GENERATING FUNCTIONS FOR A CLASS OF q-POLYNOMIALS

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ABSTRACT

Some simple ideas are used here to prove a theorem on generating functions for a certain class of q-polynomials. This general theorem is then applied to derive a fairly large number of known as well as new generating functions for the familiar q-analogues of various polynomial systems including, for example, the classical orthogonal polynomials of Hermite, Jacobi, and Laguerre. A number of other interesting consequences of the theorem are also discussed.

1. INTRODUCTION, NOTATIONS, AND THE MAIN RESULT

A great surge of activities in the theory of q-series and q-polynomials has been witnessed in recent years. Various q-extensions of well-known hypergeometric identities and quadratic transformations have recently been obtained by several workers. These q-extensions are known to have important applications in many areas of pure as well as applied mathematics, physics, and engineering. Workers in the field of q-series and q-polynomials are realizing the need of extending all the important results involving special functions to hold for their q-analogues. With this objective in mind, we prove a general theorem on generating functions for an important class of q-polynomials, and then apply this theorem not only to derive q-extensions of several familiar generating functions, but also to deduce (for example) Jackson's q-Pfaff transformation [8] which Andrews [3, p. 527] used to prove q-analogues of Kummer's summation theorem and Gauss's second theorem, Hahn's q-analogue [7] of Kummer's first formula, and Jackson's q-analogue [9] of the celebrated Pfaff-Saalschütz theorem.

For real or complex q, $|q| < 1$, let

$$(1.1) \qquad (\lambda;q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right)$$

for arbitrary λ and μ, so that

$$(1.2)\quad \begin{cases} (\lambda;q)_0 = 1;\ (\lambda;q)_n = (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}),\ \forall n \in \{1,2,3,\ldots\}, \text{ and} \\ (\lambda;q)_\infty = \prod_{j=0}^{\infty} (1-\lambda q^j). \end{cases}$$

Define, as usual, a generalized basic (or q-) hypergeometric function by (cf. [11, Chapter 3]; see also [13, p. 347, Equation (272)])

$$(1.3)\quad {}_{p+1}\Phi_{p+r}\left[\begin{matrix}\alpha_1, \ldots, \alpha_{p+1}; \\ \beta_1, \ldots, \beta_{p+r};\end{matrix}\ q, z\right]$$

$$= \sum_{n=0}^{\infty} (-1)^{rn} q^{\frac{1}{2}rn(n-1)} \frac{(\alpha_1;q)_n \cdots (\alpha_{p+1};q)_n}{(\beta_1;q)_n \cdots (\beta_{p+r};q)_n} \frac{z^n}{(q;q)_n},$$

where, for convergence, $|q| < 1$ and $|z| < \infty$ when r is a positive integer, or $|z| < 1$ when $r = 0$, provided that no zeros appear in the denominator.

We shall also need the Gaussian polynomial (or q-binomial coefficient) defined, for all non-negative integers n and k, by (see, e.g., [4, p. 35])

$$(1.4)\quad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{j=1}^{k} \left(\frac{1-q^{n-j+1}}{1-q^j}\right), & \text{if } 1 \leqq k \leqq n, \\ 0, & \text{if } k > n. \end{cases}$$

For a non-negative integer m, the familiar q-binomial theorem (cf. [4, p. 17, Theorem 2.1])

$$(1.5)\quad {}_1\Phi_0\left[\begin{matrix}\lambda; \\ -;\end{matrix}\ q, t\right] \equiv \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n} t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}, \quad |t| < 1, \quad |q| < 1$$

can be rewritten at once as

$$(1.6)\quad \sum_{n=0}^{\infty} \frac{(\lambda;q)_{m+n}}{(q;q)_n} t^n = \frac{(\lambda;q)_m}{(\lambda t;q)_m} \frac{(\lambda t;q)_\infty}{(t;q)_\infty}, \quad |t| < 1, \quad |q| < 1,$$

which, in view of (1.2), yields (1.5) when $m = 0$ (or when λ is replaced by λq^{-m}). Making use of (1.6), we shall prove the following

