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Article

Local Fractional Integral Hölder-Type Inequalities and Some Related Results

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Abstract: This paper is devoted to establishing some functional generalizations of Hölder and reverse Hölder's inequalities with local fractional integral introduced by Yang. Then, based on the obtained results, we derive some related inequalities including local fractional integral Minkowski-type and Dresher-type inequalities, which are some extensions of several existing local fractional integral inequalities.

Keywords: local fractional integral; Hölder-type inequality; Minkowski-type inequality; Dresher-type inequality

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1. Introduction

Let $\varphi(u), \phi(u) \in C(a,b)$. Then the famous Hölder inequality and its reverse version are as follows.

(1) If $s > 1, 1/s + 1/t = 1$, then one has Hölder inequality (see [1]):

$$\int_a^b |\varphi(u)\phi(u)|du \leq \left(\int_a^b |\varphi(u)|^s du\right)^{1/s} \left(\int_a^b |\phi(u)|^t du\right)^{1/t}. \tag{1}$$

(2) If $0 < s < 1, 1/s + 1/t = 1$, then one has the reverse Hölder inequality (see [2]):

$$\int_a^b |\varphi(u)\phi(u)|du \geq \left(\int_a^b |\varphi(u)|^s du\right)^{1/s} \left(\int_a^b |\phi(u)|^t du\right)^{1/t}, \tag{2}$$

where $C(a,b)$ denotes the space of continuous functions defined on the interval (a,b).

The above two inequalities play an important role not only in pure mathematics but also in applied mathematics. A variety of generalizations and refinements have been studied in the literature; the reader can be referred to [3–12] and the cited references therein.

Nowadays, the theory of local fractional calculus, revised and perfected by Yang [13], has become a very important and popular tool to deal with various non-differentiable problems that appear in many fields such as image processing, cipher securityljio, chaos theory, theoretical physics, engineering sciences.

On the other hand, local fractional calculus also has gained important application in pure mathematics (see [13–31]). In particular, the local fractional calculus is utilized to establish some new inequalities which are extensions of classical real inequalities on certain fractal spaces. For example, Mo et al. [32] derived generalized Jensen's inequality and generalized Hermite–Hadamard's inequality on fractal space, Khan et al. [33] obtained generalized trapezium-type inequalities in the settings of fractal sets, Sun [29] given generalization of some inequalities for generalized harmonically convex functions via local fractional integrals. More recent results in this direction can be found in [13–17,27,29,30,32,33].

Recently, Yang [13] obtained the following local fractional integral Hölder inequality.

Assume that $\varphi(u),\ \phi(u) \in C_\lambda(a,b)$, $0 < \lambda \le 1$, $1/s + 1/t = 1$ with $s > 1$, then we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b |\varphi(u)\phi(u)|(du)^\lambda \\ &\le \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b |\varphi(u)|^s(du)^\lambda\right)^{1/s}\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b |\phi(u)|^t(du)^\lambda\right)^{1/t},\end{aligned} \tag{3}$$

where $C_\lambda(a,b)$ denotes the fractal space, which consists of local fractional continuous functions defined on the interval (a,b).

Very recently, Chen [14] established a reserve form of (3) as follows.

Assume that $\varphi(u),\ \phi(u) \in C_\lambda(a,b)$, $0 < \lambda \le 1$, $1/s + 1/t = 1$ with $0 < s < 1$, then one has

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b |\varphi(u)\phi(u)|(du)^\lambda \\ &\ge \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b |\varphi(u)|^s(du)^\lambda\right)^{1/s}\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b |\phi(u)|^t(du)^\lambda\right)^{1/t}.\end{aligned} \tag{4}$$

It is needed to point out that if $\lambda = 1$, then local fractional integral inequalities (3) and (4) reduce to inequalities (1) and (2), respectively. For more results on variations and generalizations of (3) and (4), please refer to [15,16].

Motivated by the above discussion, we shall give some new generalizations of (3) and (4). More precisely, the main purpose of this paper is to establish some functional generalizations of local fractional integral Hölder and reverse Hölder's inequalities on fractal space. Moreover, the obtained results will be applied to establish several related local fractional integral inequalities, which are functional generalizations of some known results such as local fractional integral Minkowski's inequality, Dresher's inequality, and their corresponding reverse versions.

The reset of this paper is planned as follows. In Section 2, we recall some basic definitions and lemmas involving local fractional calculus which are necessary in the sequel. In Section 3, we first state the main results, and then applying the obtained results to establish local fractional integral Minkowski-type inequality and its reverse form. In Section 4, we give a subdividing of local fractional integral Hölder-type inequality. In Section 5, an improvement of local fractional integral Minkowski-type inequality and its reverse version are obtained. In Section 6, local fractional integral Dresher-type inequality and its reverse version are established. In Section 7, we present the conclusion.

2. Preliminaries

For the sake of convenience, we list some basic concepts and notions with respect to local fractional calculus as follows.

Definition 1 (see [13,18,19])**.** *A non-differentiable function* $\varphi : \mathbb{R} \to \mathbb{R}^\lambda$ $(0 < \lambda \le 1)$, $u \to \varphi(u)$ *is said to be local fractional continuous at* u_0, *if for any* $\varepsilon > 0$, *there exists a positive constant* δ, *such that* $|\varphi(u) - \varphi(u_0)| < \varepsilon^\lambda$ *whenever* $|u - u_0| < \delta$. *If the function* $\varphi(u)$ *is local fractional continuous on the interval* (a,b), *then we denote* $\varphi(u) \in C_\lambda(a,b)$.

