SOME REDUCTION FORMULAS FOR MULTIPLE HYPERGEOMETRIC SERIES

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ABSTRACT

The authors prove a general theorem on the reducibility of a certain multiple series and apply it to derive numerous reduction formulas for hypergeometric functions of several variables. Some applications of this theorem, involving Jacobi polynomials, are also presented.

1. INTRODUCTION AND THE MAIN RESULT

For Appell's double hypergeometric function F_4 defined by (*cf.*, *e.g.*, [1, p. 14])

$$F_4[\alpha,\beta;\gamma,\delta;x,y] = \sum_{\ell,m=0}^{\infty} \frac{(\alpha)_{\ell+m}\,(\beta)_{\ell+m}}{(\gamma)_\ell\,(\delta)_m} \frac{x^\ell}{\ell!}\frac{y^m}{m!}, \tag{1.1}$$

where, and throughout this paper,

$$(\lambda)_\ell = \frac{\Gamma(\lambda+\ell)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \ell = 0, \\ \lambda(\lambda+1)\cdots(\lambda+\ell-1), & \forall \ell \in \{1,2,3,\ldots\}, \end{cases} \tag{1.2}$$

Bailey [3, p. 42, Equation (4.2)] proved the reduction formula:

$$F_4\left[\alpha,\beta;\beta,\alpha; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right] = (1-x)^\alpha\,(1-y)^\beta\,(1-xy)^{-1}. \tag{1.3}$$

Formula (1.3) is known to play a rather crucial rôle in one of many interesting proofs of Jacobi's generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n = 2^{\alpha+\beta}\,R^{-1}(1-t+R)^{-\alpha}\,(1+t+R)^{-\beta}, \tag{1.4}$$

$$R = (1-2xt+t^2)^{1/2}$$

for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined, in terms of the Gaussian hypergeometric function, by

$$(1.5)\qquad P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n}\, {}_2F_1\left[\begin{matrix} -n,\ \alpha+\beta+n+1; \\ \alpha+1; \end{matrix}\ \frac{1-x}{2}\right].$$

See, for example, Rainville [8, p. 269, Section 140] and Carlitz [4]; see also Askey [2, p. 246] and Srivastava [9, p. 201].

An interesting extension of Bailey's formula (1.3) was given recently by Cohen [6, p. 272, Theorem 1] who indeed applied his result to deduce several new or known generating functions for Jacobi polynomials including, for instance, a remarkable special case $(\lambda = 0,\ \mu = s + 1)$ of the following generalization of (1.4) due to Srivastava and Singhal [12, p. 749, Equation (8)]:

$$(1.6)\qquad \sum_{n=0}^{\infty} P_n^{(\alpha-\lambda n,\beta-\mu n)}(x)\,t^n = (1+\xi)^{\alpha+1}\ (1+\eta)^{\beta+1}[1+\lambda\xi+\mu\eta-(1-\lambda-\mu)\xi\eta]^{-1},$$

where ξ and η are functions of x and t defined implicitly by

$$(1.7)\qquad (x+1)^{-1}\ \xi = (x-1)^{-1}\ \eta = \frac{1}{2}\,t(1+\xi)^{1-\lambda}\ (1+\eta)^{1-\mu}.$$

The object of the present paper is to prove a multidimensional extension of Cohen's theorem and to show how our general result can be applied with a view to deriving, for example, reduction formulas for multiple hypergeometric functions. Our main result is contained in the following

THEOREM. _Let_ $r_1, \ldots, r_n$ _and_ $\alpha_1, \ldots, \alpha_n$ _denote arbitrary complex numbers_. _Also_, _for convenience_, _let_

$$(1.8)\qquad k_{n+1} = k_1,$$

where n _is a fixed positive integer_.

Then

$$(1.9)\qquad \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{i=1}^{n} \left\{ \frac{(\alpha_i + r_i k_{i+1})_{k_i}}{(1-z_i)^{k_i + r_i k_{i+1}}} \frac{(-z_i)^{k_i}}{k_i!} \right\}$$

$$= \frac{(1-z_1)^{\alpha_1} \cdots (1-z_n)^{\alpha_n}}{1 - (-1)^n r_1 z_1 \cdots r_n z_n},$$

<u>provided that</u>

$$|z_1| < 1, \ \ldots, \ |z_n| < 1, \ \underline{\text{and}} \ |r_1 z_1 \cdots r_n z_n| < 1.$$

2. PROOF OF THE THEOREM

To prove the multidimensional reduction formula (1.9), we begin by considering the multiple series:

$$(2.1)\qquad \Omega(x; z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!}$$

$$\cdot\, D_x^{m_1}\left\{ x^{\alpha_1 + m_1 - 1} (1-x^{r_1})^{m_2} \right\} \cdots D_x^{m_n}\left\{ x^{\alpha_n + m_n - 1} (1-x^{r_n})^{m_1} \right\},$$

where, as usual, $D_x = d/dx$.

