SPECTRAL FLOW IN TYPE I AND II FACTORS

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SPECTRAL FLOW IN TYPE I AND II FACTORS – A NEW APPROACH

by

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Introduction

We denote the space of self-adjoint (Breuer-) Fredholm operators in a semifinite factor, which are neither essentially positive nor essentially negative, by $\mathcal{F}_*^{sa}$. If the factor is of type I_∞, Atiyah and Lusztig [APS1,3] have defined the *spectral flow* of a continuous path in $\mathcal{F}_*^{sa}$ to be the net number of eigenvalues (counted with multiplicities) which pass through 0 in the positive direction as one moves from the initial point of the path to the final point. This definition is appealing geometrically as an "intersection number" but is a little difficult to make precise (see [BW] and [Ph] for different approaches to this). More importantly, there is no obvious generalization of this definition if the factor is of type II_∞, where the spectrum of a self-adjoint Breuer-Fredholm operator is not discrete in a neighbourhood of zero. J. Kaminker has described this as the problem of counting "moving globs of spectrum".

In his 1993 Ph.D. thesis, V.S. Perera [P1,P2] gave a definition of the spectral flow of a *loop* in $\mathcal{F}_*^{sa}$ for a II_∞ factor, N. He showed that the space, $\Omega(\mathcal{F}_*^{sa})$, of loops based at a unitary $(2P-1)$ in $\mathcal{F}_*^{sa}$, is homotopy equivalent to the space, $\mathcal{F}$, of all Breuer-Fredholm operators in the II_∞ factor, PNP. Since Breuer [B1,B2] showed that the index map $\mathcal{F} \longrightarrow \mathbf{R}$ classifies the connected components of $\mathcal{F}$, Perera defines spectral flow as the composition $sf : \Omega(\mathcal{F}_*^{sa}) \longrightarrow \mathcal{F} \longrightarrow \mathbf{R}$ and so obtains the isomorphism $\pi_1(\mathcal{F}_*^{sa}) \cong \mathbf{R}$. He also showed that this gives the "heuristically correct" answer for a simple family of loops.

While this is an important and elegant result, it has a couple of weaknesses. Firstly, since the map sf is not defined directly and constructively on individual loops it is not clear why spectral flow is counting "moving globs of spectrum". Secondly, the map does not extend to paths which are not loops, in any obvious way.

We now outline our solution to these problems. Let χ denote the characteristic function of the interval $[0, \infty)$. If $\{B_t\}$ is any continuous path in $\mathcal{F}_*^{sa}$, then $\{\chi(B_t)\}$ is a discontinuous path of projections whose discontinuities arise precisely because of spectral flow. For example, if $t_1 < t_2$ are neighbouring path parameters and the projections $P_i = \chi(B_{t_i})$ commute, then the spectral flow from t_1 to t_2 should be $\text{trace}(P_2 - P_1 P_2) =$ (amount of nonnegative spectrum gained) minus $\text{trace}(P_1 - P_1 P_2) =$ (amount of nonnegative spectrum lost). However, this is clearly the index of the operator $P_1P_2 : P_2(H) \to P_1(H)$. If these projections do not commute then one can still make sense of this index provided $\pi(P_1) = \pi(P_2)$ in the Calkin algebra. This notion was called *essential codimension* by Brown, Douglas and Fillmore [BDF] in the type I_∞ case and denoted by $ec(P_1, P_2)$. Perera [P1,P2] defined the obvious extension of this concept to II_∞ factors and used it to explain why his definition of spectral flow gives the "right" answer in a representative family of simple loops. Our new ingredient here is the fact that the operator $P_1P_2 : P_2(H) \to P_1(H)$ is always a (Breuer-) Fredholm operator provided $\|\pi(P_1) - \pi(P_2)\| < 1$. Since we can (easily) show that the mapping $t \mapsto \pi(\chi(B_t))$ is continuous, we can partition the parameter interval $a = t_0 < t_1 < \cdots < t_k = b$ so that on each small subinterval the projections $\pi(\chi(B_t))$ are all close. Letting $P_i = \chi(B_{t_i})$ for $i = 0, 1, \cdots, k$ we then define:

$$sf(\{B_t\}) = \sum_{i=1}^{k} \mathrm{ind}(P_{i-1}P_i).$$

With a little effort this works equally well in both the type I_∞ and II_∞ settings and agrees with all previous definitions of spectral flow where they exist. A simple lemma is the key to showing that sf is well-defined and (path-) homotopy invariant. Defining $\mathrm{Hom}(\mathcal{F}_*^{sa})$ to be the homotopy groupoid of $\mathcal{F}_*^{sa}$, we prove the following theorem.

Theorem: *If N is a factor of type I_∞ (respectively II_∞) then sf as defined above is a homomorphism from $\mathrm{Hom}(\mathcal{F}_*^{sa})$ to $\mathbf{Z}$ (respectively, $\mathbf{R}$) which restricts to an isomorphism of $\pi_1(\mathcal{F}_*^{sa})$ with $\mathbf{Z}$ (respectively $\mathbf{R}$).*

We note that to show that sf is one-to-one on $\pi_1(\mathcal{F}_*^{sa})$ we must rely on Perera's result that $\Omega(\mathcal{F}_*^{sa}) \simeq \mathcal{F}$.

We remark that in paragraphs 7, 8 and 9 of the introduction to [APS3] the authors appear to be hinting at the existence of a notion of spectral flow (for paths of self-adjoint Breuer-Fredholm operators in a II_∞ factor) to be used as a possible tool in an alternate proof of their index theorem for flat bundles. We would be delighted if this paper helps in any way with this program. This may be more than just wishful thinking: in their recent work on the Maslov Index and Spectral Flow, Booβ-Bavnbek and Furutani [BF] have used in an essential way the ideas of the author's previous work on spectral flow, [Ph]. Our point is that any new approach to spectral flow is potentially useful.

Notation

Throughout this paper we will consider N, a semifinite factor (of type I_∞ or II_∞) acting on a separable Hilbert space (usually not named). We will denote by Tr a fixed faithful, normal semifinite trace on N (with the usual normalization if N is of type I_∞). The norm-closed 2-sided ideal in N generated by the elements of finite trace will be denoted by $\mathcal{K}_N$. The quotient algebra $N/\mathcal{K}_N$ will be denoted by $\mathcal{Q}_N$ and will be called the (generalized) Calkin algebra. We will let π denote the quotient mapping $N \to \mathcal{Q}_N$.

We will let $\mathcal{F}$ denote the space of all (Breuer-) Fredholm operators in N, *i.e.*,

$$\mathcal{F} = \{T \in N \mid \pi(T) \text{ is invertible in } \mathcal{Q}_N\}.$$

The nontrivial component of the space of self-adjoint (Breuer-) Fredholm operators in N will be denoted by $\mathcal{F}_*^{sa}$, *i.e.*,

$$\mathcal{F}_*^{sa} = \{T \in \mathcal{F} \mid T = T^* \text{ and } \pi(T) \text{ is neither positive nor negative}\}.$$

We will let $\chi = \chi_{[0,\infty)}$ denote the characteristic function on $\mathbf{R}$ of the interval $[0,\infty)$, so that if T is a self-adjoint operator, then $\chi(T)$ is the spectral projection for T corresponding to the set $sp(T) \cap [0,\infty)$.

Other specific notations will be introduced as needed, but we refer the reader to the appendix for our notations for index.

§1. Essential Codimension

If P and Q are infinite projections in the semifinite factor N we wish to define the *essential codimension* of P in Q whenever $\|\pi(P) - \pi(Q)\| < 1$, where $\pi : N \to \mathcal{Q}_N$ is the Calkin map. Once we show that the operator $PQ \in PNQ$ is a (Breuer-) Fredholm operator in the sense of the appendix, then we will define the essential codimension of P in Q to be $\mathrm{ind}_{(P\text{-}Q)}(PQ)$.

