# Ramsey Theory: Avoiding Trees with Few Colours

University of Victoria

Andrew Lane, Department of Mathematics and Statistics

September 8, 2022

This research was supported by the Valerie Kuehne Undergraduate Research Awards, University of Victoria. Supervised by Dr. Natasha Morrison

## Graphs

A **graph** is a network of **vertices** and **edges** between pairs of vertices. We define a graph as a pair $(V, E)$, where $V$, the vertex set, is any set of objects, and $E$, the edge set, is a set of pairs from $V$.

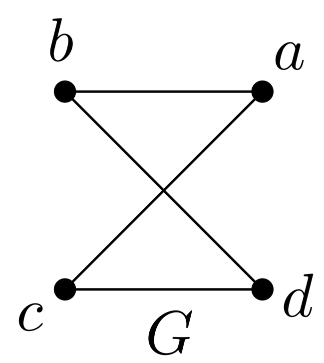


Figure 1. Example of a graph. $G$ is defined by $V = \{a, b, c, d\}$ and $E = \{ab, bd, dc, ca\}$.

In a graph $G$, a **path** is a sequence of distinct vertices $x_1, \ldots, x_k$, where there is an edge between each vertex $x_i$ and the next vertex $x_{i+1}$ in the sequence. A graph is **connected** if every pair of vertices has a path between them.

There are some special classes of graphs. **Complete graphs** have edges between every pair of vertices. A **tree** with $n$ vertices is connected and has $n-1$ edges. **Path** graphs and **stars** are special types of trees (see Figure 2).

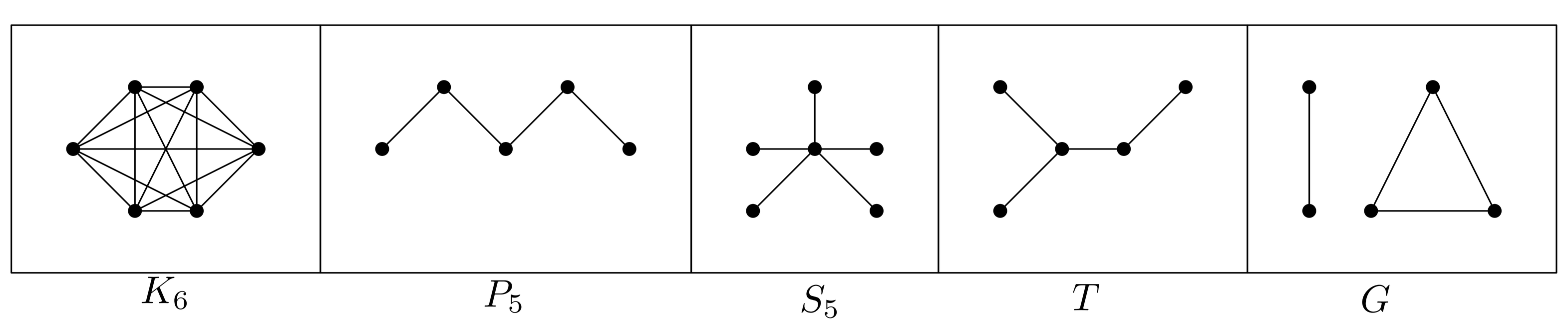


Figure 2. Examples of special (unlabelled) graphs. $K_6$ is the complete graph on 6 vertices. $P_5$ is the path on 5 vertices. $S_5$ is the star on 5 edges. $T$ is an arbitrary tree. $G$ is *not* a tree because it is not connected.

## The Classic Ramsey Problem

How many people must a party have so that it always has either a group of 3 friends or 3 strangers (or both)?

We can restate the above question as the following: if we colour the complete graph $K_n$ with two colours, red and blue, how many vertices must $K_n$ have so that it always has a red or blue triangle? We can call this the **Ramsey number** $R(3,3)$. In fact, $R(3,3) = 6$ (see Figure 3). Many Ramsey numbers, like $R(5,5)$ (the size of graphs that always contain a red or blue $K_5$) are not known exactly.

