Graph Partitions and the Bichromatic Number

by

Dennis D.A. Epple

Diplom, Freie Universität Berlin, 2005

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Abstract

A (k, l)-colouring of a graph is a partition of its vertex set into k independent sets and l cliques. The bichromatic number χ^b of a graph is the minimum r such that the graph is (k, l)-colourable for all $k + l = r$. The bichromatic number is related to the cochromatic number, which can also be defined in terms of (k, l)-colourings.

The bichromatic number is a fairly recent graph parameter that arises in the study of extremal graphs related to a classical result of Erdős, Stone and Simonovits, and in the study of the edit distance of graphs from hereditary graph classes. While the cochromatic number has been well studied in the literature, there are only few known structural results for the bichromatic number. A main focus of this thesis is to establish a foundation of knowledge about the bichromatic number. The secondary focus is on (k, l)-colourings of certain interesting graph classes.

Two known bounds for the bichromatic number are $\chi^b \leq \chi + \theta - 1$, where χ is the chromatic number and θ the clique covering number of the graph, and $\chi^b \geq \sqrt{n}$, where n the number of vertices of the graph. We give a complete characterization of all graphs for which equality holds in the first bound, and show that the second bound is best possible by constructing graphs for square numbers n such that equality holds in the bound. We investigate graphs for which the bichromatic number equals the cochromatic number and prove a Brooks-type theorem for the bichromatic number.

Regarding (k,l)-colourings, we find a new algorithm for calculating the (k,l)-colourability of cographs and show that cographs have a particularly nice representation with regard to (k,l)-colourings. For proper circular arc graphs, we provide a method for (k,l)-colouring if $l \geq 1$, and establish an algebraic characterization for all maximally $(k,0)$-colourable proper circular arc graphs.

Finally, we investigate the bichromatic number and cochromatic with respect to lexicographic products and show several nice bounds.

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Chapter 1

Introduction

1.1 Colouring variations

A k-colouring of a graph is a partition of its vertex set into k independent sets. The study of graph colourings arose from the famous four colour problem and has been one of the major areas in discrete mathematics for well over fifty years. Of particular interest has been the chromatic number of a graph, the minimum k such that there exists a k-colouring of the graph. Closely related to colourings are clique coverings. An l-clique-covering of a graph is a partition of its vertex set into l cliques, which is equivalent to an l-colouring of the complement of the graph.

The idea of partitioning the vertex set of a graph into independent sets and cliques was first considered by Lesniak and Straight [45] and by Földes and Hammer [30] in 1977. The latter defined the class of split graphs - graphs, whose vertex set can be partitioned into one independent set and one clique. Lesniak and Straight defined the cochromatic number of a graph as the minimum number that is required to partition the vertex set of a graph into that many sets, each of which being either an independent set or a clique. Since then, both split graphs and the cochromatic number have received a lot of attention. A good collection of references of results up to 1994 can be found in [35], while more recent articles include [1, 20].

While having been explicitly used in the study of the cochromatic number (for example [60, 9, 35]), the concept of partitioning a graph into fixed numbers of independent sets and cliques first appeared independent of the cochromatic number in a paper by Brandstädt [7] in 1996. A (k, l)-colouring of a graph is a partition of its vertex set into k-independent sets and l cliques. Colourings, clique coverings, split graphs and the cochromatic number can all be described in terms of (k, l)-colourings. A k-colouring is equivalent to a $(k, 0)$-colouring, whereas an l-clique-covering corresponds to a $(0, l)$-colouring. Split graphs are precisely the $(1, 1)$-colourable graphs, while the cochromatic number of a graph is the minimum $k + l$ such that the graph is (k, l)-colourable.