1.1 Lemma: *If P and Q are infinite projections in the semifinite factor N and if $\|\pi(P) - \pi(Q)\| < 1$ where $\pi : N \to \mathcal{Q}_N$ is the Calkin map, then $PQ \in PNQ$ is $(P\text{-}Q)$-Fredholm in the sense of the appendix.*

Proof. Since

$$\|\pi(PQP) - \pi(P)\| \leq \|\pi(Q) - \pi(P)\| < 1$$

and

$$\pi(P)(N/\mathcal{K}_N)\,\pi(P) = (PNP)/\mathcal{K}_{PNP},$$

we see that PQP is a Breuer-Fredholm operator in the semifinite factor, PNP. Thus, *in the notation of the appendix*, $\ker_P(QP) \subseteq \ker_P(PQP)$ and so $[\ker_P(QP)] \leq [\ker_P(PQP)]$ where the latter is a finite projection in PNP. Similarly, $[\ker_Q(PQ)]$ is a finite projection in QNQ. Since the range of PQ contains the range of PQP, and since this latter operator is Breuer-Fredholm in PNP, there is a projection $P_1 \leq P$ so that $Tr(P - P_1) < \infty$ and the range of P_1 is contained in the range of PQ. That is, PQ is (P-Q)-Fredholm. ∎

1.2 Definition: If P and Q are infinite projections in the semifinite factor N and if $\|\pi(P) - \pi(Q)\| < 1$ then the *essential codimension of P in Q*, denoted $ec(P,Q)$, is the number $\mathrm{ind}_{(P\text{-}Q)}(PQ)$. If $P \leq Q$ it is exactly the codimension of P in Q.

1.3 Lemma: *If P_1, P_2 and P_3 are infinite projections in the semifinite factor N and if $\|\pi(P_1) - \pi(P_2)\| < \frac{1}{2}$ and $\|\pi(P_2) - \pi(P_3)\| < \frac{1}{2}$ then $ec(P_1,P_3) = ec(P_1,P_2) + ec(P_2,P_3)$.*

Proof. Since we also have $\|\pi(P_1) - \pi(P_3)\| < 1$, the terms in the equation are all defined. Translating the equation into the language of index and using Proposition A5 of the appendix we see that it suffices to prove that $\mathrm{ind}_{P_3NP_3}((P_1P_3)^*(P_1P_2P_3)) = 0$. But,

$$\begin{aligned}
&\|\pi((P_1P_3)^*(P_1P_2P_3)) - \pi(P_3)\| \\
&= \|\pi(P_3P_1P_2P_3) - \pi(P_3)\| \\
&\leq \|\pi(P_1P_2) - \pi(P_3)\| \\
&\leq \|\pi(P_1P_2) - \pi(P_2)\| + \|\pi(P_2) - \pi(P_3)\| \\
&\leq \|\pi(P_1) - \pi(P_2)\| + \|\pi(P_2) - \pi(P_3)\| < 1.
\end{aligned}$$

Thus, there is a (Breuer-) compact operator k in P_3NP_3 with $\|P_3P_1P_2P_3 + k - P_3\| < 1$. Hence, $\mathrm{ind}(P_3P_1P_2P_3) = \mathrm{ind}(P_3P_1P_2P_3 + k) = 0$ as this latter operator is invertible in P_3NP_3. ∎

1.4 Remark: If P and Q are infinite projections in N with $\|P-Q\|<1$, then $ec(P,Q)=0$. To see this, note that $\|PQP-P\|\leq\|Q-P\|<1$ so that PQP is invertible in PNP and hence range $P\supseteq$ range $PQ\supseteq$ range $PQP=$ range P. Thus, range $PQ=$ range P and similarly range $QP=$ range Q so the (P-Q) index of PQ is 0.

§2. Spectral Flow

As mentioned above, we let $\chi=\chi_{[0,\infty)}$ denote the characteristic function of the interval $[0,\infty)$ so that if T is any self-adjoint operator in a von Neumann algebra $\mathcal{A}$ then $\chi(T)$ is a projection in $\mathcal{A}$.

2.1 Lemma: *If $\mathcal{A}$ is a von Neumann algebra, $\mathcal{J}$ is a closed 2-sided ideal in $\mathcal{A}$, T is a self-adjoint operator in $\mathcal{A}$ and $\pi(T)$ is invertible in $\mathcal{A}/\mathcal{J}$ (where $\pi:\mathcal{A}\to\mathcal{A}/\mathcal{J}$ is the quotient mapping), then $\chi(\pi(T))=\pi(\chi(T))$.*

Proof. Since 0 is not in the spectrum of $\pi(T)$, the left hand side is a well-defined element of the C^*-algebra $\mathcal{A}/\mathcal{J}$. Choose $\epsilon>0$ so that $[-\epsilon,\epsilon]$ is disjoint from $sp(\pi(T))$. Let $f_1\geq f_2$ be the following piecewise linear continuous functions on $\mathbf{R}$:

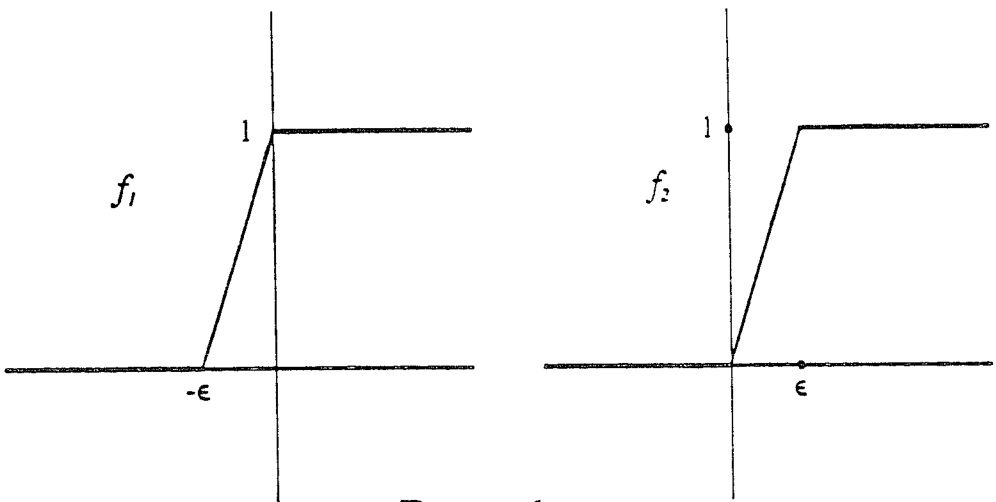

FIGURE 1

Now, $f_1\geq\chi\geq f_2$ on $\mathbf{R}$, but all three functions are equal on $sp(\pi(T))$. Thus,

$$\chi(\pi(T))=f_1(\pi(T))=\pi(f_1(T))\geq\pi(\chi(T))\geq\pi(f_2(T))=f_2(\pi(T))=\chi(\pi(T)).$$

Hence,

$$\chi\left(\pi(T)\right) = \pi\left(\chi(T)\right). \qquad \blacksquare$$

2.2 Definition: Let N be a semifinite factor with fixed semifinite, faithful, normal trace, Tr. Let $\mathcal{F}_*^{sa}$ denote the space of all self-adjoint (Breuer-) Fredholm operators in N which are neither essentially positive nor essentially negative. Let $\{B_t\}$ be any continuous path in $\mathcal{F}_*^{sa}$ (indexed by some interval $[a,b]$). Then $\{\chi(B_t)\}$ is a (generally discontinuous) path of infinite projections in N. By Lemma 2.1, $\pi\left(\chi(B_t)\right) = \chi\left(\pi(B_t)\right)$ and since the spectra of $\pi(B_t)$ are bounded away from 0, this latter path is continuous. By compactness we can choose a partition $a = t_0 < t_1 < \cdots < t_k = b$ so that for each $i = 1, 2, \cdots, k$

$$\left\|\pi\left(\chi(B_t)\right) - \pi\left(\chi(B_s)\right)\right\| < \tfrac{1}{2} \quad \text{for all } t, s \text{ in } [t_{i-1}, t_i].$$

Letting $P_i = \chi(B_{t_i})$ for $i = 0, 1, \cdots, k$ we define the *spectral flow of the path* $\{B_t\}$ to be the number:

$$sf\left(\{B_t\}\right) = \sum_{i=1}^{k} ec\left(P_{i-1}, P_i\right).$$

To see that this definition is independent of the partition, it suffices to see that it is invariant under adding a single point to the partition. However, this is exactly the content of Lemma 1.3.

2.3 Remark: If $\{B_t\}$ is a path in $\mathcal{F}_*^{sa}$ and if $t \mapsto \chi(B_t)$ is continuous, then by Remark 1.4, $sf\left(\{B_t\}\right) = 0$. That is, as expected heuristically, spectral flow can be nontrivial only when the path $t \mapsto \chi(B_t)$ has discontinuities.