Definition 2 (see [13,18,19])**.** *Assume that $\varphi(u) \in C_\lambda(a,b)$. Local fractional derivative of $\varphi(u)$ of order λ at $u = u_0$ is defined by*

$$\varphi^{(\lambda)}(u_0) = \frac{d^\lambda \varphi(u)}{du^\lambda}|_{u=u_0} = \lim_{u \to u_0} \frac{\Gamma(1+\lambda)(\varphi(u) - \varphi(u_0))}{(u-u_0)^\lambda},$$

where $\Gamma(1+\lambda) = \int_0^\infty e^{-x} x^{\lambda-1} dx$, $x > 0$.

Definition 3 (see [13,18,19])**.** *Assume that $\varphi(u) \in C_\lambda(a,b)$, then local fractional integral of $\varphi(u)$ in the interval $[a,b]$ is defined as follows.*

$${}_aI_b^\lambda \varphi(u) = \frac{1}{\Gamma(1+\lambda)} \int_a^b \varphi(u)(du)^\lambda = \frac{1}{\Gamma(1+\lambda)} \lim_{\lambda \to 0} \sum_{i=1}^{N-1} \varphi(u_i)(\Delta u_i)^\lambda,$$

where $\Delta u_j = u_{j+1} - u_j$, $\Delta u = \max\{\Delta u_1, \ldots, \Delta u_j, \ldots\}$, and $[u_j, u_{j+1}]$, $j = 1, 2, \ldots, N-1$, $u_0 = a$, $u_N = b$ is a partition of the interval $\alpha_{kj} = -t/p_k$. Here, we denote ${}_aI_b^\lambda \varphi(u) = 0$ if $a = b$, ${}_aI_b^\lambda \varphi(u) = -{}_bI_a^\lambda \varphi(u)$ if $a < b$.

Definition 4 (See [13,18,19])**.** *Assume that $\varphi(u) = F^{(\lambda)}(u) \in C_\lambda(a,b)$, then one has*

$${}_aI_b^\lambda \varphi(u) = F(b) - F(a).$$

Lemma 1 (See [13,18,19])**.** *Assume that $\varphi(u)$ and $\phi(u)$ local fractional continuous on $[a,b]$.*

(1) *If $\varphi(u) \geq 0$ for all $u \in [a,b]$, then $\frac{1}{\Gamma(1+\lambda)} \int_a^b \varphi(u)(du)^\lambda \geq 0$.*

(2) *If $0 \leq \varphi(u) \leq \phi(u)$ for all $u \in [a,b]$, then*

$$\frac{1}{\Gamma(1+\lambda)} \int_a^b \varphi(u)(du)^\lambda \leq \frac{1}{\Gamma(1+\lambda)} \int_a^b \phi(u)(du)^\lambda.$$

(3) *If $\varphi(u) \geq 0$ for all $u \in [a,b]$, then $\varphi(u) = 0$ if and only if $\frac{1}{\Gamma(1+\lambda)} \int_a^b \varphi(u)(du)^\lambda = 0$.*

3. Main Results

In this section, we give our main results.

Theorem 1 (Hölder-type inequality)**.** *Assume that $s > 1$, $1/s + 1/t = 1$. Let $\Xi_l(\psi_1, \ldots, \psi_l) > 0$, $\Psi_m(\varphi_1, \ldots, \varphi_m)$ and $Z_k(\phi_1, \ldots, \phi_k)$ be three arbitrary functions of l, m and k variables, respectively. Assume that $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ and $\{\psi_i(u)\}_{i=1}^l$ are local fractional continuous functions on $[a,b]$, then*

$$\begin{aligned} &\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l)|\Psi_m(\varphi_1, \ldots, \varphi_m) Z_k(\phi_1, \ldots, \phi_k)|(du)^\lambda \\ \leq &\left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l)|\Psi_m(\varphi_1, \ldots, \varphi_m)|^s (du)^\lambda \right)^{1/s} \\ &\left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l)|Z_k(\phi_1, \ldots, \phi_k)|^t (du)^\lambda \right)^{1/t}. \end{aligned} \tag{5}$$

Moreover, the equality in (5) holds if and only if $A|\Psi_m(\varphi_1, \ldots, \varphi_m)|^s = B|Z_k(\phi_1, \ldots, \phi_k)|^t$, where A and B are constant.

Proof. Denoting $\Theta(\Xi_l, \Upsilon_m, Z_k)$ by the right-hand side of (5). By item 3 of Lemma 1, it is easy to see that if $\Theta(\Xi_l, \Upsilon_m, Z_k) = 0$, then $\Upsilon_m(\varphi_1, \ldots, \varphi_m) = 0$ or $Z_k(\phi_1, \ldots, \phi_k) = 0$. Hence, we may assume that $\Theta(\Xi_l, \Upsilon_m, Z_k) \neq 0$ and take

$$f(u) = \frac{(\Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)))^{1/s} |\Upsilon_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)) |\Upsilon_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|^s (du)^\lambda}$$

and

$$g(u) = \frac{(\Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)))^{1/t} |\Upsilon_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)) |\Upsilon_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|^t (du)^\lambda}.$$