The bichromatic number, a graph parameter intricately related to (k, l)-colourings and the cochromatic number, arose independently of those concepts in extremal graph theory out of articles by Prömel and Steger [50, 51] in 1992 under the name τ. Recently, the bichromatic number has been connected to the edit distance of graphs from hereditary classes of graphs. The edit distance between two graphs on the same number of vertices is the minimum number of edge additions and deletions required to transform one graph into the other. Generalizing this, the edit distance between a graph and a class of graphs is the minimum number of edge additions and deletions required to transform the graph into a member of the graph class. Finally, we let $ex^*(n, H)$ be the maximum edit distance over all graphs on n vertices to the class of graphs that do not contain an induced copy of a graph H. Prömel and Steger [51] showed that this parameter can be expressed in terms of the bichromatic number. With the bichromatic number $\chi^b(H)$ being defined as the minimum integer r such that H is (k, l)-colourable for all k, l with $k + l = r$, it is

$$ex^*(n, H) = \left(1 - \frac{1}{\chi^b(H) - 1}\right) \frac{n^2}{2} + o(n^2).$$

If we replace $\chi^b(H)$ by the chromatic number $\chi(H)$ and $ex^*(n, H)$ by $ex(n, H)$, the maximum number of edges a graph can have without containing H as a (not necessarily induced) subgraph, we obtain a classical result by Erdős, Stone and Simonovits [26, 25]. Furthermore, Prömel and Steger could show [50] that the number

$Forb^*(n, H)$ of graphs on n vertices not containing H as an induced subgraph essentially only depends on the bichromatic number of H, namely

$$Forb^*(n, H) = 2^{ex^*(n,H)(1+o(1))},$$

which is a direct analogue of a result by Erdős, Frankl and Rödl [22]. Axenovich, Kézdy and Martin [3] (using the term binary chromatic number) gave some better lower and upper bounds for $ex^*(n, H)$ and gave a couple of bounds for the bichromatic number, as well as calculating the bichromatic number for a few graph families. Bollobás and Thomason [6] (calling the bichromatic number the coloring number) generalized the maximum edit distance to general graph classes, in particular hereditary graph classes, that is, classes of graphs that are closed under taking induced subgraphs. Alon and Stav [2] expanded on this idea and provided a bound for the maximum edit distance to the class of (k, l)-colourable graphs and a bound for the maximum edit distance of the random graph to an arbitrary hereditary graph class that involves the bichromatic number. Regarding the name, we use bichromatic number as a shorter version of binary chromatic number, which also nicely complements the term cochromatic number.

In this thesis, the main focus lies on establishing foundational results about the bichromatic number and (k, l)-colourings, though the cochromatic number appears frequently in connection to the bichromatic number. Some basic properties of the three concepts are shown in Chapter 2, while Chapter 3 is dedicated to connections between the bichromatic number and other better known graph parameters, like the chromatic number, clique covering number, cochromatic number, number of vertices and a variation of the maximum degree.

1.2 Special graph classes

In Chapter 4, we will consider (k, l)-colourings of a few select classes of graphs (cographs, chordal graphs and proper circular arc graphs). Since determining whether an arbitrary graph is (k, l)-colourable is NP-complete if either of k or l is at least three [7], most of the research concerning (k, l)-colourings is on graph classes for which the (k, l)-colouring problem can be solved in polynomial time. Necessarily, the $(k, 0)$-colouring problem (that is, the k-colouring problem) and $(0, l)$-colourings problem (the l-clique covering problem) need to be polynomial time solvable. In fact this is also sufficient, as was shown in [29].

Overall, perfect graphs have garnered the most attention with regard to (k, l)-colourings. A perfect graph is a graph for which the chromatic number equals the order of a maximum clique and this property holds for every induced subgraph as well. Lovász [46] showed that the complement of a perfect graph is also perfect. Therefore, a perfect graph is k-colourable if and only if it does not contain the complete graph on $k+1$ vertices, and l-clique-coverable if and only if it does not contain the edgeless graph on $l+1$ vertices. It was first shown in [43] that the (k, l)-colouring problem is polynomial time solvable for perfect graphs. Further results about perfect graphs can be found in [27].

Among the various subclasses of perfect graphs, permutation graphs were the first to be investigated with regard to (k, l)-colourings. In fact, the first occurrence of the (k, l)-colouring problem is to our knowledge the article by Brandstädt and Kratsch [9] from 1986 about permutation graphs. A permutation graph on n vertices arises from a permutation on the numbers 1 to n (given in list form) by defining the edge set of the graph to be the set of inversions of the permutation. With that definition, an independent set in the permutation graph corresponds to an increasing subsequence of the permutation, while a clique corresponds to a decreasing subsequence. Thus the minimum number of monotone subsequences needed to cover a permutation is equal to the cochromatic number of its permutation graph [60, 9]. For the (k, l)-colouring problem on permutation graphs, see also [43].