2.4 Remark: For $T \in \mathcal{F}_*^{sa}$, let

$$N(T) = \left\{S \in \mathcal{F}_*^{sa} \mid \left\|\pi\left(\chi(S)\right) - \pi\left(\chi(T)\right)\right\| < \tfrac{1}{4}\right\}.$$

Then $N(T)$ is open in $\mathcal{F}_*^{sa}$ since $S \mapsto \pi\left(\chi(S)\right) = \chi\left(\pi(S)\right)$ is continuous on $\mathcal{F}_*^{sa}$. Moreover, if $S_1, S_2 \in N(T)$, then by the definition of spectral flow, all paths from S_1 to S_2 lying entirely in $N(T)$ have the same spectral flow, namely, $ec\left(\chi(S_1), \chi(S_2)\right)$.

2.5 Proposition: *Spectral flow is homotopy invariant, that is, if* $\{B_t\}$ *and* $\{B'_t\}$ *are two continuous paths in* $\mathcal{F}_*^{sa}$ *with* $B_0 = B'_0$ *and* $B_1 = B'_1$ *which are homotopic in* $\mathcal{F}_*^{sa}$ *via a homotopy leaving the endpoints fixed, then* $sf\left(\{B_t\}\right) = sf\left(\{B'_t\}\right)$.

Proof. Let $H : I \times I \to \mathcal{F}_*^{sa}$ be a homotopy from $\{B_t\}$ to $\{B'_t\}$. That is, H is continuous, $H(t,0) = B_t$ for all t, $H(t,1) = B'_t$ for all t, $H(0,s) = B_0 = B'_0$ for all s, and $H(1,s) = B_1 = B'_1$ for all s. By compactness we can cover the image of H by a finite number of open sets $\{N_1, \cdots, N_k\}$ as in Remark 2.4. The inverse images of these open sets, $\{H^{-1}(N_1), \cdots, H^{-1}(N_k)\}$ is a finite cover of $I \times I$. Thus, there exists $\epsilon_0 > 0$ (the Lebesgue number of the cover) so that any subset of $I \times I$ of diameter $\leq \epsilon_0$ is contained in some element of this finite cover of $I \times I$. Thus, if we partition $I \times I$ into a grid of squares of diameter $\leq \epsilon_0$, then the image of each square will lie entirely within some N_i. Effectively, this breaks H up into a finite sequence of "short" homotopies by restricting H to $I \times J_i$ where J_i are subintervals of I (of length $\leq \epsilon_0/\sqrt{2}$). These short homotopies have the added property that for fixed J_i we can choose a single partition of I so that for each subinterval J_ℓ of the partition, $H(J_\ell \times J_i)$ is contained in one of $\{N_1, \cdots, N_k\}$. By concentrating on the ith " short homotopy" and relabelling $N_1, \cdots, N_k$ if necessary we can assume H has the form indicated in Figure 2:

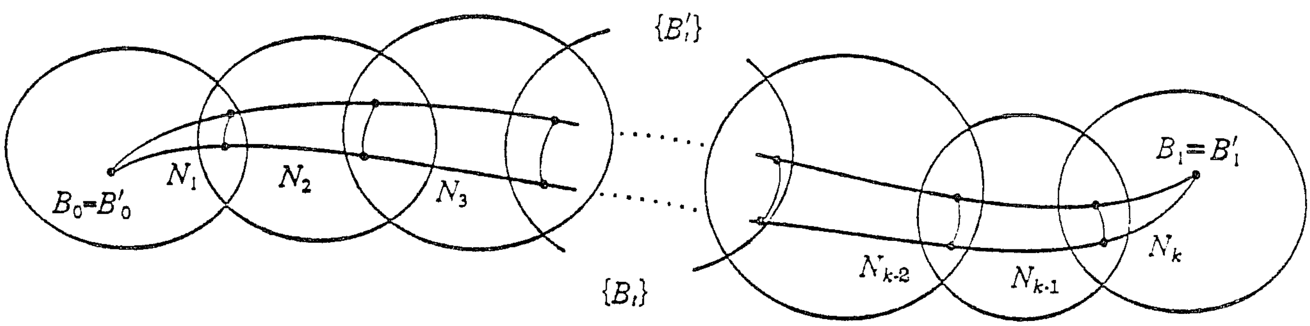

FIGURE 2

By the observation in Remark 2.4, in each of the following pairs of short paths in Figure 3, the spectral flow of the upper path equals the spectral flow of the lower path.

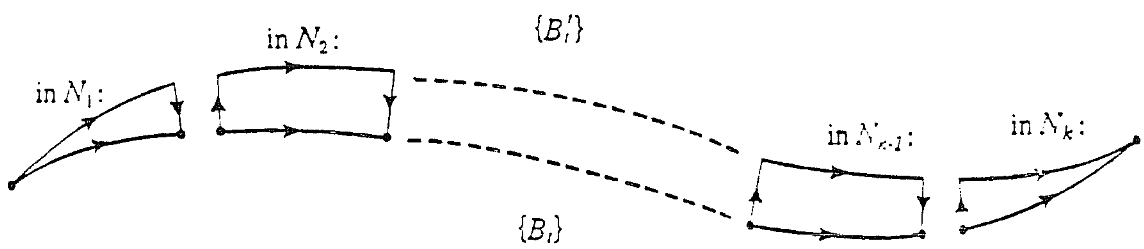

FIGURE 3

By definition, the sum of the spectral flows of the lower paths is $sf(\{B_t\})$. Since the spectral flows of the vertical paths cancel in pairs, the sum of the spectral flows of the upper paths equals $sf(\{B_t'\})$ and hence $sf(\{B_t\}) = sf(\{B_t'\})$. ∎

2.6 Examples: If N is a II_∞ factor, then it is well-known (and not difficult to prove) that N contains an abelian von Neumann subalgebra isomorphic to $L^\infty(\mathbf{R})$ with the property that the restriction of the trace to $L^\infty(\mathbf{R})$ coincides with the usual trace on $L^\infty(\mathbf{R})$ given by Lebesgue integration. We construct our first examples inside this sublagebra. Let B_0 in $L^\infty(\mathbf{R})$ be the continuous function:

$$B_0(t) = \begin{cases} 1 & \text{if } t \geq 1 \\ t & \text{if } t \in [-1,1] \\ -1 & \text{if } t \leq -1. \end{cases}$$

Let r be any fixed real number. Then for $t \in [0,1]$ let B_t be defined by $B_t(s) = B_0(s + tr)$ for all $s \in \mathbf{R}$. Clearly $\{B_t\}$ is a continuous path in $\mathcal{F}_*^{sa}$. Moreover, $\chi(B_t) = \chi_{[-tr,\infty)}$ which differs from $\chi_{[0,\infty)}$ by the finite projection $\chi_{[-tr,0)}$ if $r > 0$ (or, $\chi_{[0,-tr)}$ if $r < 0$). Thus, $\pi(\chi(B_t))$ is *constant* in $\mathcal{Q}_N$. Hence,

$$P_0 = \chi(B_0) = \chi_{[0,\infty)},$$

$$P_1 = \chi(B_1) = \chi_{[-r,\infty)}$$

and

$$sf(\{B_t\}) = ec(P_0, P_1) = \text{ind}_{(P_0\text{-}P_1)}(P_0P_1)$$
$$= Tr(P_1 - P_0P_1) - Tr(P_0 - P_0P_1)$$
$$= r.$$

We note that for these examples the spectral pictures are constant! That is, $sp(B_t) = [0,1]$ for all t and $sp(\pi(B_t)) = \{-1,1\}$ for all t. Thus, one cannot tell from the spectrum alone (even knowing the multiplicities) what the spectral flow will be.

One might feel (momentarily) uneasy about these examples since there exists a (strong-operator topology) continuous path of unitaries $\{U_t\}$ so that $B_t = U_tB_0U_t^*$. However, there cannot exist a *norm*-continuous path of such unitaries as this would imply that the path $t \mapsto \chi(B_t)$ is a norm-continuous path of projections which it is not since $\|\chi(B_t) - \chi(B_s)\| = 1$ if $s \neq t$.

On the other hand, it is not hard to prove that there is a unitary U_1 in N so that $B_1 = U_1B_0U_1^*$. Since the unitary group of N is connected in the norm topology we can find in N a norm continuous path $\{U_t\}$ of unitaries for $t \in [1,2]$ so that U_1 is as above and $U_2 = I$. Then we can extend $\{B_t\}$ to a continuous loop based at B_0 by defining $B_t = U_tB_0U_t^*$ for $t \in [1,2]$. Since the second half of the loop satisfies $t \mapsto \chi(B_t)$ is norm continuous, its spectral flow is 0 and so $sf\left(\{B_t\}_{[0,2]}\right) = r$.