From the Young's inequality, it follows that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)} \int_a^b f(u) g(u) (du)^\lambda \\
&\leq \frac{1}{\Gamma(1+\lambda)} \int_a^b [\frac{f^s(u)}{s} + \frac{g^t(u)}{t}] (du)^\lambda \\
&= \frac{1}{s} \frac{1}{\Gamma(1+\lambda)} \int_a^b \frac{\Xi_l(\psi_1(u), \ldots, \psi_l(u)) |\Upsilon_m(\varphi_1(u), \ldots, \varphi_m(u))|^s}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \ldots, \psi_l(u)) |\Upsilon_m(\varphi_1(u), \ldots, \varphi_m(u))|^s (du)^\lambda} (du)^\lambda \\
&+ \frac{1}{t} \frac{1}{\Gamma(1+\lambda)} \int_a^b \frac{\Xi_l(\psi_1(u), \ldots, \psi_l(u)) |Z_k(\phi_1(u), \ldots, \phi_k(u))|^t}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \ldots, \psi_l(u)) |Z_k(\phi_1(u), \ldots, \phi_k(u))|^t (du)^\lambda} (du)^\lambda \\
&= \frac{1}{s} + \frac{1}{t} = 1.
\end{aligned}$$

Therefore, we conclude that the desired inequality holds. □

Theorem 2 (reverse Hölder-type inequality)**.** *Assume that* $0 < s < 1$, $1/s + 1/t = 1$. *Let* $\Xi_l(\psi_1, \ldots, \psi_l) > 0$, $\Upsilon_m(\varphi_1, \ldots, \varphi_m)$ *and* $Z_k(\phi_1, \ldots, \phi_k)$ *be three arbitrary functions of l, m and k variables, respectively. Assume that* $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ *and* $\{\psi_i(u)\}_{i=1}^l$ *are local fractional continuous functions on* $[a, b]$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |\Upsilon_m(\varphi_1, \ldots, \varphi_m) Z_k(\phi_1, \ldots, \phi_k)| (du)^\lambda \\
&\geq \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |\Upsilon_m(\varphi_1, \ldots, \varphi_m)|^s (du)^\lambda \right)^{1/s} \\
&\left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |Z_k(\phi_1, \ldots, \phi_k)|^t (du)^\lambda \right)^{1/t}.
\end{aligned} \tag{6}$$

Moreover, the equality holds in (6) *if and only if* $A|\Upsilon_m(\varphi_1, \varphi_2, \ldots, \varphi_m)|^s = B|Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^t$, *where A and B are constants.*

Proof. Let $\Theta(\Xi_l, \Upsilon_m, Z_k)$ be the right-hand side of the inequality (6). If $\Theta(\Xi_l, \Upsilon_m, Z_k) = 0$, then $\Upsilon_m(\varphi_1, \varphi_2, \ldots, \varphi_m) = 0$ or $Z_k(\phi_1, \phi_2, \ldots, \phi_k) = 0$ and the conclusion follows by item 3 of Lemma 1. If $\Theta(\Xi_l, \Upsilon_m, Z_k) \neq 0$, then set

$$F(u) = \frac{(\Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)))^{1/s} |\Upsilon_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)) |\Upsilon_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|^s (du)^\lambda}$$

and

$$G(u) = \frac{(\Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)))^{1/t} |\Psi_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \psi_2(u), \ldots, \psi_l(u)) |\Psi_m(\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u))|^t (du)^\lambda}.$$

Thanks to the reverse Young's inequality (see [34])

$$XY \geq \tfrac{1}{s} X^s + \tfrac{1}{t} Y^t, \; X, Y > 0, \; 0 < s < 1, \; 1/s + 1/t = 1$$

with equality holds if and only if $X = Y$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)} \int_a^b F(u) G(u) (du)^\lambda \\
&\geq \frac{1}{\Gamma(1+\lambda)} \int_a^b [\frac{F^s(u)}{s} + \frac{G^t(u)}{t}] (du)^\lambda \\
&= \frac{1}{s} \frac{1}{\Gamma(1+\lambda)} \int_a^b \frac{\Xi_l(\psi_1(u), \ldots, \psi_l(u)) |\Psi_m(\varphi_1(u), \ldots, \varphi_m(u))|^s}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \ldots, \psi_l(u)) |\Psi_m(\varphi_1(u), \ldots, \varphi_m(u))|^s (du)^\lambda} (du)^\lambda \\
&\quad + \frac{1}{t} \frac{1}{\Gamma(1+\lambda)} \int_a^b \frac{\Xi_l(\psi_1(u), \ldots, \psi_l(u)) |Z_k(\phi_1(u), \ldots, \phi_k(u))|^t}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1(u), \ldots, \psi_l(u)) |Z_k(\phi_1(u), \ldots, \phi_k(u))|^t (du)^\lambda} (du)^\lambda \\
&= \frac{1}{s} + \frac{1}{t} = 1.
\end{aligned}$$

Therefore, we obtain the desired inequality. □

It should be pointed out that if $\Xi_l(\psi_1, \ldots, \psi_l) = 1$, $\Psi_m(\varphi_1, \ldots, \varphi_m) = \varphi$ and $Z_k(\phi_1, \ldots, \phi_k) = \phi$, then (5) and (6) reduce to (3) and (4), respectively.