The concept of (k, l)-colourings has been extensively studied for the class of cographs [17, 16, 28, 11]. A cograph is a graph that does not contain P_4 as an induced subgraph [54] (see [15] for a good overview and a variety of characterizations, a selection of which are mentioned in Theorem 3.1.2). Cographs are perfect and the complement of a cograph is a cograph itself. The interest in cographs stems from the fact that many graph problems can be solved efficiently for cographs [15].

Another interesting class of graphs are chordal graphs. Chordal graphs are graphs that do not contain an induced cycle of length greater than 3 [38]. Chordal graphs are perfect [4, 19]. The complement of a chordal graph is not necessarily a chordal graph itself, however. With regard to (k, l)-colourings, chordal graphs have the nice property that for each pair (k, l) there is only one chordal graph that is not (k, l)-colourable and minimal in that respect [41]. The class of perfect graphs has that property only if either k or l is zero.

As for classes of non-perfect graphs, (k, l)-colourings of line graphs were considered in [18]. The line graph of a graph has as vertex set the edge set of the graph and as edge set the incidence set of the edges. It was shown in [18] that the problem of finding the minimum l such that a graph is (k, l)-colourable is NP-complete even for line graphs of bipartite graphs.

A class of non-perfect graphs that has not been considered with regard to (k, l)-colourings is the class of proper circular arc graphs. A proper circular arc graph can be defined as the intersection graphs of an inclusion-free set of circular arcs on a fixed circle [57]. Proper circular arc graphs can be seen as nearly perfect graphs in the sense that there exist cliques in the graph whose removal produces a perfect graph. As such, they are a natural class to investigate if one wants to find a class of non-perfect graphs for which the (k, l)-colouring problem might be polynomial time solvable.

Two single graphs receive special attention in this thesis, as common examples and counterexamples. They are P_4 and C_5, the path on four vertices and the cycle on five vertices. We mention a few properties of these graphs that justify their special status in this thesis.

The path P_4 is the smallest graph that is not a cograph (which play a major role in this thesis). In fact, cographs are precisely those graphs that do not contain

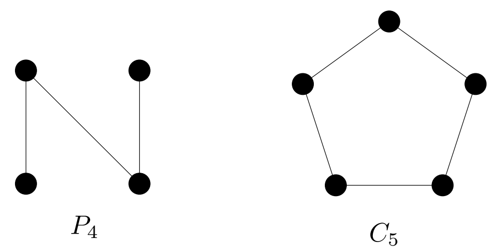

Figure 1.1: P_4 and C_5.

P_4 as an induced subgraph. Also, P_4 is a perfect graph and the smallest nontrivial self-complementary graph. Self-complementary graphs are of interest, since both the cochromatic number and the bichromatic number are the same for a graph and its complement. All of the colouring parameters considered in this thesis, the chromatic, clique covering, cochromatic and bichromatic numbers as well as the clique and independence numbers of P_4 are equal to 2.

The cycle C_5 is self-complementary and the smallest graph that is not perfect. The chromatic, clique covering, cochromatic and bichromatic numbers of C_5 are all equal to 3 (the clique and independence numbers equal to 2).

1.3 Other topics

The lexicographic product of graphs, first considered by Harary [39], is one of the four standard graph products (see [42] for a good introduction to graph products). The lexicographic product of graphs G and H can be obtained by replacing each vertex of G by a copy of H and connecting two vertices in different copies if and only if the corresponding vertices of G are adjacent. The chromatic number of lexicographic products has been extensively investigated (see [34, 32]) and has been shown to have connections to the fractional chromatic number [44].

The fractional chromatic number of a graph arises from the chromatic number by formulating the chromatic number as a solution to an integer program and then relaxing the program to a linear program [37]. See [53] for an introduction to fractional graph parameters.

