $\{x_k,y_k\}\subseteq\ell\in U$, then one of i,j (say i) is distinct from k. Then $\{x_i,x_k\}\subseteq\ell$ contradicting the private neighbour property of y_k. (Similar proof for $\{y_i,y_j\}\subseteq\ell$.) ∎

Proposition 2. *Let ℓ be any line. If $\{x_i,x_j\}\subseteq\ell$, then $\{y_k,y_\ell\}\not\subseteq\ell$ and conversely.*

Proof. Suppose $\{x_i,x_j,y_k,y_\ell\}\subseteq\ell$. Clearly the private neighbour property is contradicted. ∎

Proposition 3. *Let ℓ_i be the line defined by x_i,y_i. Then none of the other six lines which contain x_i,y_i are in U.*

Proof. Suppose $m\neq\ell_i$ contains x_i and $m\in U$. Then for some j, $\{x_j,y_j,x_i\}\subseteq m$, contrary to the private neighbour property of y_j. (Similar proof for $y_i\in m$.) ∎

Proposition 4. *If $v\in A$, then $\{r(v),\ c(v),\ d(v),\ s(v)\}\cap U=\emptyset$.*

Proof. Suppose $v\in A$ is on line $\ell\in U$. Then for some i, $\{x_i,y_i,v\}\subseteq\ell$, contrary to the private neighbour property. ∎

Proposition 5. *If $v\in A$, then $N(v)\subseteq Z$.*

Proof. Vertex v is isolated in $G[I]$ and is not adjacent to $y\in Y$ (for otherwise y is not a private neighbour). ∎

Proposition 6. *For each $i=1\ldots,t$, $\ell_i-\{x_i,y_i\}\subseteq Z$.*

Proof. Let $w_i\in\ell_i-\{x_i,y_i\}$. If $w_i=x\in I$, then both x and x_i are adjacent to y_i. If $w_i=y_j\in Y$, then x_i is adjacent to both y_i and y_j. In each case the private neighbour property is contradicted. ∎

Now suppose that U contains r, c, s, d rows, columns, sum-diagonals and difference-diagonals respectively.

Proposition 7. *If $r+\alpha \geq n-4$ (or $c+\alpha \geq n-4$), then $|I| \leq 3n$.*

Proof. Since A is independent, the rows occupied by vertices of A are distinct and the r rows of U are distinct. Further by Proposition 4, these two sets of rows are disjoint. By Propositions 5 and 6, Z contains $(n-2)$ elements of r rows of U and $(n-1)$ elements from α additional rows. Since $|Z| = n^2 - 2t - \alpha$,

$$r(n-2) + \alpha(n-1) \leq n^2 - 2t - \alpha.$$

Therefore

$$\begin{aligned} 2t &\leq n^2 - (r+\alpha)n + 2r \\ &\leq n^2 - (n-4)n + 2r \\ &= 4n + 2r. \end{aligned}$$

Hence $|I| = t + \alpha \leq 2n + (r+\alpha) \leq 3n$. ∎

We now establish the upper bound for $IR(Q_n)$.

Theorem 8. *For $n \geq 6$, $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n+3} \rfloor$.*

Proof. If $r+\alpha$ (or $c+\alpha$) $\geq n-4$, then by Proposition 7, $|I| \leq 3n \leq 6n+6-8\sqrt{n+3}$ for $n \geq 6$. Hence assume $r+\alpha \leq n-5$ and $c+\alpha \leq n-5$. Assume, without loss of generality, that $d \leq s$ and re-label X, Y so that $\ell_1, \ldots, \ell_s$ are sum-diagonals. Let $r_1, \ldots, r_s$ (respectively $r'_1, \ldots, r'_s$) be the rows occupied by $x_1, \ldots, x_s$ ($y_1, \ldots, y_s$). Note that there may be repetitions among $r_1, \ldots, r_s$ and among $r'_1, \ldots, r'_s$, but no r_i is equal to an r'_j ($r_i \neq r'_i$ since x_i, y_i are on a sum-diagonal and $r_i = r'_j$ ($j \neq i$) contradicts the private neighbour property).