Theorem 3 (Minkowski-type inequality)**.** *Assume that* $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ *and* $\{\psi_i(u)\}_{i=1}^l$ *are local fractional continuous functions on* $[a, b]$. *Let* $\Xi_l(\psi_1, \ldots, \psi_l) > 0$, $\Psi_m(\varphi_1, \ldots, \varphi_m)$ *and* $Z_k(\phi_1, \ldots, \phi_k)$ *be three arbitrary functions of* l, m *and* k *variables, respectively. Let* $s > 1$. *Then*

$$\begin{aligned}
&\left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |\Psi_m(\varphi_1, \ldots, \varphi_m) + Z_k(\phi_1, \ldots, \phi_k)|^s (du)^\lambda \right)^{1/s} \\
&\leq \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |\Psi_m(\varphi_1, \ldots, \varphi_m)|^s (du)^\lambda \right)^{1/s} + \\
&\quad \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |Z_k(\phi_1, \ldots, \phi_k)|^s (du)^\lambda \right)^{1/s}.
\end{aligned} \tag{7}$$

Moreover, the equality holds if in (7) $\Psi_m(\varphi_1, \ldots, \varphi_m)$ *and* $Z_k(\phi_1, \ldots, \phi_k)$ *are proportional.*

Proof. It follows from the triangle inequality that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |\Psi_m(\varphi_1, \ldots, \varphi_m) + Z_k(\phi_1, \ldots, \phi_k)|^s (du)^\lambda \\
&\leq \frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |\Psi_m(\varphi_1, \ldots, \varphi_m)| |\Psi_m(\varphi_1, \ldots, \varphi_m) + Z_k(\phi_1, \ldots, \phi_k)|^{s-1} (du)^\lambda \\
&\quad + \frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \ldots, \psi_l) |Z_k(\phi_1, \ldots, \phi_k)| |\Psi_m(\varphi_1, \ldots, \varphi_m) + Z_k(\phi_1, \ldots, \phi_k)|^{s-1} (du)^\lambda.
\end{aligned}$$

Next, applying local fractional integral Hölder's inequality (5) with $t = s/(s-1)$ to the above inequality, it can be derived that the following inequality holds.

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda \\
\leq&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^s(du)^\lambda\right)^{1/s} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^{t(s-1)}(du)^\lambda\right)^{1/t} \\
&+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{1/s} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^{t(s-1)}(du)^\lambda\right)^{1/t} \\
=&\left\{\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^s(du)^\lambda\right)^{1/s}\right. \\
&\left.+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{1/s}\right\} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^{t(s-1)}(du)^\lambda\right)^{1/t}.
\end{aligned}$$

Dividing the two sides of the above inequality by

$$\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^{t(s-1)}(du)^\lambda\right)^{1/t},$$

we obtain the desired inequality. □

Theorem 4 (reverse Minkowski-type inequality). *Let* $0 < s < 1$, $\Xi_l(\psi_1,\ldots,\psi_l) > 0$, $\Psi_m(\varphi_1,\ldots,\varphi_m)$ *and* $Z_k(\phi_1,\ldots,\phi_k)$ *be three arbitrary functions of* l, m *and* k *variables, respectively. Assume that* $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ *and* $\{\psi_i(u)\}_{i=1}^l$ *are local fractional continuous functions on* $[a,b]$, *then*

$$\begin{aligned}
&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{1/s} \\
\geq&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^s(du)^\lambda\right)^{1/s}+ \\
&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{1/s}.
\end{aligned} \tag{8}$$

Moreover, the equality holds if in (8) $\Psi_m(\varphi_1,\ldots,\varphi_m)$ *and* $Z_k(\phi_1,\ldots,\phi_k)$ *are proportional.*

Proof. Let

$$P = \frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^s(du)^\lambda,$$

$$Q = \frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda,$$

$$\begin{aligned}
\Pi =&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^s(du)^\lambda\right)^{1/s}+ \\
&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{1/s}.
\end{aligned}$$

By local fractional integral reverse Hölder inequality (6), in view of $0 < s < 1$, we have

$$\begin{aligned}\Pi &= \frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)(|\Psi_m(\varphi_1,\dots,\varphi_m)|^s P^{1/s-1} + |Z_k(\phi_1,\dots,\phi_k)|^s Q^{1/s-1})(du)^\lambda \\ &\le \frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m) + Z_k(\phi_1,\dots,\phi_k)|^s (P^{1/s} + Q^{1/s})^{1-s}(du)^\lambda \\ &= \Pi^{1-s}\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m) + Z_k(\phi_1,\dots,\phi_k)|^s(du)^\lambda.\end{aligned} \tag{9}$$

From inequality (9), it follows that local fractional integral reserve Minkowski's inequality holds and the theorem is completely proved. □

It is needed to mentioned that if $\Xi_l(\psi_1,\dots,\psi_l) = 1$, $\Psi_m(\varphi_1,\dots,\varphi_m) = \varphi$ and $Z_k(\phi_1,\dots,\phi_k) = \phi$, then (7) and (8) reduce to the results of [13] and [14], respectively.

4. A Subdividing of Local Fractional Integral Hölder-Type Inequality

In this section, we establish a subdividing of local fractional integral Hölder-type inequality (5) which an extension of the result in [16], its reverse form also is presented.