THEOREM. <u>In terms of a bounded complex sequence</u> $\{S_{n,q}\}_{n=0}^{\infty}$ <u>generated by</u>

$$F_{\omega}(\lambda,\mu,q,t) = \sum_{n=0}^{\infty} \frac{(\lambda;q)_{\omega n}}{(\lambda\mu;q)_{\omega n}(q;q)_{\omega n}} S_{n,q}\, t^n, \tag{1.7}$$

<u>define a family of basic</u> (<u>or</u> q-) <u>polynomials</u> $\{f_{n,N}(x;q)\}_{n=0}^{\infty}$ <u>by</u>

$$f_{n,N}(x;q) = \sum_{k=0}^{[n/N]} \begin{bmatrix} n \\ Nk \end{bmatrix} S_{k,q}\, x^k \qquad (n = 0,1,2,\ldots), \tag{1.8}$$

<u>where</u> N <u>is a positive integer</u>.

<u>Then</u>

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n} f_{n,N}(x;q)\, t^n = \frac{(\lambda t;q)_{\infty}}{(t;q)_{\infty}} F_N(\lambda,t,q,xt^N), \tag{1.9}$$

<u>provided that each side exists</u>, $|t| < 1$, <u>and</u> $|q| < 1$.

2. PROOF OF THE THEOREM

Denote, for convenience, the left-hand side of our assertion (1.9) by $\Omega(t)$. Substituting for $f_{n,N}(x;q)$ from the definition (1.8) into $\Omega(t)$, and inverting the order of summation, we have

$$\Omega(t) = \sum_{k=0}^{\infty} S_{k,q} \frac{(xt^N)^k}{(q;q)_{Nk}} \sum_{n=0}^{\infty} \frac{(\lambda;q)_{n+Nk}}{(q;q)_n} t^n,$$

provided that the series involved converge absolutely.

Now sum the inner series by appealing to (1.6) with $m = Nk$, and we find for $|t| < 1$ and $|q| < 1$ that

$$\Omega(t) = \frac{(\lambda t;q)_{\infty}}{(\lambda;q)_{\infty}} \sum_{k=0}^{\infty} \frac{(\lambda;q)_{Nk}}{(\lambda t;q)_{Nk}(q;q)_{Nk}} S_{k,q}\, (xt^N)^k.$$

Interpreting this last expression by means of the generating relation (1.7), we are led immediately to the theorem.

REMARK. For substantially more general classes of q-generating functions, and for their multivariable extensions, the reader should refer to Section 3 of a recent paper by Srivastava [12].

3. APPLICATIONS

We begin by applying our theorem to derive generating functions for the q-analogues of many of the classical orthogonal polynomials. Setting

$$S_{n,q} = \frac{q^{n(n-1)}}{(\alpha q;q)_n}$$

in our theorem, we find from (1.8) that

$$f_{n,1}(x;q) = {}_1\Phi_1\left[\begin{matrix} q^{-n}; \\ \\ \alpha q; \end{matrix}\; q,\ xq^n\right] = \frac{(q;q)_n}{(\alpha q;q)_n} L_n^{(\alpha)}(x;q),$$

where $L_n^{(\alpha)}(x;q)$ denotes the q-Laguerre polynomial defined by (cf. [6])

$$\text{(3.1)} \qquad L_n^{(\alpha)}(x;q) = \frac{(\alpha q;q)_n}{(q;q)_n}\, {}_1\Phi_1\left[\begin{matrix} q^{-n}; \\ \\ \alpha q; \end{matrix}\; q,\ xq^n\right].$$

Thus our theorem yields the following generating function for the q-Laguerre polynomials:

$$\text{(3.2)} \qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\alpha q;q)_n} L_n^{(\alpha)}(x;q)\, t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\, {}_1\Phi_2\left[\begin{matrix} \lambda; \\ \\ \alpha q,\ \lambda t; \end{matrix}\; q,\ xt\right],$$

which provides a q-extension of a well-known generating function for Laguerre polynomials [14, p. 132, Equation (5)].