In Chapter 5, we consider (k, l)-colourings, the cochromatic number and the bichromatic number with regards to lexicographic products and give fractional versions of the three concepts.

1.4 Terminology

For notation and terminology, we mostly follow [61].

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a set of unordered pairs of distinct vertices from $V(G)$. The set $V(G)$ is called the *vertex set* of G and its elements *vertices*, while $E(G)$ is called the *edge set* of G and its elements *edges*. For simplicity, an edge $\{v, w\}$ is denoted by vw. The cardinality of $V(G)$ is called the *order of* G.

Two vertices v, w of a graph G are *adjacent* if $vw \in E(G)$. The *open neighbourhood* $N(v)$ of a vertex v is the set of all vertices that are adjacent to v. The *closed neighbourhood* $N[v]$ is defined as $N(v) \cup \{v\}$. The *degree* $d(v)$ of a vertex v is defined as $|N(v)|$. The *maximum degree* of G is denoted by $\Delta(G)$, whereas the *minimum degree* of G is denoted by $\delta(G)$. A graph G is *regular* if $\Delta(G) = \delta(G)$. An *independent set* of G is a subset of the vertex set that does not contain two adjacent vertices. A *clique* of G is a set of pairwise adjacent vertices. The cardinality of an independent set or a clique is called the *order* of the set. The maximum order over all independent sets of G is called the *independence number* of G and is denoted by $\alpha(G)$, whereas the maximum order over all cliques of G is called the *clique number* of G and is denoted by $\omega(G)$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is an *induced subgraph* of a graph G if H is induced by some $V' \subseteq V(G)$. The *complement* of G is the graph $\overline{G}$ with the same vertex set as G and the property that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

For two vertices v, w of G, a *path from v to w* is a sequence of distinct vertices

$v = v_0, v_1, \ldots, v_m = w$ such that $v_i v_{i+1} \in E(G)$ for all $0 \leq i \leq m-1$. A graph G is *connected* if for any two vertices v and w, there exists a path from v to w. A maximal connected induced subgraph of a graph G is called a *component* of G.

A *k-colouring* of a graph G is a partition $S_1, S_2, \ldots, S_k$ of the vertex set of G such that each S_i is an independent set. The sets S_i are called *colour classes*. A graph G is *k-colourable* if it has a k-colouring. The minimum number k for which G is k-colourable is the *chromatic number* of G and is denoted by $\chi(G)$. An *l-clique-covering* of G is a partition $C_1, C_2, \ldots, C_l$ of the vertex set of G such that each C_j is a clique. A graph G is *l-clique-coverable* if it has an l-clique-covering. The minimum number l, for which G is l-clique-coverable is the *clique covering number* of G and is denoted by $\theta(G)$.

If G and H are graphs on disjoint vertex sets, then the *disjoint union* of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ and is denoted by $G + H$. The disjoint union of m copies of G is also denoted by mG. The *join* of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{vw \mid v \in V(G), w \in V(H)\}$ and is denoted by $G \vee H$.

The *complete graph* K_n is the graph on n vertices such that $V(K_n)$ is a clique. The *edgeless graph* $\overline{K_n}$ is the complement of K_n. The *n-cycle* C_n is the unique connected graph on n vertices such that each vertex has degree 2. The *n-path* P_n is the unique connected graph on n vertices with two vertices of degree 1 and all other vertices of degree 2. The *complete multipartite graph* $K_{m_1, m_2, \ldots, m_s}$ is the graph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_s$ such that the V_i are disjoint, $|V_i| = m_i$ for all i, and the property that vw is an edge of $K_{m_1, m_2, \ldots, m_s}$ if and only if v and w are not contained in the same set V_i. If $s = 2$, the graph is called *complete bipartite*.

A *tree* is a connected graph that does not contain C_n as a subgraph for any $n \geq 3$. A *rooted tree* is a tree with one special vertex, called the *root* of the tree. If v, w are vertices such that the unique path from the root to w also contains v, then w is called a *descendant* of v and v an *ancestor* of w. If v and w are adjacent and w is a descendant of v, then v is called the *parent* of w and w a *child* of v. A vertex of degree 1 in a tree is called a *leaf*.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. A graph is a *cograph* if it does not contain P_4 as an induced subgraph. A graph is *chordal* if it does not contain C_n as an induced subgraph for any $n \geq 4$. A graph is *bipartite* if it is 2-colourable. A graph is a *split graph* if its vertex set can be partitioned into an independent set and a clique.