Theorem 5. *Let $\Xi_l(\psi_1,\dots,\psi_l) > 0$, $\Psi_m(\varphi_1,\dots,\varphi_m)$ and $Z_k(\phi_1,\dots,\phi_k)$ be three arbitrary functions of l, m and k variables, respectively. Assume that $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ and $\{\psi_i(u)\}_{i=1}^l$ are local fractional continuous functions on $[a,b]$, and $s,t \in \mathbb{R}$, and let $p = (s-t)/(1-t)$, $q = (s-t)/(s-1)$,*

(1) *if $s < 1 < t$ or $s > 1 > t$, then*

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)Z_k(\phi_1,\dots,\phi_k)|(du)^\lambda \\ \le& \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)|^{sp}(du)^\lambda\right)^{1/p^2} \\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^{tq}(du)^\lambda\right)^{1/q^2} \\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)|^{tp}(du)^\lambda\right)^{1/pq} \\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^{sq}(du)^\lambda\right)^{1/pq}\end{aligned} \tag{10}$$

with equality if and only if $\Psi_m(\varphi_1,\dots,\varphi_m)$ and $Z_k(\phi_1,\dots,\phi_k)$ are proportional.

(2) *if $s > t > 1$ or $s < t < 1$;$t > s > 1$ or $t < s < 1$, then*

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)Z_k(\phi_1,\dots,\phi_k)|(du)^\lambda \\ \ge& \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)|^{sp}(du)^\lambda\right)^{1/p^2} \\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^{tq}(du)^\lambda\right)^{1/q^2} \\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)|^{tp}(du)^\lambda\right)^{1/pq} \\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^{sq}(du)^\lambda\right)^{1/pq}\end{aligned} \tag{11}$$

with equality if and only if $\Psi_m(\varphi_1,\dots,\varphi_m)$ and $Z_k(\phi_1,\dots,\phi_k)$are proportional.

Proof. (1) Let $p = \frac{s-t}{1-t}$ and in view of $s < 1 < t$ or $s > 1 > t$. Then one has

$$p = \frac{s-t}{1-t} > 1,$$

by local fractional integral Hölder's inequality (3) with indices $\frac{s-t}{1-t}$ and $\frac{s-t}{s-1}$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|(du)^\lambda \\
&= \frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)[|\Psi_m Z_k|^s]^{(1-t)/(s-t)}[|\Psi_m Z_k|^t]^{(s-1)/(s-t)}(du)^\lambda \\
&\leq \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{(1-t)/(s-t)} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda\right)^{(s-1)/(s-t)}
\end{aligned} \tag{12}$$

with equality if and only if $|\Psi_m Z_k|^s$ and $|\Psi_m Z_k|^t$ are proportional.

On the other hand, from local fractional integral Hölder's inequality again for $p = \frac{s-t}{1-t} > 1$, it follows that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda \\
&\leq \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^{s(s-t)/(1-t)}(du)^\lambda\right)^{(1-t)/(s-t)} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^{s(s-t)/(s-1)}(du)^\lambda\right)^{(s-1)/(s-t)}
\end{aligned} \tag{13}$$

with equality if and only if $|\Psi_m|^{s(s-t)/(1-t)}$ and $|Z_k|^{s(s-t)/(s-1)}$ are proportional, and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda \\
&\leq \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^{t(s-t)/(1-t)}(du)^\lambda\right)^{(1-t)/(s-t)} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^{t(s-t)/(s-1)}(du)^\lambda\right)^{(s-1)/(s-t)}
\end{aligned} \tag{14}$$

with equality if and only if $|\Psi_m|^{t(s-t)/(1-t)}f^{t(s-t)/(1-t)}$ and $|Z_k|^{t(s-t)/(s-1)}$ are proportional.

From (12)–(14), the case (1) of Theorem 5 is proved.

(2) Let $p = \frac{s-t}{1-t}$ and in view of $s > t > 1$ or $s < t < 1$, it is obvious that

$$0 < p = \frac{s-t}{1-t} < 1.$$

Since $t > s > 1$ or $t < s < 1$, one has $\frac{s-t}{1-t} < 0$. Based on local fractional integral reverse Hölder's inequality (4) with indices $\frac{s-t}{1-t}$ and $\frac{s-t}{s-1}$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|(du)^\lambda \\
&= \frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)[|\Psi_m Z_k|^s]^{(1-t)/(s-t)}[|\Psi_m Z_k|^t]^{(s-1)/(s-t)}(du)^\lambda \\
&\geq \left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{(1-t)/(s-t)} \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda\right)^{(s-1)/(s-t)}
\end{aligned} \tag{15}$$

with equality if and only if $|\Psi_m Z_k|^s$ and $|\Psi_m Z_k|^t$ are proportional.

On the other hand, by local fractional integral reverse Hölder's inequality (4) again for $0 < p = \frac{s-t}{1-t} < 1$ or $p = \frac{s-t}{1-t} < 0$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda \\
\geq&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^{s(s-t)/(1-t)}(du)^\lambda\right)^{(1-t)/(s-t)} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^{s(s-t)/(s-1)}(du)^\lambda\right)^{(s-1)/(s-t)}
\end{aligned} \tag{16}$$

with equality if and only if $|\Psi_m|^{s(s-t)/(1-t)}$ and $|Z_k|^{s(s-t)/(s-1)}$ are proportional, and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda \\
\geq&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^{t(s-t)/(1-t)}(du)^\lambda\right)^{(1-t)/(s-t)} \\
&\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^{t(s-t)/(s-1)}(du)^\lambda\right)^{(s-1)/(s-t)}
\end{aligned} \tag{17}$$

with equality if and only if $|\Psi_m|^{t(s-t)/(1-t)}f^{t(s-t)/(1-t)}$ and $|Z_k|^{t(s-t)/(s-1)}$ are proportional.