Next we consider the little q-Jacobi polynomials defined by (cf. [6])

$$(3.3) \qquad P_n^{(\alpha,\beta)}(x;q) = \frac{(\alpha q;q)_n}{(q;q)_n}\, {}_2\Phi_1\left[\begin{matrix} q^{-n},\ \alpha\beta q^{n+1}; \\ \alpha q; \end{matrix}\ q,\ qx\right],$$

and our theorem with $N = 1$, and

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha\beta q;q)_n}{(\alpha q;q)_n}$$

gives us the generating function:

$$(3.4) \qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\alpha q;q)_n} P_n^{(\alpha,\beta q^{-n})}(xq^n;q)\,t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\, {}_2\Phi_2\left[\begin{matrix} \lambda,\ \alpha\beta q; \\ \alpha q,\ \lambda t; \end{matrix}\ q,\ xqt\right].$$

For $\lambda = 0$, (3.4) reduces immediately to

$$(3.5) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta q^{-n})}(xq^n;q)\, \frac{t^n}{(\alpha q;q)_n} = \frac{1}{(t;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix} \alpha\beta q; \\ \alpha q; \end{matrix}\ q,\ xqt\right],$$

which is a q-extension of a known generating function for Jacobi polynomials ([1, p. 159, Equation (3.5)]; see also [14, p. 170, Problem 19(i)]).

Setting

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha\beta q;q)_n(\nu q/x;q)_n}{(\beta q;q)_n(\nu q;q)_n},$$

we observe from (1.8) that

$$f_{n,1}(x;q) = \frac{(\alpha^{-1};q)_n}{(\beta q;q)_n} Q_n(\alpha x;\alpha q^{-n},\beta,\nu|q)$$

in terms of the q-Hahn polynomials defined by

$$(3.6) \qquad Q_n(x;\alpha,\beta,\nu|q) = {}_3\Phi_2\left[\begin{matrix} q^{-n},\ \alpha\beta q^{n+1},\ x; \\ \alpha q,\ \nu q; \end{matrix}\ q,\ q\right]$$

or, equivalently, by

$$(3.7)\qquad Q_n(x;\alpha,\beta,\nu|q) = \frac{(\beta q;q)_n}{(1/\alpha q^n;q)_n}\ {}_3\Phi_2\left[\begin{matrix} q^{-n},\ \alpha\beta q^{n+1},\ \nu q/x; \\ \beta q,\ \nu q; \end{matrix}\ q,\ \frac{x}{\alpha}\right].$$

Our theorem when applied to the q-Hahn polynomials yields the generating function:

$$(3.8)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n(\alpha^{-1};q)_n}{(\beta q;q)_n(q;q)_n} Q_n(x;\alpha q^{-n},\beta,\nu|q)t^n$$

$$= \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\ {}_3\Phi_3\left[\begin{matrix} \lambda,\ \alpha\beta q,\ \nu q/x; \\ \beta q,\ \nu q,\ \lambda t; \end{matrix}\ q,\ \frac{xt}{\alpha}\right].$$

Similarly, for the q-Meixner polynomials defined by

$$(3.9)\qquad M_n(x;\beta,\gamma|q) = (\beta;q)_n\ {}_2\Phi_1\left[\begin{matrix} q^{-n},\ x; \\ \beta; \end{matrix}\ q,\ \frac{q^{n+1}}{\gamma}\right],$$

we obtain the generating function

$$(3.10)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\beta;q)_n(q;q)_n} M_n(x;\beta,\gamma|q)t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\ {}_2\Phi_2\left[\begin{matrix} \lambda,\ x; \\ \beta,\ \lambda t; \end{matrix}\ q,\ \frac{qt}{\gamma}\right].$$

In particular, (3.10) with $\lambda = \beta$ yields

$$(3.11)\qquad \sum_{n=0}^{\infty} M_n(x;\beta,\gamma|q)\frac{t^n}{(q;q)_n} = \frac{(\beta t;q)_\infty}{(t;q)_\infty}\ {}_1\Phi_1\left[\begin{matrix} x; \\ \beta t; \end{matrix}\ q,\ \frac{qt}{\gamma}\right],$$

which provides a q-extension of a known generating function for the Meixner polynomials [5, p. 225, Equation 10.24(13)].

