THE CONNECTIVITY OF THE BLOCK–INTERSECTION GRAPHS OF DESIGNS

by

Donovan R. Hare & William McCuaig

DMS–588–IR September 1991

The Connectivity of the Block-Intersection Graphs of Designs

Donovan R. Hare*
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B. C.
Canada V5A 1S6

William McCuaig†
Department of Mathematics and Statistics
P. O. Box 3045
University of Victoria
Victoria, B.C.
Canada V8W 3P4

September 6, 1991

Abstract

It is shown that the vertex connectivity of the block-intersection graph of a balanced incomplete block design, $BIBD(v, k, 1)$, is equal to its minimum degree. A similar statement is proved for the edge connectivity of the block-intersection graph of a pairwise balanced design, $PBD(v, K, 1)$. A partial result on the vertex connectivity of these graphs is also given. Minimal vertex and edge cuts for the corresponding graphs are characterized.

*Research supported in part by a B.C. Science Council G.R.E.A.T. Scholarship.
†Research supported in part by an NSERC Postdoctoral Fellowship.

1 Introduction

Let K be a finite set of positive integers, and let λ and v be positive integers such that $v > \max K$ (here $\max K$ is the maximum element in K; similarily for $\min K$). A *pairwise balanced design*, denoted $PBD(v, K, \lambda)$, is a pair $(V, \mathcal{B})$ where V is a finite set whose elements are called *points*, $\mathcal{B}$ is a collection of subsets of V, called *blocks*, such that $|V| = v$, the blocks have their cardinalities from K and any pair of distinct points is contained in exactly λ blocks. If $K = \{k\}$, then $(V, \mathcal{B})$ is called a *balanced incomplete block design* and we denote it by $BIBD(v, k, \lambda)$. In this case each point is in the same number of blocks. This number is usually denoted by r and is called the *replication number* (a counting argument shows $r = \lambda\frac{v-1}{k-1}$). When $k = 3$ and $\lambda = 1$, the pair $(V, \mathcal{B})$ is called a *Steiner triple system.*

The reader is referred to [1] for any graph theory notation or definitions. There are many ways of defining graphs from designs. In this paper we use the following graph: the *block-intersection graph* of a $PBD(v, K, \lambda)$, $(V, \mathcal{B})$, denoted by $B(\mathcal{B})$, has vertex-set $\mathcal{B}$ and has two vertices adjacent if and only if their corresponding blocks have nonempty intersection. We restrict our attention to designs which have $\lambda = 1$. Moreover, since every pair of blocks intersects in a *symmetric* $BIBD(v, k, 1)$ $(k = r)$, we also restrict our attention to those $BIBD(v, k, 1)$ which satisfy $k > r$.

In answering a question about the hamiltonicity of the block-intersection graph of a $BIBD(v, k, 1)$, P. Horák and A. Rosa [3] showed that the vertex connectivity of such graphs is at least $\frac{v}{k}$. In this note we complete this investigation by showing that these graphs have vertex connectivity equal to their (minimum) degree (these graphs are $(k(r-1))$-regular). We also determine the edge connectivity of the block-intersection graph of a $PBD(v, K, 1)$ and give a partial result about its vertex connectivity. Characterizations of minimal vertex and edge cuts are given for the various cases.

2 Theorems

Theorem 1 *Let $(V, \mathcal{B})$ be a $BIBD(v, k, 1)$ and let $G = B(\mathcal{B})$. If $\mathcal{C}$ is a vertex cut separating vertices a and c, then $\delta = \delta(G) \leq |\mathcal{C}|$. Furthermore, we have equality if and only if $\mathcal{C}$ is the set of vertices adjacent to either a or c.*

Proof: For $i = 1, 2$, let $\mathcal{B}_i$ be the set of blocks in $\mathcal{B} \setminus \{a, c\}$ which intersect exactly i blocks of $\{a, c\}$. Let $\mathcal{C}_1$ be $\mathcal{C} \cap \mathcal{B}_1$ and let P be the set of points in V which are not in a or c. Note that $\mathcal{B}_2 \subseteq \mathcal{C} \cap \overline{\mathcal{B}_1}$