From (15)–(17), the proof of case (2) of Theorem 5 is completed. □

5. An Improvement of Local Fractional Integral Minkowski-Type Inequality

In this section, we are devoted to deriving an improvement of local fractional integral Minkowski-type inequality (7), its reverse version is also obtained.

Theorem 6. *Let $\Xi_l(\psi_1,\ldots,\psi_l) > 0$, $\Psi_m(\varphi_1,\ldots,\varphi_m)$ and $Z_k(\phi_1,\ldots,\phi_k)$ be three arbitrary functions of l, m and k variables, respectively. Assume that $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ and $\{\psi_i(u)\}_{i=1}^l$ are local fractional continuous functions on $[a,b]$, $p > 0$, $s,t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$.*

(1) *Let $p,s,t \in \mathbb{R}$ be different, such that $s,t > 1$ and $(s-t)/(p-t) > 1$. Then*

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^p(du)^\lambda \\
\leq&\left[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^s(du)^\lambda\right)^{\frac{1}{s}}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^s(du)^\lambda\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\
&\times\left[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^t(du)^\lambda\right)^{\frac{1}{t}}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^t(du)^\lambda\right)^{\frac{1}{t}}\right]^{t(p-t)/(s-t)}.
\end{aligned} \tag{18}$$

(2) *Let $p,s,t \in \mathbb{R}$ be different, such that $0 < s,t < 1$ and $(s-t)/(p-t) < 1$. Then*

$$\begin{aligned}
&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^p(du)^\lambda \\
\geq&\left[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^s(du)^\lambda\right)^{\frac{1}{s}}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^s(du)^\lambda\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\
&\times\left[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^t(du)^\lambda\right)^{\frac{1}{t}}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^t(du)^\lambda\right)^{\frac{1}{t}}\right]^{t(p-t)/(s-t)}.
\end{aligned} \tag{19}$$

Proof. (1) According to $(s-t)/(p-t) > 1$, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^p(du)^\lambda\\ =&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(|\Psi_m+Z_k|^s)^{(p-t)/(s-t)}(|\Psi_m+Z_k|^t)^{(s-p)/(s-t)}(du)^\lambda.\end{aligned}$$

By using local fractional integral Hölder's inequality (3) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, it can be concluded that

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^p(du)^\lambda\\ \le&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{(p-t)/(s-t)}\\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda\right)^{(s-p)/(s-t)}.\end{aligned}\tag{20}$$

On the other hand, by using local fractional integral Minkowski's inequality (7) for $s > 1$ and $t > 1$, respectively, we obtain

$$\begin{aligned}&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{\frac{1}{s}}\\ \le&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^s(du)^\lambda\right)^{\frac{1}{s}}\\ &+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{\frac{1}{s}}\end{aligned}\tag{21}$$

and

$$\begin{aligned}&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda\right)^{\frac{1}{t}}\\ \le&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)|^t(du)^\lambda\right)^{\frac{1}{t}}\\ &+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda\right)^{\frac{1}{t}}.\end{aligned}\tag{22}$$

From (20)–(22), it follows that the desired result holds.

(2) Based on $(s-t)/(p-t) < 1$, a direct computation yields

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^p(du)^\lambda\\ =&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(|\Psi_m+Z_k|^s)^{(p-t)/(s-t)}(|\Psi_m+Z_k|^t)^{(s-p)/(s-t)}(du)^\lambda.\end{aligned}$$

By using local fractional integral reverse Hölder's inequality (4) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^p(du)^\lambda\\ \ge&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^s(du)^\lambda\right)^{(p-t)/(s-t)}\\ &\times\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\ldots,\psi_l)|\Psi_m(\varphi_1,\ldots,\varphi_m)+Z_k(\phi_1,\ldots,\phi_k)|^t(du)^\lambda\right)^{(s-p)/(s-t)}.\end{aligned}\tag{23}$$

On the other hand, in view of local fractional integral Minkowski's inequality (8) for the cases of $0 < s < 1$ and $0 < t < 1$, we obtain

$$\begin{aligned}&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)+Z_k(\phi_1,\dots,\phi_k)|^s(du)^\lambda\right)^{\frac{1}{s}}\\ \geq&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)|^s(du)^\lambda\right)^{\frac{1}{s}}\\ &+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^s(du)^\lambda\right)^{\frac{1}{s}}\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)+Z_k(\phi_1,\dots,\phi_k)|^t(du)^\lambda\right)^{\frac{1}{t}}\\ \geq&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|\Psi_m(\varphi_1,\dots,\varphi_m)|^t(du)^\lambda\right)^{\frac{1}{t}}\\ &+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^t(du)^\lambda\right)^{\frac{1}{t}}.\end{aligned} \tag{25}$$

By (23)–(25), we conclude that the desired result holds. □

Remark 1. *(1) For Theorem 6, for* $p > 1$, *taking* $s = p + \varepsilon$, $t = p - \varepsilon$, *when* p, s, t *are different,* $s, t > 1$, *and* $(s - t)/(p - t)/2 > 1$, *and taking* $\varepsilon \to 0$, *it follows that* (18) *is transformed into* (7)*.*

(2) For Theorem 6, for $0 < p < 1$, *taking* $s = p + \varepsilon$, $t = p - \varepsilon$, *when* p, s, t *are different,* $0 < s, t < 1$, *and* $0 < (s - t)/(p - t)/2 < 1$, *and taking* $\varepsilon \to 0$, *it follows that* (19) *is transformed into* (8)*.*

(3) If $\Xi_l(\psi_1,\dots,\psi_l) = 1$, $\Psi_m(\varphi_1,\dots,\varphi_m) = \varphi$ *and* $Z_k(\phi_1,\dots,\phi_k) = \phi$, *then Theorem 6 becomes to the results of [17].*

6. Dresher-Type Inequality and Its Reverse Form

The aim of this section is to establish a functional generalization of local fractional integral Dresher's inequality obtained by Chen [14], its reverse form is also derived.