Applying the binomial theorem and operating upon the resulting sums term-by-term, we find from (2.1) that

$$(2.2)\qquad \Omega(x; z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} z_1^{m_1} \cdots z_n^{m_n}$$

$$\cdot \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} \prod_{i=1}^{n} \left\{ (-1)^{k_i} \frac{(\alpha_i + r_i k_{i+1})_{m_i}}{k_i!(m_i - k_i)!} x^{\alpha_i + r_i k_{i+1} - 1} \right\},$$

where we have followed the convention (1.8).

Now we set $x = 1$ in (2.1) and (2.2). Obviously, (2.1) contributes only when

$$m_1 = \ldots = m_n,$$

and we have

$$(2.3) \qquad \Omega(1;z_1,\ldots,z_n) = \sum_{m=0}^{\infty} (-1)^{mn} (r_1 z_1 \cdots r_n z_n)^m$$

$$= [1 - (-1)^n \, r_1 z_1 \cdots r_n z_n]^{-1},$$

provided that

$$\left| r_1 z_1 \cdots r_n z_n \right| < 1.$$

On the other hand, by inverting the order of summation in (2.2) with $x = 1$, we readily observe that

$$(2.4) \qquad \Omega(1;z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{i=1}^{n} \left\{ (\alpha_i + r_i k_{i+1})_{k_i} \frac{(-z_i)^{k_i}}{k_i!} \right\}$$

$$\cdot \prod_{i=1}^{n} \left\{ \sum_{m_i=0}^{\infty} (\alpha_i + k_i + r_i k_{i+1})_{m_i} \frac{z_i^{m_i}}{m_i!} \right\}.$$

Since

$$(2.5) \qquad \sum_{m=0}^{\infty} (\lambda)_m \frac{z^m}{m!} = (1-z)^{-\lambda}, \qquad |z| < 1,$$

we find from (2.4) that

$$(2.6) \qquad \Omega(1;z_1,\ldots,z_n) = (1-z_1)^{-\alpha_1} \cdots (1-z_n)^{-\alpha_n}$$

$$\cdot \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{i=1}^{n} \left\{ \frac{(\alpha_i + r_i k_{i+1})_{k_i}}{(1-z_i)^{k_i + r_i k_{i+1}}} \frac{(-z_i)^{k_i}}{k_i!} \right\},$$

provided that

$$\max\{|z_1|,\ldots,|z_n|\} < 1.$$

The reduction formula (1.9) would follow immediately upon equating the two expressions for

$$\Omega(1;z_1,\ldots,z_n)$$

given by (2.3) and (2.6), thus completing the proof of the theorem.

3. APPLICATIONS

The case $n = 2$ of the reduction formula (1.9) was given earlier by Cohen [6, p. 272, Theorem 1], and Bailey's formula (1.3) follows immediately from (1.9) upon setting $n = 2$ and $r_1 = r_2 = 1$. We shall show how our general result (1.9) can be applied, when $n > 2$, in order to derive various new or known results involving multiple hypergeometric functions and Jacobi polynomials.

I. For $r_1 = \ldots = r_n = 1$, (1.9) can at once be rewritten as a reduction formula for the hypergeometric function $H_B^{(n)}$ of n variables, and we are thus led to a known result due to Khichi ([7, p. 23, Equation (3.2)]; see also Srivastava and Karlsson [10, p. 308, Equation (122)]) who incidentally derived it first for $n = 3$ by appealing to the proof of Bailey's formula (1.3) by Bailey [3, pp. 41-42] and Rainville [8, pp. 266-267] mutatis mutandis. It should be remarked in passing that $H_B^{(2)}$ is the Appell function F_4 defined by (1.1), and $H_B^{(3)}$ is Srivastava's function H_B [10, p. 43, Equation (12)].

II. For the triple hypergeometric series F_K of Lauricella's set (see, e.g., [12, p. 42, Equation (4)]), our theorem with $n = 4$, $r_1 = r_3 = r_4 = 1$, and $r_2 = \rho$ yields

$$(3.1)\qquad \sum_{\ell=0}^{\infty} \frac{(c)_\ell}{\ell!}\left\{-\frac{z}{(1-z)(1-y)^\rho}\right\}^\ell F_K\left[b+\rho\ell,\ d,\ d,\ a,\ c+\ell,\ a;\ a,\ c,\ d;\right.$$

$$\left. -\frac{y}{(1-y)(1-x)},\ -\frac{t}{(1-t)(1-z)},\ -\frac{x}{(1-x)(1-t)}\right]$$

$$= (1-x)^a\ (1-y)^b\ (1-z)^c\ (1-t)^d\ (1-\rho xyzt)^{-1}.$$

Some further consequences of (3.1) are worthy of mention. First of all, since F_K is expressible as an infinite series of products of two Gaussian hypergeometric ${}_2F_1$ functions (cf. [11, p. 243, Equation (13)]), each of which can be rewritten fairly easily as a Jacobi polynomial in view of (1.5), setting

$$a = \alpha + 1,\ b = -\beta,\ c = \gamma + 1,\ d = -\delta,$$

and

$$\frac{y}{1-y} = -\frac{1}{2}(1-x)(1-u),\ \frac{t}{1-t} = -\frac{1}{2}(1-z)(1-v),$$