The definitions (3.3) and (3.9) imply the following relationship between q-Meixner polynomials and the little q-Jacobi polynomials:

$$(3.12)\qquad M_n(x;\beta,\gamma|q) = (q;q)_n\ p_n^{(\beta/q,\,x/\beta q^n)}\left(\frac{q^n}{\gamma}\right),$$

which can be used to show that the generating functions (3.4) and (3.10), and indeed also (3.5) and (3.11), are essentially the same.

Now we turn to the q-Charlier polynomials defined by

$$c_n(x;\alpha|q) = {}_2\Phi_1\left[\begin{matrix} q^{-n}, x; \\ 0; \end{matrix}\ q, -\frac{q^{n+1}}{\alpha}\right] \tag{3.13}$$

for which our theorem with $N = 1$, and

$$S_{n,q} = q^{\frac{1}{2}n(n+1)}(x;q)_n$$

readily yields the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n} c_n(x;\alpha|q)t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty} {}_2\Phi_2\left[\begin{matrix} \lambda, x; \\ \lambda t, 0; \end{matrix}\ q, -\frac{qt}{\alpha}\right]. \tag{3.14}$$

In its special case when $\lambda = 0$, (3.14) reduces immediately to

$$\sum_{n=0}^{\infty} c_n(x;\alpha|q) \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty} {}_1\Phi_1\left[\begin{matrix} x; \\ 0; \end{matrix}\ q, -\frac{qt}{\alpha}\right], \tag{3.15}$$

which is a q-extension of a known generating function for Charlier polynomials [5, p. 226, Equation 10.25(6)].

Setting

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)},$$

the definition (1.8) assumes the form:

$$f_{n,1}(x;q) = (x;q)_n,$$

and our theorem immediately yields the identity:

$${}_2\Phi_1\left[\begin{matrix} \lambda, x; \\ 0; \end{matrix}\ q, t\right] = \frac{(\lambda t;q)_\infty}{(t;q)_\infty} {}_1\Phi_1\left[\begin{matrix} \lambda; \\ \lambda t; \end{matrix}\ q, xt\right]. \tag{3.16}$$

On the other hand, in view of Heine's transformation (cf. [4, p. 19, Corollary 2.3]; see also [13, p. 348, Equation (275)])

$$(3.17)\qquad {}_2\Phi_1\left[\begin{matrix} a,\ b; \\ c; \end{matrix}\ q,\ z\right] = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} z,\ c/b; \\ az; \end{matrix}\ q,\ b\right],$$

the first member of (3.16) can also be expressed as

$$(3.18)\qquad {}_2\Phi_1\left[\begin{matrix} \lambda,\ x; \\ 0; \end{matrix}\ q,\ t\right] = \frac{(x;q)_\infty (\lambda t;q)_\infty}{(t;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} t,\ 0; \\ \lambda t; \end{matrix}\ q,\ x\right].$$

Comparing (3.16) and (3.18), we readily obtain [7, p. 374, Equation (10.2)]

$$(3.19)\qquad {}_2\Phi_1\left[\begin{matrix} a,\ 0; \\ b; \end{matrix}\ q,\ z\right] = \frac{1}{(z;q)_\infty}\ {}_1\Phi_1\left[\begin{matrix} b/a; \\ b; \end{matrix}\ q,\ az\right],$$

which is a q-extension of Kummer's first formula for the confluent hypergeometric function [10, p. 125, Theorem 42].