Theorem 7 (Dresher's inequality)**.** *Let* $0 < r \leq 1 \leq p$, $\Xi_l(\psi_1,\dots,\psi_l) > 0$, $\Psi_m(\varphi_1,\dots,\varphi_m)$ *and* $Z_k(\phi_1,\dots,\phi_k)$ *be three arbitrary functions of* l, m *and* k *variables, respectively. Assume that* $\{\varphi_i(u)\}_{i=1}^m$, $\{\phi_i(u)\}_{i=1}^k$ *and* $\{\psi_i(u)\}_{i=1}^l$ *are local fractional continuous functions on* $[a, b]$*, then*

$$\begin{aligned}&\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)+Z_k(\phi_1,\phi_2,\dots,\phi_k)|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)+Z_k(\phi_1,\phi_2,\dots,\phi_k)|^r(du)^\lambda}\right)^{1/(p-r)}\\ \leq&\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|^r(du)^\lambda}\right)^{1/(p-r)}\\ &+\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|Z_k(\phi_1,\phi_2,\dots,\phi_k)|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|Z_k(\phi_1,\phi_2,\dots,\phi_k)|^r(du)^\lambda}\right)^{1/(p-r)},\end{aligned} \tag{26}$$

where the equality holds true if and only if the functions $|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|$ *and* $|Z_k(\phi_1,\phi_2,\dots,\phi_k)|$ *are proportional.*

Proof. It is clear that

$$\begin{aligned}&\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)+Z_k(\phi_1,\phi_2,\dots,\phi_k)|^p(du)^\lambda\right)^{1/(p-r)}\\ &\le\left(\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^p(du)^\lambda\right)^{1/p}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^p(du)^\lambda\right)^{1/p}\right)^{1/(p-r)},\end{aligned}\tag{27}$$

by local fractional intehral Minkowski's inequality (7). Next, by the right-hand side of above inequality we have

$$\begin{aligned}&\left(\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^p(du)^\lambda\right)^{1/p}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^p(du)^\lambda\right)^{1/p}\right)^{1/(p-r)}\\ &=\left(\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda}\right)^{1/p}\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda\right)^{1/p}\right.\\ &\quad\left.+\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda}\right)^{1/p}\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda\right)^{1/p}\right)^{1/(p-r)}.\end{aligned}$$

By using local fractional integral Hölder's inequality with $q=p/(p-1)$ to the above equality, we have

$$\begin{aligned}&\left(\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda}\right)^{1/p}\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda\right)^{1/p}\right.\\ &\left.+\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda}\right)^{1/p}\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda\right)^{1/p}\right)^{1/(p-r)}\\ &\le\left(\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda}\right)^{1/(p-r)}+\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda}\right)^{1/(p-r)}\right)\\ &\times\left(\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda\right)^{1/r}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda\right)^{1/r}\right)^{r/(p-r)}.\end{aligned}\tag{28}$$

From local fractional integral reverse Minkowski's inequality with $0<r<1$, it follows that

$$\begin{aligned}&\left(\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda\right)^{1/r}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda\right)^{1/r}\right)^r\\ &\le\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)+Z_k(\phi_1,\phi_2,\dots,\phi_k)|^r(du)^\lambda.\end{aligned}\tag{29}$$

From (27)–(29), it is concluded that the desired inequality holds. □

Theorem 8 (reverse Dresher's inequality)**.** *Let* $p \leq 0 \leq r \leq 1$, $\Xi_l(\psi_1, \ldots, \psi_l) > 0$, $\Psi_m(\varphi_1, \ldots, \varphi_m)$ *and* $Z_k(\phi_1, \ldots, \phi_k)$ *be three arbitrary functions of l; m and k variables, respectively. Assume that* $\{\varphi_i(u)\}_{i=1}^{m}$, $\{\phi_i(u)\}_{i=1}^{k}$ *and* $\{\psi_i(u)\}_{i=1}^{l}$ *are local fractional continuous functions on* $[a, b]$, *then*

$$\begin{aligned} &\left(\frac{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m) + Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^p (du)^\lambda}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m) + Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^r (du)^\lambda} \right)^{1/(p-r)} \\ \geq &\left(\frac{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m)|^p (du)^\lambda}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m)|^r (du)^\lambda} \right)^{1/(p-r)} \\ + &\left(\frac{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^p (du)^\lambda}{\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^r (du)^\lambda} \right)^{1/(p-r)}, \end{aligned} \tag{30}$$

where the equality holds true if and only if the functions $|\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m)|$ *and* $|Z_k(\phi_1, \phi_2, \ldots, \phi_k)|$ *are proportional.*

Proof. Let $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\eta_1 > 0$, and $\eta_2 > 0$, and $-1 < \alpha < 0$, applying the Radon's inequality (see [3])

$$\sum_{i=1}^{n} \frac{x_i^p}{y_i^{p-1}} < \frac{(\sum\limits_{i=1}^{n} x_i)^p}{(\sum\limits_{i=1}^{n} y_i)^{p-1}}, \; x_k \geq 0, \; y_k > 0, 0 < p < 1,$$

we have

$$\frac{\lambda_1^{\alpha+1}}{\eta_1^{\alpha}} + \frac{\lambda_2^{\alpha+1}}{\eta_2^{\alpha}} \leq \frac{(\lambda_1 + \lambda_2)^{\alpha+1}}{(\eta_1 + \eta_2)^{\alpha}}, \tag{31}$$

where the equality holds true if and only if (λ) and (η) are proportional.