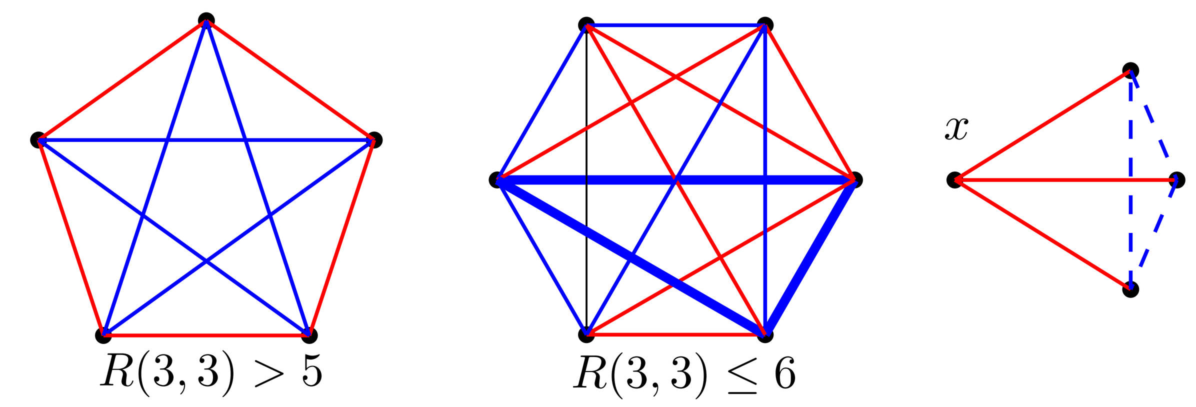


Figure 3. The $K_5$ has no red or blue triangle, so $R(3,3) > 5$. $K_6$ always has a red or blue triangle, so $R(3,3) \leq 6$. The proof uses the fact that at least 3 edges from each vertex are the same colour.

## The Generalized Ramsey Problem

Surprisingly, the Ramsey number $R(s,t)$ exists for arbitrarily large $s$ and $t$. We can always find a patch of "order" (few colours) in enough "disorder". But if we add in more colours, it becomes harder to ensure there is a patch of "order". The question is, how many colours must we add such that we *cannot* find a specific substructure with few colours?

**Definition.** Let $f(n, H, q)$ be the minimum number of colours needed to colour the edges of $K_n$ so that every copy of the graph $H$ has at least $q$ colours.

This is related to the classic Ramsey problem. Consider $f(n, K_3, 2)$. As in Figure 3, we can colour $K_5$ with only 2 colours so that every $K_3$ (triangle) has at least 2 colours, but we cannot colour $K_6$ in that way using only 2 colours—we would have to add another colour. So $f(5, K_3, 2) = 2$, but $f(6, K_3, 2) > 2$.

If we colour every edge in $K_n$ with a different colour, we can easily make every copy of a graph $H$ have many colours. $K_n$ has $n(n-1)/2$ edges, so the number of colours would increase **quadratically** with $n$. We can potentially do much better, though. If we find a colouring that uses only a number of colours roughly proportional to the number of vertices in $K_n$, we can say that $f(n, H, q)$ is **linear** (or sub-linear) in $n$.

## Existing Results

The problem of the function $f(n, K_p, q)$ was first posed by Erdős and Shelah[2]. The **linear and quadratic thresholds** are the values for $q$ at which $f(n, H, q)$ grows at least linearly and quadratically with $n$, respectively. Erdős and Gyárfás determined the linear and quadratic thresholds for $f(n, K_p, q)$, the **complete graph** case[3]. Axenovich, Füredi, and Mubayi adapted this to another class of graphs known as complete bipartite graphs[1]. Krueger opened this up to general graphs, focusing on **paths**[4].

## Results for General Trees

In this project, I focused on $f(n, T, q)$ for general trees $T$. In some cases, I found exact values: for example, $f(n, T_5, 3) = 2\lceil \frac{n}{2} \rceil - 1$, where $T_5$ is the tree in Figure 4 below.