When N is a type I_∞ factor we can use a similar construction with $\ell^\infty(\mathbf{Z})$ in place of $L^\infty(\mathbf{R})$ to obtain paths with any given integer as their spectral flow. Of course, these examples will not have a constant spectral picture.

2.7 Remark: It is clear from the above definition that spectral flow does not change under reparametrization of intervals and is additive when we compose paths by concatenation. Hence, spectral flow defines a groupoid homomorphism from the homotopy groupoid, $\text{Hom}(\mathcal{F}_*^{sa})$ to $\mathbf{Z}$ in the type I_∞ case (respectively, to $\mathbf{R}$ in the type II_∞ case). By the examples just constructed these homomorphisms are surjective, even when restricted to paths based at a point B_0 in F_*^{sa}, *i.e.*, $sf : \pi_1(\mathcal{F}_*^{sa}) \to \mathbf{Z}$ (respectively, $\mathbf{R}$) is surjective. To see that this group homomorphism is one-to-one seems to require the homotopy equivalence $\mathcal{F}_*^{sa} \simeq U(\infty)$ of [AS] in the type I_∞ case or the homotopy equivalence $\Omega(\mathcal{F}_*^{sa}) \simeq \mathcal{F}$ of

[P1,P2] in the type II_∞ case: in fact, both results only need the somewhat weaker result, $\Omega(\mathcal{F}_*^{sa}) \simeq \mathcal{F}$.

2.8 Spectral flow is one-to-one on $\pi_1(\mathcal{F}_*^{sa})$: We briefly outline Perera's result as we need it for our calculation. Let $Gr(\mathcal{Q}_N)$ be the space of projections in $\mathcal{Q}_N$ which are neither 0 nor 1. Then,

$$B \mapsto \pi\left(\chi(B)\right) : \mathcal{F}_*^{sa} \to Gr(\mathcal{Q}_N)$$

is a homotopy equivalence: the homotopy inverse is easily constructed from a Bartle-Graves selection and the homotopy in $\mathcal{F}_*^{sa}$ connecting the composite map with the identity is the linear homotopy. Now, fix an infinite and co-infinite projection P_0 in N and define a fibration $\alpha : U(N) \to Gr(\mathcal{Q}_N)$ *via* $\alpha(U) = \pi(UP_0U^*)$. Since $U(N)$ is contractible and the fiber

$$\alpha^{-1}\left(\pi(P_0)\right) = \left\{ U \in U(N) \mid U = \begin{bmatrix} X & a \\ b & Y \end{bmatrix}, \quad a, b \text{ in } \mathcal{K}_N \right.$$
$$\left. \text{relative to the decomposition } 1 = P_0 + (1 - P_0) \right\},$$

we see that the space of loops in $Gr(\mathcal{Q}_N)$ based at $\pi(P_0)$, namely $\Omega\left(Gr(\mathcal{Q}_N)\right)$, is homotopy equivalent to $\alpha^{-1}\left(\pi(P_0)\right)$. By a result of [CP], the map $\begin{bmatrix} X & a \\ b & y \end{bmatrix} \mapsto X$ is a homotopy equivalence from $\alpha^{-1}\left(\pi(P_0)\right)$ to $\mathcal{F}$, $(= \mathcal{F}_{P_0NP_0})$ the space of all Fredholm operators in the semifinite factor P_0NP_0. Perera's spectral flow is then the composition:

$$\Omega(\mathcal{F}_*^{sa}) \longrightarrow \Omega\left(Gr(\mathcal{Q}_N)\right) \longrightarrow \alpha^{-1}\left(\pi(P_0)\right) \longrightarrow \mathcal{F} \xrightarrow{\text{-ind}} \mathbf{R}$$

(or $\mathbf{Z}$ in case N is type I_∞).

To be completely explicit about this map, we take a loop in $\mathcal{F}_*^{sa}$ based at B_0, say $\{B_t\}$. We push it to a loop, $\{\pi\left(\chi(B_t)\right)\}$ in $Gr(\mathcal{Q}_N)$ based at $\pi(P_0) = \pi\left(\chi(B_0)\right)$. Since α is a fibration we can lift this to a path $\{V_t\}$ in $U(N)$ with $V_0 = 1$ and $\alpha(V_t) = \pi\left(V_t P_0 V_t^*\right) = \pi\left(\chi(B_t)\right)$ for all t. The final point of this path $\alpha(V_1)$ lies in the fiber $\alpha^{-1}\left(\pi(P_0)\right)$ so that $\alpha(V_1) = \begin{bmatrix} X & a \\ b & Y \end{bmatrix}$. Then, this map sends the loop $\{B_t\}$ to -ind(X).

Since this map classifies connected components in $\Omega(\mathcal{F}_*^{sa})$ and since our definition of spectral flow is constant on connected components, it suffices to show that the two definitions agree on the set of examples in 2.6. To this end, let $r \in \mathbf{R}$ and let $\{B_t\}$

$t \in [0,2]$ be the loop constructed in 2.6 with $sf(\{B_t\}) = r$. We define the lifted path $\{V_t\}$ in $U(N)$ as follows:

$$\text{for } t \in [0,1] \quad \text{define} \quad V_t = 1;$$
$$\text{for } t \in [1,2] \quad \text{define} \quad V_t = U_t U_1^*$$

where $\{U_t\}$ is the path chosen in 2.6. Then $\{V_t\}$ is certainly continuous in $U(N)$. Now, for $t \in [0,1]$

$$\alpha(V_t) = \alpha(1) = \pi(P_0) = \pi(\chi(B_t))$$

since this path is constant in $Gr(\mathcal{Q}_N)$; for $t \in [1,2]$ we compute:

$$\alpha(V_t) = \pi(V_t P_0 V_t^*) = \pi(V_t)\,\pi(P_0)\,\pi(V_t^*)$$
$$= \pi(V_t)\,\pi(P_1)\,\pi(V_t^*) = \pi(U_t U_1^* P_1 U_1 U_t^*)$$
$$= \pi(U_t U_1^* \chi(B_1) U_1 U_t^*) = \pi(\chi(U_t U_1^* B_1 U_1 U_t^*))$$
$$= \pi(\chi(U_t B_0 U_t^*)) = \pi(\chi(B_t))$$

as required. Now, since $U_2 = 1$ we have

$$U_1^* = U_2 U_1^* = V_2 = \begin{bmatrix} X & a \\ b & Y \end{bmatrix}$$

where $X = P_0 U_1^* P_0 \in \mathcal{F}$. Since $U_1 P_0 U_1^* = P_1$ we see that $P_0 U_1^* = U_1^* P_1$ so that

$$X = P_0 U_1^* P_0 = U_1^* P_1 P_0 = (U_1^* P_1)(P_1 P_0).$$

Letting $W = U_1^* P_1$ we have $X = W(P_1 P_0)$ where

$$WW^* = P_0 U_1^* U_1 P_0 = P_0$$

and

$$W^* W = P_1 U_1 U_1^* P_1 = P_1$$

so that W is $(P_0\text{-}P_1)$-Fredholm with 0 index and so by proposition A5,

$$\operatorname{ind}_{(P_0\text{-}P_0)} X = \operatorname{ind}_{(P_0\text{-}P_1)}(W) + \operatorname{ind}_{(P_1 - P_0)}(P_1 P_0)$$
$$= 0 + (-r) = -r = -sf(\{B_t\})$$

as required. This completes the proof of the main theorem. ∎

2.9 Theorem. *If N is a factor (on separable Hilbert space) of type I_∞ (respectively, II_∞) then $sf : \mathrm{Hom}(\mathcal{F}_*^{sa}) \to \mathbf{Z}$ (respectively, $sf : \mathrm{Hom}(\mathcal{F}_*^{sa}) \to \mathbf{R}$) is a groupoid homomorphism which restricts to an isomorphism of $\pi_1(\mathcal{F}_*^{sa})$ with $\mathbf{Z}$ (respectively, $\mathbf{R}$).*

2.10 Remark. Once one has established the ontological status [MPX, p. 165] of a *well-defined* groupoid homomorphism, $sf : \mathrm{Hom}(\mathcal{F}_*^{sa}) \to \mathbf{Z}$ (or $\mathbf{R}$) extending the isomorphism $\pi_1(\mathcal{F}_*^{sa}) \to \mathbf{Z}$ (or $\mathbf{R}$), its uniqueness follows from the property mentioned in 2.3. That is, if χ is continuous on a path, its spectral flow is 0.