Let

$$\lambda_1 = \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m)|^p (du)^\lambda \right)^{1/p}, \tag{32}$$

$$\eta_1 = \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |\Psi_m(\varphi_1, \varphi_2, \ldots, \varphi_m)|^r (du)^\lambda \right)^{1/r}, \tag{33}$$

$$\lambda_2 = \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^p (du)^\lambda \right)^{1/p}, \tag{34}$$

$$\eta_2 = \left(\frac{1}{\Gamma(1+\lambda)} \int_a^b \Xi_l(\psi_1, \psi_2, \ldots, \psi_l) |Z_k(\phi_1, \phi_2, \ldots, \phi_k)|^r (du)^\lambda \right)^{1/r}, \tag{35}$$

and set $\alpha = \frac{r}{p-r}$. In term of (31)–(35), one has

$$\begin{aligned}
\frac{\lambda_1^{\alpha+1}}{\eta_1^{\alpha}}+\frac{\lambda_2^{\alpha+1}}{\eta_2^{\alpha}}=&\frac{\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|^p(du)^\lambda\right)^{(\alpha+1)/p}}{\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|^r(du)^\lambda\right)^{\alpha/r}}\\
&+\frac{\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|Z_k(\phi_1,\phi_2,\dots,\phi_k)|^p(du)^\lambda\right)^{(\alpha+1)/p}}{\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|Z_k(\phi_1,\phi_2,\dots,\phi_k)|^r(du)^\lambda\right)^{\alpha/r}}\\
=&\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\psi_2,\dots,\psi_l)|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|^r(du)^\lambda}\right)^{1/(p-r)}\\
&+\left(\frac{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^p(du)^\lambda}{\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l(\psi_1,\dots,\psi_l)|Z_k(\phi_1,\dots,\phi_k)|^r(du)^\lambda}\right)^{1/(p-r)}\le\frac{(\lambda_1+\lambda_2)^{\alpha+1}}{(\eta_1+\eta_2)^{\alpha}}\\
=&\frac{[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^p(du)^\lambda\right)^{1/p}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^p(du)^\lambda\right)^{1/p}]^{p/(p-r)}}{[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m|^r(du)^\lambda\right)^{1/r}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k|^r(du)^\lambda\right)^{1/r}]^{r/(p-r)}}.
\end{aligned}\tag{36}$$

□

Since $-1<\alpha=\frac{r}{p-r}<0$, we may assume $p<0<r$, and by local fractional integral Minkowski inequality for $p<0$ and $0<r\le 1$, we obtain, respectively

$$\begin{aligned}
&\left[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m(\varphi_1,\dots,\varphi_m)|^p(du)^\lambda\right)^{\frac{1}{p}}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k(\phi_1,\dots,\phi_k)|^p(du)^\lambda\right)^{\frac{1}{p}}\right]^p\\
\ge&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m(\varphi_1,\dots,\varphi_m)+Z_k(\phi_1,\dots,\phi_k)|^p(du)^\lambda
\end{aligned}\tag{37}$$

with equality if and only if $|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|$ and $|Z_k(\phi_1,\phi_2,\dots,\phi_k)|$ are proportional, and

$$\begin{aligned}
&\left[\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m(\varphi_1,\dots,\varphi_m)|^r(du)^\lambda\right)^{\frac{1}{r}}+\left(\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|Z_k(\phi_1,\dots,\phi_k)|^r(du)^\lambda\right)^{\frac{1}{r}}\right]^r\\
\le&\frac{1}{\Gamma(1+\lambda)}\int_a^b \Xi_l|\Psi_m(\varphi_1,\dots,\varphi_m)+Z_k(\phi_1,\dots,\phi_k)|^r(du)^\lambda,
\end{aligned}\tag{38}$$

where the equality holds true if and only if $|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|$ and $|Z_k(\phi_1,\phi_2,\dots,\phi_k)|$ are proportional,

From equality conditions for (31), (37) and (38), it follows that the sign of equality in (30) holds if and only if $|\Psi_m(\varphi_1,\varphi_2,\dots,\varphi_m)|$ and $|Z_k(\phi_1,\phi_2,\dots,\phi_k)|$ are proportional.

From (36)–(38), we arrive to desired local fractional integral reverse Dresher's inequality.

7. Conclusions

In this paper, with the help of local fractional integral theory, which is used in various problems involving continuously nondifferentiable functions and fractals, we establish some functional generalizations of local fractional integral Hölder-type inequality and its reverse form. Besides that, we apply obtained inequalities to derive Minkowski-type and Dresher-type inequalities, as well as their reverse forms. It also is shown that many existing local fractional integral inequalities are special cases of the main results which are obtained in this work. In future work, we will continue to consider the applications of the obtained results.

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