Let x be in P. For ℓ in $\{a, c\}$, let $\mathcal{S}_\ell(x)$ be the set of blocks in $\mathcal{B}$ which contain x and a point in ℓ. Suppose there exists b_1 in $\mathcal{S}_a(x) \setminus \mathcal{S}_c(x)$ and b_2 in $\mathcal{S}_c(x) \setminus \mathcal{S}_a(x)$ such that $\mathcal{C}$ does not contain b_1 or b_2. But then ab_1b_2c is a (a, c)-path in $G - \mathcal{C}$. Hence, $\mathcal{S}_a(x) \setminus \mathcal{S}_c(x) \subseteq \mathcal{C}$ or $\mathcal{S}_c(x) \setminus \mathcal{S}_a(x) \subseteq \mathcal{C}$. Also,

$$\begin{aligned} |\mathcal{S}_a(x) \setminus \mathcal{S}_c(x)| &= |\mathcal{S}_a(x)| - |\mathcal{S}_a(x) \cap \mathcal{S}_c(x)| \\ &= k - |\mathcal{S}_a(x) \cap \mathcal{S}_c(x)| \\ &= |\mathcal{S}_c(x)| - |\mathcal{S}_a(x) \cap \mathcal{S}_c(x)| \end{aligned}$$

$$= \quad |\mathcal{S}_c(x) \setminus \mathcal{S}_a(x)|.$$

Therefore, at least half of the blocks in $\mathcal{B}_1$ containing x are in $\mathcal{C}_1$.

The previous paragraph now implies

$$\begin{aligned} 2|\mathcal{C}_1|(k-1) &= 2\sum_{b\in\mathcal{C}_1} |\{(x,b) : x \in P, x \in b\}| \\ &= 2\sum_{x\in P} |\{(x,b) : b \in \mathcal{C}_1, x \in b\}| \\ &\geq \sum_{x\in P} |\{(x,b) : b \in \mathcal{B}_1, x \in b\}| \\ &= \sum_{b\in\mathcal{B}_1} |\{(x,b) : x \in P, x \in b\}| \\ &= |\mathcal{B}_1|\,(k-1). \end{aligned}$$

Hence, $|\mathcal{C}| \geq |\mathcal{C}_1| + |\mathcal{B}_2| \geq \frac{1}{2}|\mathcal{B}_1| + |\mathcal{B}_2| = \frac{1}{2}(d_G(a) + d_G(c)) \geq \delta$.

Now suppose $\delta(G) = |\mathcal{C}|$. Then we have equality in both equations of the last paragraph. Hence, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{B}_2$. Thus, if b is a block in a vertex cut $\mathcal{C}$ of size δ which separates blocks a' and c', then b intersects a' or c'. Equality also implies that for every point x in P, exactly half the blocks in $\mathcal{B}_1$ containing x are in $\mathcal{C}$. Thus, the set of blocks in $\mathcal{C}_1$ containing x is either $\mathcal{S}_a(x)$ or $\mathcal{S}_c(x)$, for every point x in P.

Every point in a is in $r-(k+1)$ blocks in $\mathcal{B}_1$. Thus, if $r = k+1$, then $\mathcal{C} = \mathcal{B}_2$ and hence $\mathcal{C} = N_G(a) = N_G(c)$. Therefore suppose $r > k+1$. Let p_1 and p_2 be distinct points in a and let p_i be in a block b_i which is in $\mathcal{B}_1$, $i = 1, 2$. Suppose $b_1 \in \mathcal{C}$ and $b_2 \notin \mathcal{C}$. Since b_1 is a block in a vertex cut $\mathcal{C}$ of size δ which separates blocks b_2 and c, b_1 intersects b_2 or c as seen in the previous paragraph. Since b_1 and a intersect and $b_1 \in \mathcal{B}_1$, b_1 and c do not intersect. Hence, b_1 and b_2 intersect in some point x. But then x is in P and $\mathcal{S}_a(x)$ has blocks in both $\mathcal{C}_1$ and $\mathcal{B}_1 \setminus \mathcal{C}_1$. Therefore, we can conclude that the set of blocks in $\mathcal{B}_1$ which are adjacent to a is either contained in $\mathcal{C}$ or disjoint from $\mathcal{C}$. It now follows that $\mathcal{C}$ is $N_G(a)$ or $N_G(c)$. ∎