2.11 Proposition: *If G is a group and $\phi : \pi_1(\mathcal{F}_*^{sa}, B_0) \to G$ is a homomorphism then there exists a unique extension $\tilde{\phi} : \mathrm{Hom}(\mathcal{F}_*^{sa}) \to G$ satisfying $\tilde{\phi}[\{F_t\}] = 0$ whenever $t \mapsto \chi(F_t)$ is continuous.*

Proof. **Existence.** Let $\mathbf{A}$ denote $\mathbf{Z}$ or $\mathbf{R}$ depending on whether N is of type I_∞ or type II_∞. Since $sf : \pi_1(\mathcal{F}_*^{sa}, B_0) \to \mathbf{A}$ is an isomorphism, we have a homomorphism $\psi : \mathbf{A} \to G$ so that the following diagram commutes:

$$
\begin{array}{ccccc}
\pi_1(\mathcal{F}_*^{sa}, B_0) & = & \pi_1(\mathcal{F}_*^{sa}, B_0) & \xrightarrow{\ \phi\ } & G \\
{\scriptstyle i}\big\downarrow & & {\scriptstyle sf}\big\downarrow & & {\scriptstyle \psi}\big\uparrow \\
\mathrm{Hom}(\mathcal{F}_*^{sa}) & \xrightarrow[\ sf\ ]{} & \mathbf{A} & = & \mathbf{A}
\end{array}
$$

where $i : \pi_1(\mathcal{F}_*^{sa}, B_0) \to \mathrm{Hom}(\mathcal{F}_*^{sa})$ is the inclusion map. It follows from 2.3 that $\psi \circ sf : \mathrm{Hom}(\mathcal{F}_*^{sa}) \to G$ is the required extension $\tilde{\phi}$.

Uniqueness. We first observe that any pair T_0, T_1 in $\mathcal{F}_*^{sa}$ can be joined by a path $\{T_t\}$ for which the map $t \mapsto \chi(T_t)$ is continuous. To see this we first note that any T in $\mathcal{F}_*^{sa}$ can be joined to its corresponding symmetry $(2\chi(T) - 1)$ by the straight line segment $(1-t)T + t\,(2\chi(T) - 1)$ and that χ is *constant* on this path. We apply this to both T_0 and T_1 and then join their corresponding symmetries $S_0 = (2\chi(T_0) - 1)$ and $S_1 = (2\chi(T_1) - 1)$ *via* a path $\{U_t S_0 U_t^*\}$ where $t \mapsto U_t \in U(N)$ is norm continuous. This is possible since S_0

and S_1 are unitarily equivalent in N and $U(N)$ is arcwise connected. For this path we also have

$$t \mapsto \chi(U_t S_0 U_t^*) = U_t\,\chi(S_0)\,U_t^*$$

is continuous.

Now, if $\{F_t\}$ is any path in $\mathcal{F}_*^{sa}$ then we join B_0 to F_0 and F_1 to B_0 with paths on which χ is continuous, thereby obtaining a loop based at B_0. $\tilde{\phi}$ of this loop is determined by ϕ as $\tilde{\phi}$ extends ϕ. Also, $\tilde{\phi}$ of the paths from B_0 to F_0 and F_1 to B_0 must be 0 by hypothesis. Thus, $\tilde{\phi}\,[\{F_t\}]$ equals ϕ $([\{B_0 \bullet\!-\!\bullet F_0 \bullet\!-\!\bullet F_1 \bullet\!-\!\bullet B_0\}])$ is completely determined. ∎

2.12 Final Remark. While one can "axiomatize" spectral flow in this way, the true value of any definition will lie in its "computability" or "applicability". We hope our definition is of value.

§3. Finitely Summable Fredholm Modules and an Integral Formula for Spectral Flow

Let A be a unital $*$-algebra and let (H, F_0) be an odd Fredholm module for A. Let $u \in U(A)$ and let $F_1 = uF_0u^*$ so that $F_1 - F_0 = [u, F_0]\,u^*$ is compact. Then, $F_t^u = (1-t)F_0 + t\,F_1$ is a path in $\mathcal{F}_*^{sa}$. Since $\pi(F_t) = \pi(F_0)$ for all t, we have that $sf\,(\{F_t\}) = \mathrm{ind}_{(P_0\text{-}P_1)}(P_0P_1)$ where $P_i = \chi(F_i) = \frac{1}{2}(F_i + 1)$. But, $P_1 = uP_0u^*$ so that $u^*P_1 = P_0u^*$ is an isometry from P_1 to P_0 and therefore is $(P_0\text{-}P_1)$-Fredholm of index 0. Thus

$$sf\,(\{F_t^u\}) = \mathrm{ind}_{(P_0\text{-}P_1)}(P_0P_1) = \mathrm{ind}_{(P_0\text{-}P_1)}(P_0u\,P_0u^*) = \mathrm{ind}_{(P_0\text{-}P_0)}(P_0u\,P_0),$$

the index of the "Toeplitz" operator $T_u = P_0u\,P_0$. Thus, the spectral flow of the straight line path from F_0 to uF_0u^* recovers the usual pairing of the odd Fredholm module $(H,\,F_0)$ with u in $U(A)$. In the usual way we can jack this up to the mapping $\mathcal{K}_1(A) \to \mathbf{Z}$ induced by the odd Fredholm module $(H,\,F_0)$.

As in [CP] one can consider odd Breuer-Fredholm modules $(N,\,F_0)$ where F_0 is a symmetry in the II_∞ factor N and $A \to N$ is a $*$-representation which commutes with F_0

modulo $\mathcal{K}_N$. Then, as above we get $sf(\{F_t^u\}) = \mathrm{ind}(T_u) \in \mathbf{R}$ where spectral flow of the path $\{F_t^u\}$ now makes sense with our new definition.

In [G], Ezra Getzler considers odd θ-summable Fredholm modules (H, D) over a unital Banach $*$-algebra A. For $u \in U(A)$ he considers the straight line path from D to uDu^*. Here, D is an unbounded self-adjoint operator on H satisfying:

(1) There is a positive constant C so that $\|[D,a]\| \le C\|a\|$ for all $a \in A$, and

(2) for $t > 0$, $e^{-tD^2}$is trace class.

Letting $D_t = (1-t)\,D + t(uDu^*)$ for $t \in [0,1]$ one shows that $sf(\{D_t\})$ makes sense and Getzler proves that

$$sf(\{D_t\}) = \frac{1}{\sqrt{\pi}} \int_0^1 Tr\left(\frac{d}{dt}(D_t)\, e^{-D_t^2}\right) dt.$$

Here, of course, $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2}\, dt$.

Our point here is to prove an analogous spectral flow formula for the pairing of $\mathcal{K}_1(A)$ with finitely summable Fredholm modules which works equally well in both the type I_∞ and type II_∞ situations. First we prove a more general result which yields the theorem for finitely summable (Breuer-) Fredholm modules as an easy corollary. The proof of the general result depends on some (easy) technical lemmas which we leave to the end of the section.

3.1 Theorem. *Let P, Q be infinite and co-infinite projections in the semifinite factor N. Let $p > 0$ and suppose $(P-Q)$ is p-summable, that is, $Tr(|P-Q|^p) < +\infty$. Let n be a positive integer so that $2n \ge p$. Then, $B_0 = 2P-1$ and $B_1 = 2Q-1$ are both in $\mathcal{F}_*^{sa}$ as is the path $B_t = (1-t)\,B_0 + t\,B_1$, and in this case,*

$$\mathrm{sf}(\{B_t\}) = \frac{1}{C_n} \int_0^1 Tr\left(\frac{d}{dt}(B_t)(1-B_t^2)^n\right) dt$$

where

$$C_n = \int_{-1}^{1} (1-t^2)^n\, dt = \frac{n!\, 2^{n+1}}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

Proof: Since $\pi(P) = \pi(Q)$, $\pi(\chi(B_t))$ is constant and so, by definition

$$sf(\{B_t\}) = \mathrm{ind}_{(P\text{-}Q)}(PQ).$$

Now, $\frac{d}{dt}(B_t) = B_1 - B_0 = 2(Q-P)$ and also, $(1-B_t^2) = 2^2(t-t^2)(Q-P)^2$. Hence,

$$\frac{d}{dt}(B_t)(1-B_t^2)^n = 2^{2n+1}(t-t^2)^n(Q-P)^{2n+1}$$

which is trace class since $2n+1 > p$.