A *digraph* D is an ordered pair $(V(D), A(D))$, where $V(D)$ is a finite set and $A(D)$ is a set of ordered pairs of distinct vertices from $V(D)$. The set $V(D)$ is called the *vertex set* of D and its elements *vertices*, while $A(D)$ is called the *arc set* of D and its elements *arcs*. An arc (v, w) is also denoted by vw, when there is no risk of confusion. All the digraphs in this thesis are oriented graphs, that is, they have the property that for any two vertices v, w, at most one of vw and wv is an element of the arc set. The cardinality of $V(D)$ is called the *order* of D.

Two vertices v, w of a digraph D are *adjacent* if either vw or wv is an arc of D. If $vw \in A(D)$ for two vertices v, w, then w is called an *outneighbour* of v and v an *inneighbour* of w. The *open outneighbourhood* $N^+(v)$ of a vertex v is the set of outneighbours of v. The *closed outneighbourhood* $N^+[v]$ is defined as $N^+(v) \cup \{v\}$. The *open inneighbourhood* $N^-(v)$ is the set of all inneighbours of v. The *closed inneighbourhood* $N^-[v]$ is defined as $N^-(v) \cup \{v\}$. The *outdegree* $d^+(v)$ of a vertex v is defined as $|N^+(v)|$, whereas the *indegree* $d^-(v)$ is defined as $|N^-(v)|$. A *clique* of a digraph D is a set of mutually adjacent vertices of D. A clique is *transitive* if for any three vertices u, v, w of the clique with $uv, vw \in A(D)$, it is $uw \in A(D)$.

1.5 Glossary of notation

Notation that is introduced in this thesis is marked with a reference to the Section, where it first appears.

$\mathscr{B}(r, s)$	set of box cographs of dimension r times s (3.1.2)
C_n	cycle on n vertices

$\mathcal{C}(G)$	family of cliques of G
$\mathcal{C}(G, v)$	family of cliques of G containing a vertex v
$d(v)$	degree of a vertex v
$d^+(v)$	outdegree of a vertex v
$d^-(v)$	indegree of a vertex v
$\overline{G}$	complement of G
$G + H$	disjoint union of graphs G and H
$G \vee H$	join of graphs G and H
$G - S$	subgraph of G obtained from G by deleting $S \subset V(G)$
$G - v$	subgraph of G obtained from G by deleting $v \in V(G)$
$G[H]$	lexicographic product of G with H
$G^{[m]}$	m-fold lexicographic product of G with itself (5.2.2)
mG	disjoint union of m copies of G
K_n	complete graph on n vertices
$K_{m_1,m_2,\ldots,m_s}$	complete multipartite graph on vertex sets of orders $m_1, \ldots, m_s$
$N(v)$	open neighbourhood of a vertex v
$N[v]$	closed neighbourhood of a vertex v
$N^+(v)$	open outneighbourhood of a vertex v
$N^+[v]$	closed outneighbourhood of a vertex v
$N^-(v)$	open inneighbourhood of a vertex v
$N^-[v]$	closed inneighbourhood of a vertex v
P_n	path on n vertices
$R(a, b)$	Ramsey number
$\mathcal{S}(G)$	family of independent sets of G
$\mathcal{S}(G, v)$	family of independent sets of G containing a vertex v
T_G	cotree of a cograph G
$\mathbb{N}$	set of nonnegative integers
$V(G)$	vertex set of G
$\mathbb{Z}$	set of integers