Suppose L is the set of lines which are neither in U nor pass through any vertex of A. Let λ be the largest multiplicity of a row in the sequence $r_1, \ldots, r_s$. Then there are at least $\lceil s/\lambda \rceil$ distinct rows in the sequence. These rows are in L (by Proposition 3 and the fact that no vertex of A is adjacent to vertices of $X \cup Y$). Further the λ vertices of X which are on the same row, occupy distinct columns and distinct difference-diagonals. These 2λ lines are also in L. Hence we have a set of lines $L_1 \subseteq L$ satisfying (using elementary calculus)

$$|L_1| = \left\lceil \frac{s}{\lambda} \right\rceil + 2\lambda \geq \frac{s}{\lambda} + 2\lambda \geq 2\sqrt{2s}.$$

Applying the same argument to the sequence $r'_1, \ldots, r'_s$, we obtain a set $L_2 \subseteq L$ with $|L_2| \geq 2\sqrt{2s}$ and $L_1 \cap L_2 = \emptyset$ (otherwise x_i, y_j are on same row, column or difference-diagonal which contradicts the private neighbour property). We conclude $|L| \geq 4\sqrt{2s}$.

The total number of lines is $6n - 2$. Hence

$$t + 4\alpha + 4\sqrt{2s} \leq 6n - 2$$

and

$$|I| = t + \alpha \leq 6n - 2 - 4\sqrt{2s} - 3\alpha.$$

Therefore

$$|I| \leq f_1(s) = 6n - 2 - 4\sqrt{2s}. \tag{1}$$

Moreover

$$\begin{aligned} |I| &= (r + \alpha) + c + s + d \\ &\leq (n - 5) + (n - 5) + 2s. \end{aligned}$$

Therefore

$$|I| \leq f_2(s) = (2n - 10) + 2s. \tag{2}$$

Hence

$$|I| \leq \max_{1 \leq s \leq 2n-3} \Big(\min\big(f_1(s), f_2(s)\big)\Big).$$

The maximum occurs where $f_1(s) = f_2(s)$.

Solving the quadratic for $\sqrt{2s}$, we find that the maximum occurs when $\sqrt{2s} = 2\sqrt{n+3} - 2$ and so from (1)

$$\begin{aligned} |I| &\leq 6n - 2 - 4\big(2\sqrt{n+3} - 2\big) \\ &= 6n + 6 - 8\sqrt{n+3}. \end{aligned}$$

∎

3. An upper bound for $IR(Q_8)$

In this section we show that the bound of Section 2 is not exact. The bound for $IR(Q_8)$ is 27. However we prove

Theorem 9. $IR(Q_8) \leq 23$.

Proof. Suppose I is an irredundant set of 24 vertices of Q_8. Then $t = 24 - \alpha$ and $|Z| = 64 - 2t - \alpha = 16 + \alpha$. Suppose $\alpha \geq 2$ and a_1, $a_2 \in A$. By Proposition 5, $N(a_1) \cup N(a_2) \subseteq Z$ and the minimum degree of Q_8 is 21. Hence

$$|Z| \geq \big|N(a_1)\big| + \big|N(a_2)\big| - \big|N(a_1) \cap N(a_2)\big|$$

$$= 42 - \big|N(a_1) \cap N(a_2)\big|.$$

But for any n and non-adjacent v_1, $v_2 \in V(Q_n)$, $\big|N(v_1) \cap N(v_2)\big| \leq 12$, hence $|Z| = 16 + \alpha \geq 30$, which is impossible since $\alpha \leq \beta(Q_n) = 8$. We have shown that $\alpha = 0$ or 1.

Suppose U contains 4 or more lines which contain 8 squares (*i.e.* 4 or more rows, columns or major diagonals). Let four of these lines be $\ell_1, \ldots, \ell_4$ and $Z_i = \ell_i - \{x_i, y_i\}$. By Proposition 6, $\bigcup_1^4 Z_i \subseteq Z$. Therefore

$$|Z| \geq \left|\bigcup_1^4 Z_i\right| \geq \sum_1^4 |Z_i| - \sum_{1 \leq i \leq j \leq 4} |Z_i \cap Z_j|$$
$$\geq 24 - 6 = 18.$$

Hence $16 + \alpha \geq 18$, a contradiction.

It follows (using $\alpha \in \{0, 1\}$) that U contains at least 20 lines from the set of sum-diagonals and difference-diagonals numbered $\pm 1, \ldots, \pm 6$.

Without losing generality, U contains at least 10 sum-diagonals from this list say $s_1, \ldots x_{10}$ and these are disjoint. By Proposition 6

$$|Z| \geq \sum_{i=1}^{10} \big|s_i - \{x_i, y_i\}\big| \geq 2(0 + 1 + 2 + 3 + 4) = 20.$$

Therefore $16 + \alpha \geq 20$, a contradiction which shows that there is no 24-vertex irredundant set. ∎

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