We compute:

$$\begin{aligned}
&\int_0^1 Tr\left(\frac{d}{dt}(B_t)(1-B_t^2)^n\right) dt \\
&\left.\begin{aligned}
&= 2^{2n+1}\, Tr\left((Q-P)^{2n+1}\right) \textstyle\int_0^1 (t-t^2)^n\, dt \\
&= 2^{2n+1}\, Tr\left((Q-P)^{2n+1}\right) \textstyle\int_{-1}^1 (1-s^2)^n\, ds\, \frac{1}{2^{2n+1}}
\end{aligned}\right\}\ s = 2t-1 \\
&= Tr\left((Q-P)^{2n+1}\right) \cdot C_n \\
&= \left[Tr\left(|(1-P)Q|^{2n}\right) - Tr\left(|(1-Q)P|^{2n}\right)\right] \cdot C_n \text{ by Lemma 3.4} \\
&= \left(\mathrm{ind}_{(P\text{-}Q)}(PQ)\right) \cdot C_n \text{ by Lemma 3.5.}
\end{aligned}$$

The claimed value of C_n is proved in Lemma 3.6. ∎

3.2 Definition: Let A be a $*$-algebra and $p > 0$ then an *odd p-summable (Breuer-) Fredholm module for A* is a pair (N, F) where N is a semifinite factor (on a separable Hilbert space), $F \in N$ is a self-adjoint unitary in $\mathcal{F}_*^{sa}$, and $A \to N$ is a $*$-representation satisfying: $[a, F]$ is p-summable for every $a \in A$. Of course, the actual module in the algebraic sense is the underlying Hilbert space.

3.3 Theorem: *Let A be a $*$-algebra and let (N, F) be an odd p-summable (Breuer-) Fredholm module for A. Let $P = \chi(F) = \frac{1}{2}(F+1)$ and let n be a positive integer with $2n \geq p$. For each $u \in U(A)$ consider the path $F_t^u = (1-t)F + t(uFu^*)$, $t \in [0,1]$. Then, $\{F_t^u\}$ is a path in $\mathcal{F}_*^{sa}$ and*

$$\mathrm{ind}(PuP^*) = sf(\{F_t^u\}) = \frac{1}{C_n}\int_0^1 Tr\left(\frac{d}{dt}(F_t^u)\left(1-(F_t^u)^2\right)^n\right) dt.$$

Proof. The first equality follows from the discussion at the beginning of this section while the second equality follows from Theorem 3.1 with $Q = \chi(uFu^*) = \frac{1}{2}(uFu^*+1)$ since $P - Q = \frac{1}{2}(F - uFu^*) = -\frac{1}{2}[u, F]\,u^*$ which is p-summable. ∎

3.4 Lemma: *Let P and Q be two projections in the same von Neumann algebra. Then*

(1) $(Q-P)^2 = (1-P)\,Q + (1-Q)\,P$

(2) $(Q-P)\,Q = (1-P)\,Q$ *and* $(P-Q)\,P = (1-Q)\,P$

(3) $(Q-P)^{2n+1} = |(1-P)\,Q|^{2n} - |(1-Q)\,P|^{2n}$ *for each positive integer n.*

Proof. (1) and (2) are trivial observations. We prove (3) by induction on n.

For $n = 1$ we have

$$\begin{aligned}(Q-P)^3 &= (Q-P)\,[(1-P)\,Q + (1-Q)\,P] \text{ by (1)}\\ &= Q(1-P)\,Q - P(1-Q)\,P\\ &= Q(1-P)(1-P)\,Q - P(1-Q)(1-Q)\,P\\ &= |(1-P)\,Q|^2 - |(1-Q)\,P|^2\end{aligned}$$

as required.

Now, suppose the formula holds for $n = k \geq 1$. Then,

$$\begin{aligned}(Q-P)^{2(k+1)+1} &= (Q-P)^{2k+1}(Q-P)^2\\ &= \left([Q(1-P)\,Q]^k - [P(1-Q)\,P]^k\right)((1-P)\,Q + (1-Q)\,P) \text{ by (1)}\\ &= \left([Q(1-P)\,Q]^k\,Q - [P(1-Q)\,P]^k\,P\right)((1-P)\,Q + (1-Q)\,P)\\ &= [Q(1-P)\,Q]^{k+1} - [P(1-Q)\,P]^{k+1}\\ &= |(1-P)\,Q|^{2(k+1)} - |(1-Q)\,P|^{2(k+1)}\end{aligned}$$

as required. ∎

3.5 Lemma: *If n is a positive integer, P and Q are projections in a semifinite factor N and $(P-Q)$ is $2n$-summable in N then PQ is $(P\text{-}Q)$-Fredholm and*

$$\mathrm{ind}_{(P\text{-}Q)}(PQ) = Tr\left[|(1-P)\,Q|^{2n}\right] - Tr\left[|(1-Q)\,P|^{2n}\right].$$

Proof. Since $(1-P)\,Q = (Q-P)\,Q$ and $(1-Q)\,P = (P-Q)\,P$ the right-hand side makes sense. Since $\pi(P) = \pi(Q)$ the left-hand side also makes sense. Now, let $V : P \to Q$

be a partial isometry in N so that $PQV \in PNP$ and PQV is (Breuer-) Fredholm with

$$\mathrm{ind}_{PNP}(PQV) = \mathrm{ind}_{(P\text{-}Q)}(PQ)$$

by Proposition A-2. We now apply Theorem A-7 with $T = PQV$, $S = V^*QP$. Then, working in PNP:

$$R_1 = P - ST = P - V^*QPQV = V^*QV - V^*QPQV$$

$$= V^*Q(Q-P)QV = V^*Q(1-P)QV = V^*|(1-P)Q|^2V$$

$$R_2 = P - TS = P(P-Q)P = |(1-Q)P|^2$$

so that both R_1 and R_2 are n-summable. Thus,

$$\mathrm{ind}_{(P\text{-}Q)}(PQ) = \mathrm{ind}_{PNP}(PQV) = Tr(R_1^n) - Tr(R_2^n)$$

$$= Tr\left(V^*|(1-P)Q|^{2n}V\right) - Tr\left(|(1-Q)P|^{2n}\right)$$

$$= Tr\left(|(1-P)Q|^{2n}\right) - Tr\left(|(1-Q)P|^{2n}\right). \quad \blacksquare$$

3.6 Lemma: *For each nonnegative integer n the following holds for all integers $k \geq 0$.*

$$\int_{-1}^{1} t^{2k}(1-t^2)^n\, dt = \frac{n!\, 2^{n+1}}{(2k+1)(2k+3)\cdots(2(k+n)+1)}.$$

In particular,

$$C_n = \int_{-1}^{1} (1-t^2)^n\, dt = \frac{n!\, 2^{n+1}}{1 \cdot 3 \cdots (2n+1)}.$$

Proof. Although we are only interested in C_n, our proof by induction on n seems to require the extra leeway afforded by the parameter k.

The case $n = 0$ is trivial. So, suppose the result holds for some $n \geq 0$. Then, for all

$k \geq 0$

$$\begin{aligned}\int_{-1}^{1} t^{2k}(1-t^2)^{n+1}\,dt &= \int_{-1}^{1} t^{2k}(1-t^2)^n(1-t^2)\,dt \\ &= \int_{-1}^{1} t^{2k}(1-t^2)^n\,dt - \int_{-1}^{1} t^{2(k+1)}(1-t^2)^n\,dt \\ &= \frac{n!\;2^{n+1}}{(2k+1)(2k+3)\cdots(2(k+n)+1)} - \frac{n!\;2^{n+1}}{(2(k+1)+1)\cdots(2(k+1+n)+1)} \\ &= \frac{n!\;2^{n+1}\left[(2(k+1+n)+1)-(2k+1)\right]}{(2k+1)(2k+3)\cdots(2(k+1+n)+1)} \\ &= \frac{n!\;2^{n+1}(2n+2)}{(2k+1)(2k+3)\cdots(2(k+1+n)+1)} \\ &= \frac{(n+1)!\;2^{n+2}}{(2k+1)(2k+3)\cdots(2(k+n+1)+1)}. \qquad \blacksquare\end{aligned}$$

3.7 Remark: Alan Carey and the author are currently investigating other integral formulas for spectral flow and their possible relations to η invariants.

Appendix

(BREUER-) FREDHOLM OPERATORS IN SEMIFINITE FACTORS

The standard references for Breuer-Fredholm operators in a II_∞ factor are [B1] and [B2]. We need a slight generalization of this theory. Let N be a semifinite factor acting on the separable Hilbert space, H. We endow N with a fixed semifinite normal trace, Tr (if N is a type I_∞ factor, we assume that the trace is normalized so that minimal projections have trace 1). If H_1 is a subspace of H, we denote the projection onto the closure of H_1 by $[H_1]$.