$\mathbb{Z}_k$	cyclic group on k elements
$\alpha(G)$	independence number of G
$\delta(G)$	minimum degree of G
$\Delta(G)$	maximum degree of G
$\chi(G)$	chromatic number of G
$\chi^c(G)$	cochromatic number of G (2.2)
$\chi^b(G)$	bichromatic number of G (2.3)
$\chi_f(G)$	fractional chromatic number of G
$\chi_f^c(G)$	fractional cochromatic number of G (5.1.2)
$\chi_f^b(G)$	fractional bichromatic number of G (5.1.3)
$\boldsymbol{\kappa}(G)$	colouring sequence $(\kappa_1(G), \kappa_2(G), \ldots, \kappa_{\theta(G)-1}(G))$ of G (2.1.1)
$\kappa_l(G)$	minimum k such that G admits a (k,l)-colouring (2.1.1)
$\boldsymbol{\lambda}(G)$	colouring sequence $(\lambda_0(G), \lambda_1(G), \ldots, \lambda_{\chi(G)-1}(G))$ (2.1.1)
$\lambda_k(G)$	minimum l such that G admits a (k,l)-colouring (2.1.1)
$\theta(G)$	clique covering number of G
$\theta_f(G)$	fractional clique covering number of G
$\omega(G)$	clique number of G

Chapter 2

Covering graphs with independent sets and cliques

In this chapter, we will introduce (k, l)-colourings of graphs and the derived concepts of the cochromatic number and bichromatic number.

Section 2.1 is concerned with some basic results about (k, l)-colourings. As a convenient shorthand for describing the values k and l for which a graph is (k, l)-colourable, the colouring sequences $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ are introduced in Subsection 2.1.1 and formulas regarding the join and disjoint union of graphs are established. In Subsection 2.1.2, the order of obstructions to (k, l)-colourability is investigated. Theorem 2.1.17 proves a tight lower bound for perfect graphs, while Propositions 2.1.19 and 2.1.20 show that this bound does not hold for nonperfect graphs for sufficiently large k and l. The complexity of determining (k, l)-colourability is briefly discussed in 2.1.3.

Section 2.2 introduces the cochromatic number of a graph and summarizes a few basic results that will be used later on.

In Section 2.3, the bichromatic number of a graph is defined. Proposition 2.3.5 provides a comparison of the bichromatic number and the cochromatic number with the chromatic number and clique covering number, while Proposition 2.3.8 proves that the bichromatic number can be bounded from below in terms of the number of vertices, implying that there are only finitely many graphs with a given bichromatic number (Corollary 2.3.9). Finally, the complexity of determining the bichromatic number is discussed.

2.1 (k, l)-colouring

The classic problem of colouring the vertices of a graph G such that adjacent vertices receive different colours is equivalent to the problem of partitioning its vertex set into independent sets. If we consider the complement graph $\overline{G}$, every independent set in G becomes a clique in $\overline{G}$. In that respect, independent sets and cliques are twin objects. A natural generalization of both the colouring problem and the clique-covering problem (partitioning the vertices of a graph into cliques) is therefore to allow a mixed partition of the vertex set into independent sets and/or cliques. The earliest two examples of such partitions appeared in 1977 in two different papers. Földes and Hammer introduced split graphs in [30], which are graphs whose vertex set can be partitioned into one independent set and one clique. Lesniak and Straight introduced the concept of the cochromatic number, the minimum size of a partition of the vertex set such that each partite set is either an independent set or a clique (see Section 2.2). Note the qualitative difference between those two concepts. For split graphs we fix the number of independent sets and cliques that we are allowed to use (one each), while for the cochromatic number we make no such restriction.

The concept of split graphs was generalized by Brandstädt in [7] to arbitrary fixed numbers of independent sets and cliques.

Definition 2.1.1. Let G be a graph and k, l be natural numbers. A (k, l)*-colouring* of G is a partition of the vertex set of G into (possibly empty) sets $S_1, S_2, \ldots, S_k$, $C_1, C_2, \ldots, C_l$ such that each set S_i is an independent set and each set C_j is a clique in G. A graph is (k, l)*-colourable* if there exists a (k, l)-colouring of G.