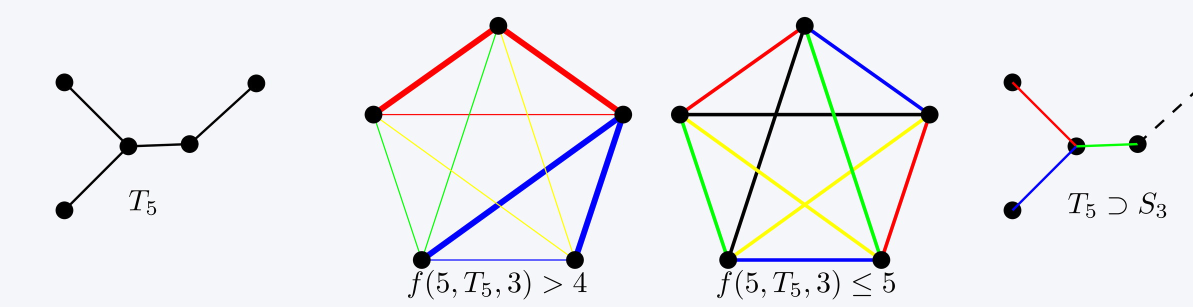


Figure 4. No matter how we colour $K_5$ with 4 colours, we always find a copy of $T_5$ (left) with only 2 colours (centre left). But with 5 colours, we can colour $K_5$ such that every copy of $T_5$ has at least 3 colours (centre right). Therefore, $f(5, T_5, 3) = 5$. The general proof follows from the fact that if the star $S_3$ contained in $T_5$ has all different colours, then $T_5$ has at least 3 colours (right).

I also studied the behavior of $f(n, T, q)$ as $n$ grows large. For instance, I fully characterized the asymptotic behavior of a class of trees called **subdivided stars**.

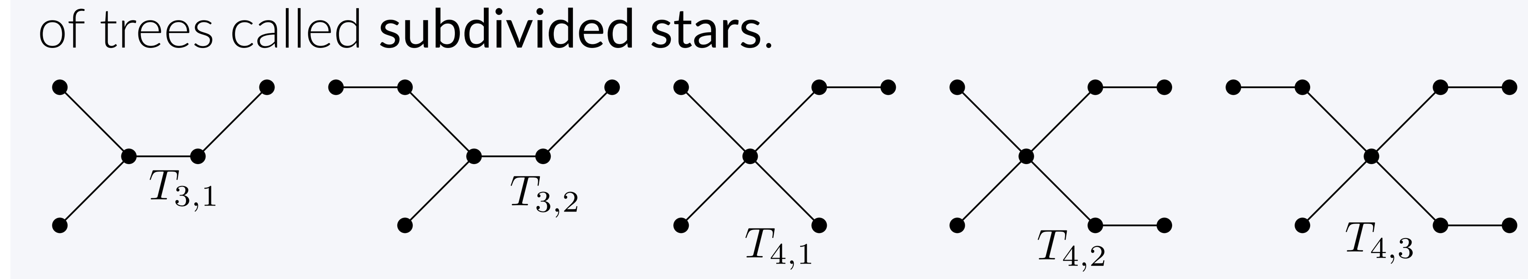


Figure 5. Examples of subdivided stars $T_{t,s}$ from proposition below.

**Proposition (Subdivided Stars).** For any $t \geq 3$ and $1 \leq s \leq t-1$, let $T_{t,s}$ be the tree obtained by replacing $s$ edges in the star $S_t$ each with paths of length 2. Then $f(n, T_{t,s}, q)$ is **linear** in $n$ for $2 \leq q \leq t$ and **quadratic** in $n$ for $t+1 \leq q \leq t+s$.

## Future Work

I hope to fully understand the asymptotic behavior of other classes of trees, including trees that contain long paths. It would be interesting to know for which classes of trees $T$ $f(n, T, q)$ is in between linear and quadratic in $n$ for some $q$.

## Acknowledgments

I would like to thank Dr. Natasha Morrison for her support in learning graph theory and her helpful feedback.

## References

[1] Maria Axenovich, Zoltán Füredi, and Dhruv Mubayi. On generalized Ramsey theory: the bipartite case. *J. Combin. Theory Ser. B*, 79(1):66–86, 2000.

[2] Paul Erdős. Problems and results on finite and infinite graphs. In *Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 183–192. (loose errata). Academia, Prague, 1975.

[3] Paul Erdős and András Gyárfás. A variant of the classical Ramsey problem. *Combinatorica*, 17(4):459–467, 1997.

[4] Robert Krueger. Generalized Ramsey numbers: forbidding paths with few colors. *Electron. J. Combin.*, 27(1):Paper No. 1.44, 10 pp., 2020.