A1. Definition: Let P and Q be infinite projections in the semifinite factor N and let $T \in PNQ$. We let $\ker_Q T = \ker\left(T_{|Q(H)}\right) = \ker T \cap Q(H)$. The operator $T \in PNQ$ is called *$(P$-$Q)$-Fredholm* if and only if

1. $[\ker_Q T]$ is finite in N,
2. $[\ker_P T^*]$ is finite in N, and
3. there exists $P_1 \leq P$ in N with $(P - P_1)$ finite in N and $P_1(H) \subseteq T(H)$.

In this case, we define the *$(P$-$Q)$-index of T* to be the number:

$$\mathrm{ind}_{(P\text{-}Q)}(T) = Tr\,[\ker_Q T] - Tr\,[\ker_P T^*]\,.$$

We observe that if $P = Q$ then this is just the definition of Breuer-Fredholm operators in the semifinite factor, QNQ, with the restriction of Tr to QNQ.

That the expected properties hold for this index, is due mainly to the following proposition.

A2. Proposition: *Let P and Q be infinite projections in the semifinite factor N with fixed trace, Tr. Let V be a partial isometry in N with $V^*V = P$ and $VV^* = Q$. Then,*

(1) *$T \in PNQ$ is $(P$-$Q)$-Fredholm if and only if VT is Breuer-Fredholm in the semifinite factor QNQ. In this case, $\mathrm{ind}_{(P\text{-}Q)}(T) = \mathrm{ind}_{QNQ}(VT)$ (where QNQ has the restricted trace), and*

(2) *$T \in PNQ$ is $(P\text{-}Q)$-Fredholm if and only if TV is Breuer-Fredholm in the semifinite factor PNP. In this case* $\mathrm{ind}_{(P\text{-}Q)}(T) = \mathrm{ind}_{PNP}(TV)$ *(where PNP has the restricted trace).*

Proof. **(1)** Since $T \in PNQ$, $T = PTQ$ so that V is an isometry on the range of T. Thus, $[\ker_Q T] = [\ker_Q VT]$. Similarly, $[\ker_P T^*] = [\ker_P V^*T^*]$. It is easy to see that $V\,[\ker_P V^*T^*]\,V^* = [\ker_Q (VT)^*]$ so that

$$[\ker_P T^*] \underset{N}{\sim} [\ker_Q (VT)^*].$$

Now, let $P_1 \leq P$ be so that $P_1(H) \subseteq \mathrm{range}\, T$. Let $Q_1 = VP_1V^* \leq VPV^* = Q$. Thus, $Q - Q_1 = V(P - P_1)V^*$ is finite if and only if $P - P_1$ is finite, and

$$Q_1(H) = VP_1V^*(H) = VP_1(H) \subseteq V(\mathrm{range}\, T) = \mathrm{range}\, VT.$$

Thus, T is $(P\text{-}Q)$-Fredholm if and only if VT is Breuer-Fredholm in QNQ and in this case $\mathrm{ind}_{(P\text{-}Q)}(T) = \mathrm{ind}_{QNQ}(VT)$.

The proof of **(2)** is similar. ∎

A3. Corollary: *Let N be a semifinite factor with trace Tr, and infinite projections P, Q. Then*

(1) *$T \in PNQ$ is $(P\text{-}Q)$-Fredholm if and only if T^* is $(Q\text{-}P)$-Fredholm and, in this case,* $\mathrm{ind}_{(Q\text{-}P)}(T^*) = -\mathrm{ind}_{(P\text{-}Q)}(T)$.

(2) *If T is $(P\text{-}Q)$-Fredholm and $S \in PNQ$ is of the form $T + k$ for some k in $\mathcal{K}_N$ then S is $(P\text{-}Q)$-Fredholm and* $\mathrm{ind}_{(P\text{-}Q)}(S) = \mathrm{ind}_{(P\text{-}Q)}(T)$.

(3) *The set of $(P\text{-}Q)$-Fredholm operators with a given index is an open connected set in PNQ.*

Proof. **(1)** If T and T^* are both Fredholm then clearly $\mathrm{ind}_{(Q\text{-}P)}(T^*) = -\mathrm{ind}_{(P\text{-}Q)}(T)$. So, suppose T is $(P\text{-}Q)$-Fredholm. Then if $V : P \to Q$ as above, VT is Breuer-Fredholm in QNQ and so $T^*V^* = (VT)^*$ is Breuer-Fredholm in QNQ and so $T^*$is $(Q\text{-}P)$-Fredholm.

(2) If $S = T + k$ for $k \in W$ then $VS = VT + Vk$ and $Vk \in QNQ \cap \mathcal{K}_N = \mathcal{K}_{QNQ}$. Since VT is Breuer-Fredholm in QNQ so is VS and thus S is $(P\text{-}Q)$-Fredholm. Hence,

$$\mathrm{ind}_{(P\text{-}Q)}(S) = \mathrm{ind}_{QNQ}(VS) = \mathrm{ind}_{QNQ}(VT) = \mathrm{ind}_{(P\text{-}Q)}(T).$$

(3) With $V: P \to Q$ as above, the map $T \mapsto VT : PNQ \to QNQ$ is an isometric bijection which takes the class of (P-Q)-Fredholm operators to the class of Breuer-Fredholm operators and preserves index. The result follows. ∎

A4. Lemma: *Let E, P, Q be infinite projections in the semifinite factor, N. If $R \in ENQ$ and $W : P \to Q$ is a partial isometry then R is (E-Q)-Fredholm if and only if RW is (E-P)-Fredholm and* $\mathrm{ind}_{(E\text{-}Q)}(R) = \mathrm{ind}_{(E\text{-}P)}(RW)$.

Proof. Let $V : E \to P$ be a partial isometry. Then $WV : E \to Q$ and so R is (E-Q)-Fredholm if and only if RWV is Breuer-Fredholm in ENE if and only if RW is (E-P)-Fredholm and

$$\mathrm{ind}_{(E\text{-}P)}(RW) = \mathrm{ind}_{ENE}(RWV) = \mathrm{ind}_{(E\text{-}Q)}(R).$$ ∎

A5. Proposition: *Let E, P, Q be infinite projections in the semifinite factor, N. If $T \in PNQ$ and $S \in ENP$ are (P-Q)- and (E-P)-Fredholm operators, respectively, then $ST \in ENQ$ is (E-Q)-Fredholm and* $\mathrm{ind}_{(E\text{-}Q)}(ST) = \mathrm{ind}_{(E\text{-}P)}(S) + \mathrm{ind}_{(P\text{-}Q)}(T)$.

Proof. Let $V : E \to P$ and $W : P \to Q$ be partial isometries. Then VS and $TW \in PNP$ are Breuer-Fredholm operators, so that $VSTW$ is also Breuer-Fredholm and

$$\begin{aligned}\mathrm{ind}_{PNP}(VSTW) &= \mathrm{ind}_{PNP}(VS) + \mathrm{ind}_{PNP}(TW) \\ &= \mathrm{ind}_{(E\text{-}P)}(S) + \mathrm{ind}_{(P\text{-}Q)}(T)\end{aligned}$$

by Proposition A2. Also, by Proposition A2, STW is (E-P)-Fredholm and

$$\mathrm{ind}_{(E\text{-}P)}(STW) = \mathrm{ind}_{PNP} VSTW$$

. Now, by Lemma A4 with $R = ST$ we have that ST is (E-Q)-Fredholm and

$$\mathrm{ind}_{(E\text{-}Q)}(ST) = \mathrm{ind}_{(E\text{-}P)}(STW) = \mathrm{ind}_{(E\text{-}P)}(S) + \mathrm{ind}_{(P\text{-}Q)}(T).$$ ∎

A6. Corollary: *If $T \in PNQ$ and $T = V|T|$ is the polar decomposition of T then $V \in PNQ$ and T is (P-Q)-Fredholm if and only if V is (P-Q)-Fredholm with* $\mathrm{ind}_{(P\text{-}Q)}(T) = \mathrm{ind}_{(P\text{-}Q)}(V)$ *and $|T|$ is (Q-Q)-Fredholm of index* 0.

Proof. One direction is immediate from Proposition A5.