This concept encompasses the classical colouring and clique covering. The $(k, 0)$-colourable graphs are by definition the k-colourable graphs and the $(0, l)$-colourable graphs are the graphs which can be covered by at most l cliques. It also generalizes the definition of split graphs, which are precisely $(1, 1)$-colourable graphs (see [30]). An example of various (k, l)-colourings of the same graph can be seen in Figure 2.1, where the independent sets are denoted by numbers, while the cliques are indicated

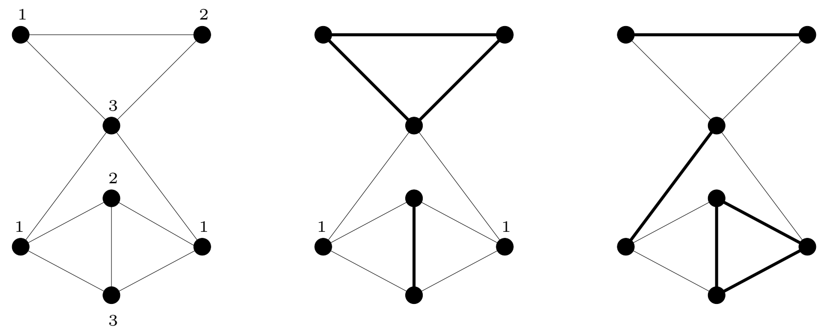

Figure 2.1: A $(3, 0)$-colouring, a $(1, 2)$-colouring and a $(0, 3)$-colouring of a graph.

by the bold edges.

The graph in Figure 2.1 is however not $(2, 1)$-colourable. If it was $(2, 1)$-colourable we could remove one clique such that the remaining graph is 2-colourable (bipartite), which is not possible. As a summary, Figure 2.2 shows all the pairs (k, l) for which the graph is not (k, l)-colourable in black and all the pairs for which the graph is (k, l)-colourable in white. Furthermore, the chromatic number and the clique covering number are shown.

For the remainder of the thesis (with the exception of Section 5.1), the letters k and l will always be nonnegative integers. The integer k will exclusively be applied to independent sets, and l to cliques. We will also reserve S_i for independent sets (the S being short for "stable set") and C_j for cliques, when considering (k, l)-colourings. We start with two very simple observations about (k, l)-colourings that we shall use without further mentioning subsequently.

Proposition 2.1.2. *If G is (k, l)-colourable, then $\overline{G}$ is (l, k)-colourable.*

Proof. Suppose that $S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$ is a (k, l)-colouring of G. Then each S_i is a clique in $\overline{G}$ and each C_j is an independent set in $\overline{G}$. Therefore $C_1, C_2, \ldots, C_l$, $S_1, S_2, \ldots, S_k$ is an (l, k)-colouring of $\overline{G}$. □

Proposition 2.1.3. *If G is (k, l)-colourable, then G is (k', l')-colourable for every $k' \geq k$, $l' \geq l$.*

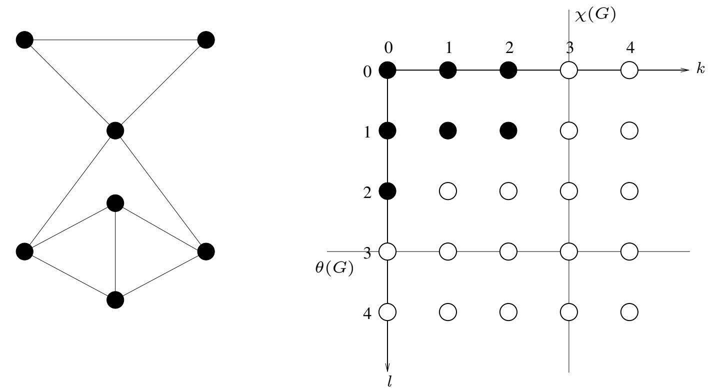

Figure 2.2: The pairs (k, l) for which the graph is not (k, l)-colourable.

Proof. Suppose that $S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$ is a (k, l)-colouring of G. Since the empty set is both an independent set and a clique, we can define the sets $S_{k+1}, S_{k+2}, \ldots, S_{k'}$ and $C_{l+1}, C_{l+2}, \ldots, C_{l'}$ as empty sets to obtain a (k', l')-colouring $S_1, S_2, \ldots, S_{k'}$, $C_1, C_2, \ldots, C_{l'}$ of G. $\square$

By this proposition, if we want to describe the set of all pairs (k, l) for which G is (k, l)-colourable, it suffices to determine, for every k, the smallest l such that G is (k, l)-colourable (or vice versa). Motivated by this, we introduce the following definition.