We now consider the vertex and edge connectivity of the block-intersection graph of the more general case where the underlying design is a $PBD(v, K, 1)$. The next theorem determines such a graph's edge connectivity and characterizes its minimal edge cuts.

Theorem 2 *Let* $(V, \mathcal{B})$ *be a* $PBD(v, K, 1)$ *and let* $G = B(\mathcal{B})$. *If* S *is an edge cut separating vertices* a *and* c, *then* $\max\{d_G(a), d_G(c)\} \leq |S|$. *Furthermore, we have equality if and only if* S *is the set of edges incident with either* a *or* c.

Proof: Let $\mathcal{A}$ and $\mathcal{C}$ be a partition of $\mathcal{B}$ such that $a \in \mathcal{A}$, $c \in \mathcal{C}$, and $S = [\mathcal{A}, \mathcal{C}]$. For every point x in V, let S_x be the set of edges bc in S such that $b \cap c = \{x\}$. (S is partitioned by $\{S_x : x \in V, S_x \neq \emptyset\}$.)

Suppose x is in V. Let r_x be the replication number of x (in this case r_x need not be the same for different x), let a_x be the number of blocks in $\mathcal{A}$ which contain x, and let c_x be the number of blocks in $\mathcal{C}$ which contain x, If $a_x \geq 1$ and $c_x \geq 1$, then $|S_x| = a_x c_x = a_x(r_x - a_x) \geq r_x - 1$.

If there exists a point x_1 which is only in blocks in $\mathcal{A}$ and there exists a point x_2 which is only in blocks in $\mathcal{C}$, then no block in $\mathcal{B}$ can contain both x_1 and x_2. Therefore, we may assume every point in a block in $\mathcal{A}$ is also in a block in $\mathcal{C}$. Hence, $|S_x| \geq r_x - 1$, for every point x which is in some block in $\mathcal{A}$. Therefore,

$$\begin{aligned} |S| &= \sum_{x\in V} |S_x| \\ &\geq \sum_{x\in a} |S_x| \\ &\geq \sum_{x\in a} (r_x - 1) \\ &= d_G(a). \end{aligned}$$

If we have equality, then S_x is empty for every point x which is not in a. Hence, every point x which is in some block in $\mathcal{A}$ is in a. Therefore, $\mathcal{A} = \{a\}$, and so S is the set of edges incident with a. ∎

We state without proof a theorem which determines the vertex connectivity of the block-intersection graph where for a fixed K, the underlying $PBD(v, K, 1)$ has 'enough' points.

Theorem 3 *Let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$ and let $G = B(\mathcal{B})$. Define $k_{\max} = \max\{k : k \in K\}$. If $v \geq \frac{1}{3}k_{\max}^4$, then the vertex connectivity of G is equal to its minimum degree.*

We conjecture that the theorem holds even without the condition $v \geq \frac{1}{3}k_{\max}^4$.

3 Acknowledgements

The first author wishes to thank the Department of Mathematics and Statistics, University of Victoria, B.C., for their hospitality during the preparation

of this paper.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory With Applications.* Elsevier Science Publishing Co., Inc., 1984.

[2] Donovan R. Hare. *The Block-Intersection Graph of Pairwise Balanced Designs.* PhD thesis, Simon Fraser University, 1991.

[3] P. Horák and A. Rosa. Decomposing Steiner triple systems into small configurations. *Ars Combinatoria*, 26:91–105, 1988.