and replacing z by y, we find from (3.1) that

$$(3.2)\qquad \sum_{\ell,m=0}^{\infty} \left\{-\frac{2x[2-(1-y)(1-v)]}{(1-x)(1+u)(1+v)}\right\}^{\ell} \left\{-\frac{2y[2-(1-x)(1-u)]^{\rho}}{(1-y)(1+u)^{\rho}(1+v)}\right\}^{m}$$

$$\cdot\ P_{\ell}^{(\alpha,\beta-\ell-\rho m)}(u)\ P_{m}^{(\gamma,\delta-\ell-m)}(v)$$

$$= \frac{(1-x)^{\alpha+1}(1-y)^{\gamma+1}[2-(1-x)(1-u)]^{\beta+1}[2-(1-y)(1-v)]^{\delta+1}}{(1+u)^{\beta}(1+v)^{\delta}[4-2\{(1-x)(1-u)+(1-y)(1-v)\}+(1-x)(1-y)(1-u)(1-v)(1-\rho xy)]},$$

which, for $x = 0$ and $\rho = s + 1$, corresponds to the aforementioned special case $(\lambda = 0,\ \mu = s + 1)$ of (1.6).

The triple hypergeometric function F_K is expressible also as an infinite series of Appell's function F_2 which can, in turn, be specialized to give a ${}_2F_1$ either directly (cf. [10, p. 305, Equation (108)]) or through Appell's function F_1 (cf [10, p. 305, Equation (106); p. 304, Equation (99)]). In this manner, (3.1) with $a = d$ and $\rho = 1$ yields an interesting reduction formula for the triple Gaussian hypergeometric series 10c or 18b [10, pp. 77 and 79], and (3.1) with $a = d$ and $\rho = 0$ leads us to a special case of a result due to Chaundy ([5, p. 62, Equation (25)]; see also [11, p. 107, Equation (14)]).

III. In terms of Horn's double hypergeometric function H_2 (see, e.g, [10, p. 24, Equation (10)]), our theorem with $n = 4$, $r_1 = -1$, $r_2 = \rho$, and $r_3 = r_4 = 1$ yields

(3.3)
$$\sum_{\ell,m=0}^{\infty} \frac{(c)_{\ell+m}\,(d)_m}{\ell!\; m!\; (c)_m} \left\{-\frac{z}{(1-z)(1-y)^{\rho}}\right\}^{\ell} \left\{-\frac{t}{(1-t)(1-z)}\right\}^{m}$$

$$\cdot\, H_2\left[a,\ d+m,\ 1-a,\ b+\rho\ell;\ d;\ -\frac{x}{(1-x)(1-t)},\ \frac{y(1-x)}{1-y}\right]$$

$$= (1-x)^{a}\,(1-y)^{b}\,(1-z)^{c}\,(1-t)^{d}\,(1+\rho xyzt)^{-1}.$$

In its special case when $\rho = 0$ and $d = a + b$, this last result (3.3) can easily be rewritten in terms of the Horn function G_2 by appealing to a known transformation [10, p. 319, Equation (170)]. If we further set $c = a + b$, we shall be led eventually to a reduction formula involving the triple Gaussian hypergeometric series 4h [10, p. 75].

We conclude by remarking that analogous results involving various other classes of multiple hypergeometric functions can be deduced, in a manner described above, for other appropriate choices of n and $r_1, \ldots, r_n$.

REFERENCES

[1] P. Appell et J. Kampé de Fériet, Fonctions Hypergéometriques et Hypersphériques; Polynômes d'Hermite, Gauthier-Villars, Paris, 1926.

[2] R. Askey, Jacobi's generating function for Jacobi polynomials, Proc. Amer. Math. Soc. **71** (1978), 243-246.

[3] W.N. Bailey, Some infinite integrals involving Bessel functions, Proc. London Math. Soc. (2) **40** (1936), 37-48.

[4] L. Carlitz, The generating function for the Jacobi polynomial, Rend. Sem. Mat. Univ. Padova **38** (1967), 86-88.

[5] T.W. Chaundy, An extension of hypergeometric functions (I), Quart. J. Math. Oxford Ser. **14** (1943), 55-78.

[6] M.E. Cohen, Generating functions for the Jacobi polynomial, Proc. Amer. Math. Soc. **57** (1976), 271-275.

[7] K.S. Khichi, A note on Srivastava's triple hypergeometric function H_B, Ganita **19** (1968), no. 1, 21-24.

[8] E.D. Rainville, Special Functions, Macmillan, New York, 1960.

[9] H.M. Srivastava, A note on Jacobi's generating function for the Jacobi polynomials, Proc. Japan Acad. Ser. A Math. Sci. 61(1985), 201-202.

[10] H.M. Srivastava and P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1985.

[11] H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[12] H.M. Srivastava and J.P. Singhal, New generating functions for Jacobi and related polynomials, J. Math. Anal. Appl. 41(1973), 748-752.

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