Now suppose T is (P-Q)-Fredholm. Since $V \in PNQ$ and $[\text{range}\, V] = [\text{range}\, T]$, $[\text{range}\, V^*] = [\text{range}\, T^*]$ and the range of V is closed we see that V is (P-Q)-Fredholm with the same index as T. But, then $|T| = V^*T$ and so $|T|$ is (Q-Q)-Fredholm of index 0. ∎

We also need the following trace formula for the index. In the type I_∞ case, this goes back to Calderón [Ca] at least for pseudo-differential operators. The first proof in the type I_∞ case for general Fredholm operators appears to be due to Hörmander [H]: see also Connes [Co] for a different proof. In the type II_∞ setting, the case $n = 1$ was proved in the special situation where T is essentially unitary and $S = T^*$ by Carey-Phillips in [CP]. Unfortunately, Connes' elegant proof [Co, Appendix 1, Proposition 6] does not seem to generalize to the II_∞ setting.

A7. Theorem: *Let N be a semifinite factor with trace Tr, and let $S, T \in N$ so that $R_1 = 1 - ST$ and $R_2 = 1 - TS$ are both n-summable for some integer $n > 0$. Then T is a Breuer-Fredholm operator and* $\text{ind}(T) = Tr(R_1^n) - Tr(R_2^n)$.

Proof. We follow the proof of Hörmander [H] in the type I_∞ case with some modifications to make it work in the type II_∞ case.

Suppose $n = 1$, that is R_1 and R_2 are trace class. If T does not have closed range then by Lemma 3.7 of [CP] there is a finite spectral projection E of $|T|$ so that $T' = T(1 - E)$ has closed range and $\text{ind}\, T = \text{ind}\, T'$. Then $T' = T - k$ where $k = TE$ is trace class. Let $R_1' = 1 - ST' = (1 - ST) + Sk = R_1 + Sk$ and let $R_2' = 1 - T'S = R_2 + kS$ which are both trace class operators. Then,

$$Tr(R_1') - Tr(R_2') = Tr(R_1) - Tr(R_2) + Tr(Sk) - Tr(kS)$$

$$= Tr(R_1) - Tr(R_2).$$

Thus, we can assume that T has closed range.

Now, if T has closed range, then we let $E_1 = \ker T$; $F_1 = (I - E_1)$; $F_2 = [\text{range}\, T]$, and $E_2 = (1 - F_2) = \ker T^*$. Moreover, $T = F_2 T F_1$ and $|T| \in F_1 N F_1$ is invertible in the algebra $F_1 N F_1$; hence, there is an X in $F_1 N F_1$ so that $|T| X = F_1 = X|T|$. If $T = V|T|$ is the polar decomposition of T then let $\tilde{T} = XV^* \in N$ so that $\tilde{T}T = (XV^*)(V|T|) = X|T| = F_1$

and $T\tilde{T} = V|T|XV^* = VF_1V^* = VV^* = F_2$. So,

$$T = F_2TF_1 \tag{1}$$

$$T\tilde{T} = F_2 \tag{2}$$

$$\tilde{T}T = F_1. \tag{3}$$

Now the following are easy to establish:

$$R_1E_1 = E_1, \tag{4}$$

$$R_2F_2 = F_2R_2F_2, \tag{5}$$

$$E_2R_2E_2 = E_2. \tag{6}$$

Thus,

$$T(F_1R_1F_1) = (TF_1)(R_1F_1) = (TR_1)F_1$$

$$= (R_2T)F_1 = R_2T = R_2(F_2T) = (F_2R_2F_2)T,$$

and so

$$F_1R_1F_1 = \tilde{T}(F_2R_2F_2)T. \tag{7}$$

Finally,

$$Tr(R_1) = Tr(R_1E_1) + Tr(R_1F_1)$$

$$= Tr(E_1) + Tr(F_1R_1F_1)$$

$$= \dim E_1 + Tr\left(\tilde{T}(F_2R_2F_2)T\right)$$

$$= \dim E_1 + Tr(F_2R_2F_2)$$

and

$$Tr(R_2) = Tr(E_2R_2E_2) + Tr(F_2R_2F_2)$$

$$= Tr(E_2) + Tr(F_2R_2F_2)$$

$$= \dim E_2 + Tr(F_2R_2F_2)$$

so

$$Tr(R_1) - Tr(R_2) = \dim E_1 - \dim E_2 = \operatorname{ind}(T).$$

The general case reduces to the case $n = 1$ exactly as in Hörmander's proof. ∎

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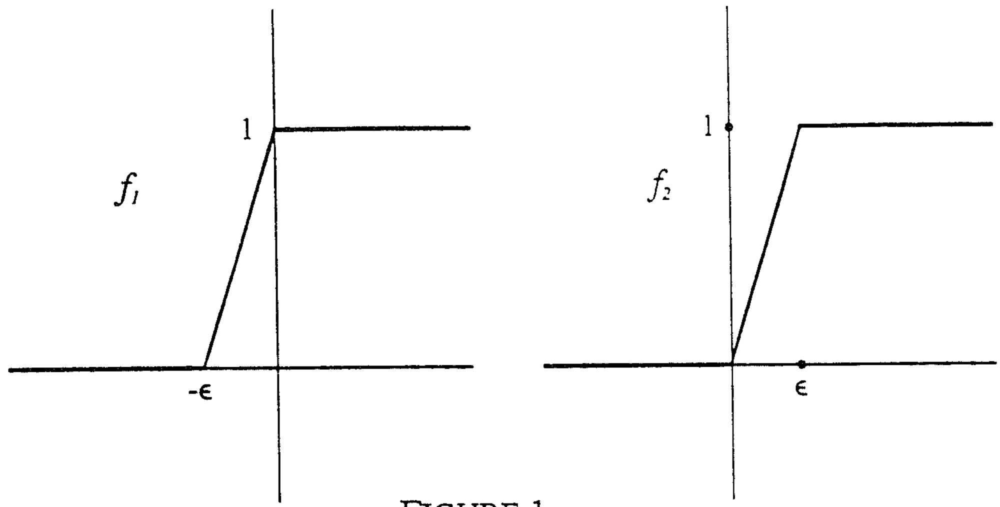

FIGURE 1

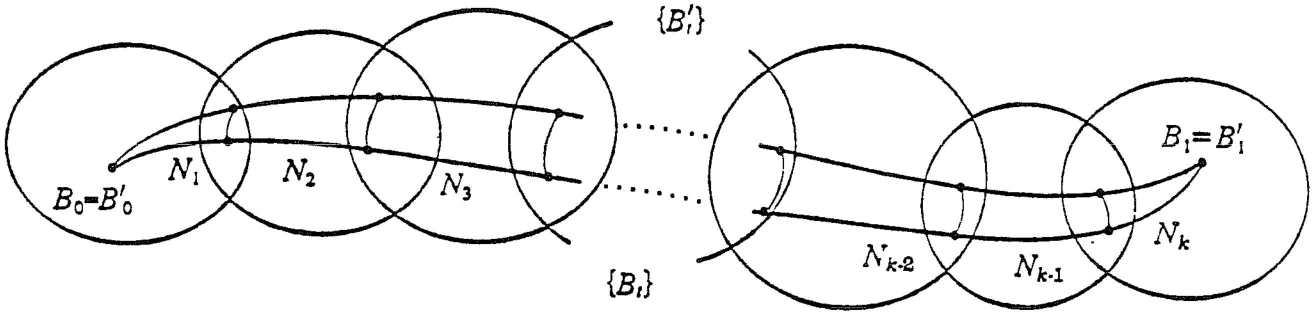

FIGURE 2

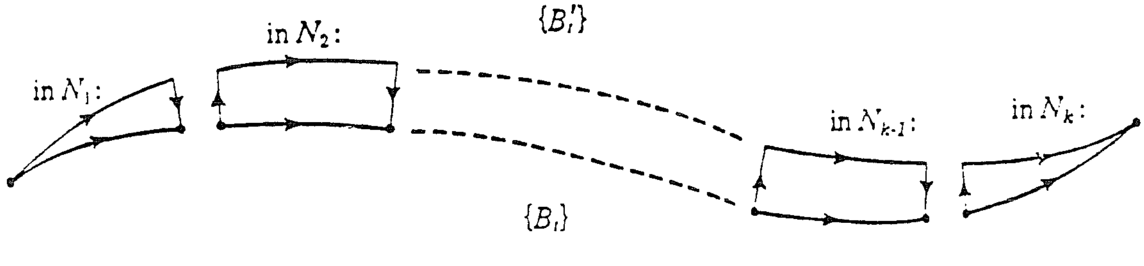

FIGURE 3