The orthogonal q-polynomials $\phi_n^{(\alpha)}(x;q)$ studied by Al-Salam and Carlitz [2, p. 48, Equation (1.11)] are precisely the polynomials defined by (1.8) with $N = 1$, and

$$S_{n,q} = (\alpha;q)_n.$$

Thus our theorem yields the following generating function for $\phi_n^{(\alpha)}(x;q)$:

$$(3.20)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n}\ \phi_n^{(\alpha)}(x;q)\,t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} \lambda,\ \alpha; \\ \lambda t; \end{matrix}\ q,\ xt\right],$$

which, for $\lambda = 0$, reduces to the following result due to Al-Salam and Carlitz [2, p. 48, Equation (1.13)]:

$$(3.21)\qquad \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x;q)\ \frac{t^n}{(q;q)_n} = \frac{(\alpha xt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}\ .$$

Setting $\alpha = 0$ in (3.20) and then applying (3.19), we have

$$(3.22)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n} H_n(x;q)\, t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty (xt;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix} t; \\ \lambda t; \end{matrix}\; q,\ \lambda xt\right],$$

where $H_n(x;q)$ denotes the q-Hermite polynomial defined by (cf. [15]; see also [4, p. 49])

$$(3.23)\qquad H_n(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k .$$

Formula (3.22) may be compared with a divergent generating function for the classical Hermite polynomials (see, e.g., [14, p. 138, Equation (7)]). On the other hand, a further special case of (3.21) when $\alpha = 0$ [that is, (3.22) with $\lambda = 0$] is a well-known result [4, p. 49, Example 3].

Yet another interesting application of our theorem with $x = \beta/\alpha$, $N = 1$, and

$$S_{n,q} = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(\alpha;q)_n}{(\beta;q)_n}$$

leads us to the generating function:

$$(3.24)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n}\, {}_2\Phi_1\left[\begin{matrix} q^{-n}, \alpha; \\ \beta; \end{matrix}\; q,\ \frac{\beta}{\alpha} q^n\right] t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\, {}_2\Phi_2\left[\begin{matrix} \lambda, \alpha; \\ \beta, \lambda t; \end{matrix}\; q,\ \frac{\beta t}{\alpha}\right].$$

In view of the q-summation formula [11, p. 97, Equation (3.3.2.6)]:

$$(3.25)\qquad {}_2\Phi_1\left[\begin{matrix} q^{-n}, b; \\ c; \end{matrix}\; q,\ \frac{c}{b} q^n\right] = \frac{(c/b;q)_n}{(c;q)_n},$$

the generating function (3.24) can be rewritten fairly easily as

$$(3.26)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n (\beta/\alpha;q)_n}{(q;q)_n (\beta;q)_n} t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\, {}_2\Phi_2\left[\begin{matrix} \lambda, \alpha; \\ \beta, \lambda t; \end{matrix}\; q,\ \frac{\beta t}{\alpha}\right]$$

or, equivalently, as Jackson's q-Pfaff transformation [8, p. 145, Equation (4)]

$$(3.27)\qquad {}_2\Phi_1\left[\begin{matrix}a,\ b;\\ \\ c;\end{matrix}\ q,\ z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}\ {}_2\Phi_2\left[\begin{matrix}a,\ c/b;\\ \\ c,\ az;\end{matrix}\ q,\ bz\right].$$

Formula (3.27) is the main lemma of Andrews [3] which he used to derive q-analogues of Kummer's summation theorem and Gauss's second theorem.

Finally, we set $x = \gamma\delta/\alpha\beta$, $N = 1$, and

$$S_{n,q} = (-1)^n\, q^{\frac{1}{2}n(n-1)}\, \frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(\delta;q)_n},$$

and our theorem yields the generating function:

$$(3.28)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n}\ {}_3\Phi_2\left[\begin{matrix}q^{-n},\ \alpha,\ \beta;\\ \\ \gamma,\ \delta;\end{matrix}\ q,\ \frac{\gamma\delta}{\alpha\beta}q^n\right]t^n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\ {}_3\Phi_3\left[\begin{matrix}\lambda,\ \alpha,\ \beta;\\ \\ \gamma,\ \delta,\ \lambda t;\end{matrix}\ q,\ \frac{\gamma\delta t}{\alpha\beta}\right].$$

The ${}_3\Phi_2$ occurring in (3.28) can be transformed by appealing to the familiar identity:

$$(3.29)\qquad {}_3\Phi_2\left[\begin{matrix}q^{-n},\ a,\ b;\\ \\ c,\ d;\end{matrix}\ q,\ \frac{cd}{ab}q^n\right] = \frac{(c/a;q)_n}{(c;q)_n}\ {}_3\Phi_2\left[\begin{matrix}q^{-n},\ a,\ d/b;\\ \\ aq^{1-n}/c,\ d;\end{matrix}\ q,\ q\right],$$

which incidentally is involved in the equivalence of (3.6) and (3.7), and we thus find from (3.28) that

$$(3.30)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n(\gamma/\alpha;q)_n}{(q;q)_n(\gamma;q)_n}\ {}_3\Phi_2\left[\begin{matrix}q^{-n},\ \alpha,\ \delta/\beta;\\ \\ \alpha q^{1-n}/\gamma,\ \delta;\end{matrix}\ q,\ q\right]t^n$$

$$= \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\ {}_3\Phi_3\left[\begin{matrix}\lambda,\ \alpha,\ \beta;\\ \\ \gamma,\ \delta,\ \lambda t;\end{matrix}\ q,\ \frac{\gamma\delta t}{\alpha\beta}\right].$$

In its special case when $\gamma = \beta$, the right-hand side of (3.30) becomes identical with the right-hand side of (3.26) with, of course, β replaced by δ. Equating the coefficients of t^n in the first members of (3.26) and (3.30), in this special case, we obtain the q-summation formula:

$$(3.31)\qquad {}_3\Phi_2\left[\begin{matrix} q^{-n},\ \alpha,\ \delta/\beta; \\ \\ \alpha q^{1-n}/\beta,\ \delta; \end{matrix}\ q,\ q\right] = \frac{(\beta;q)_n(\delta/\alpha;q)_n}{(\delta;q)_n(\beta/\alpha;q)_n}$$

or, equivalently,

$$(3.32)\qquad {}_3\Phi_2\left[\begin{matrix} a,\ b,\ q^{-n}; \\ \\ c,\ abq^{1-n}/c; \end{matrix}\ q,\ q\right] = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n},$$

which is Jackson's q-analogue of the celebrated Pfaff-Saalschütz theorem (cf. [9, p. 111, Equation (B)]; see also [11, p. 97, Equation (3.3.2.2)]). Conversely, setting $\gamma = \beta$ in (3.30) and summing the resulting ${}_3\Phi_2$ series by appealing to Jackson's result (3.32), we shall arrive at (3.26) or (3.27). Thus our formula (3.30) may also be looked upon as a generalization of the principal result employed by Andrews [3, p. 527].

We conclude by remarking that many of the q-generating functions considered in this section can alternatively be deduced from the following consequence of our theorem (see also [12, Section 3]):

$$(3.33)\qquad \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(q;q)_n}\, {}_{p+1}\Phi_p\left[\begin{matrix} q^{-n},\ \alpha_1,\ \ldots,\ \alpha_p; \\ \\ \beta_1,\ \ldots,\ \beta_p; \end{matrix}\ q,\ xq^n\right] t^n$$

$$= \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\, {}_{p+1}\Phi_{p+1}\left[\begin{matrix} \lambda,\ \alpha_1,\ \ldots,\ \alpha_p; \\ \\ \lambda t,\ \beta_1,\ \ldots,\ \beta_p; \end{matrix}\ q,\ xt\right],\quad |t| < 1,\ |q| < 1,$$

which provides a q-analogue of a well-known hypergeometric generating function (cf., e.g., [14, p. 138, Equation (8)]). Formula (3.33) can indeed be specialized also to derive generating functions for a number of q-hypergeometric polynomials in addition to those that are considered here.

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