2.1.1 The colouring sequences $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$

Definition 2.1.4. Let G be a graph. For every natural number l, the graph parameter $\kappa_l(G)$ is the minimum k such that G is (k, l)-colourable. Similarly, for every natural number k, the graph parameter $\lambda_k(G)$ is the minimum l such that G is (k, l)-colourable.

As an example, let G be the graph from Figure 2.1. We want to calculate the values of $\kappa_l(G)$ for all l. We find $\kappa_0(G) = 3$, since G is $(3, 0)$-colourable, but not $(2, 0)$-colourable. Similarly, since we cannot $(2, 1)$-colour G, we obtain $\kappa_1(G) = 3$ as well. Furthermore, we can find $\kappa_2(G) = 1$ and $\kappa_l(G) = 0$, whenever $l \geq 3$.

The knowledge of the values of $\kappa_l(G)$ (respectively $\lambda_k(G)$) can be directly used to check whether G is (k,l)-colourable. Using the above definition, a graph G is (k,l)-colourable if and only if $\kappa_l(G) \le k$ (or equivalently if and only if $\lambda_k(G) \le l$).

We make a few simple observations about κ_l and λ_k.

Proposition 2.1.5. *For any graph G,*

(i) $\kappa_0(G) = \chi(G)$ *and* $\lambda_0(G) = \theta(G)$*;*

(ii) $\kappa_{l'}(G) \le \kappa_l(G)$ *and* $\lambda_{k'}(G) \le \lambda_k(G)$ *for* $l' \ge l$ *and* $k' \ge k$*;*

(iii) $\kappa_l(G) = 0$ *if and only if* $l \ge \theta(G)$ *and* $\lambda_k(G) = 0$ *if and only if* $k \ge \chi(G)$*.*

Proof. We only show the first part of each statement, as the second can be proven similarly. For part (i), we observe that $\kappa_0(G)$ is defined as the minimum k such that G is $(k,0)$-colourable, which is precisely the definition of the chromatic number of G. Part (ii) is a direct consequence of Proposition 2.1.3. Finally, for part (iii), we note that $\kappa_l(G) = 0$ is equivalent to G being $(0,l)$-colourable, which in turn is equivalent to G being l-clique-coverable. □

Definition 2.1.6. Let G be a graph. We define the *colouring sequences* $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\lambda}(G)$ as

$$\boldsymbol{\kappa}(G) = (\kappa_0(G), \kappa_1(G), \ldots, \kappa_{\theta(G)-1}(G))$$

and

$$\boldsymbol{\lambda}(G) = (\lambda_0(G), \lambda_1(G), \ldots, \lambda_{\chi(G)-1}(G)).$$

Since $\boldsymbol{\kappa}(G)$ contains all nonzero values of $\kappa_l(G)$ by Proposition 2.1.5, the complete information about whether G is (k,l)-colourable for all k,l is encoded in $\boldsymbol{\kappa}(G)$ (and similarly in $\boldsymbol{\lambda}(G)$).

By Proposition 2.1.5 (ii), both $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ are monotonically decreasing sequences. Therefore we can represent them using a *Young diagram* (for example for $\boldsymbol{\kappa}$ by putting κ_0 dots in the first row, κ_1 dots in the second row and so on). As an example, consider once more the graph G from Figure 2.1. The Young diagram of $\boldsymbol{\kappa}(G) = (3,3,1)$ is

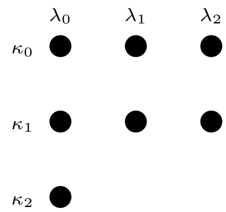

Figure 2.3: Young diagram of $\boldsymbol{\kappa}(G)$, where G is the graph from Figure 2.1.

given in Figure 2.3. We note this is precisely the diagram formed by the black dots in the chart in Figure 2.2. Furthermore, we can obtain the Young diagram of $\boldsymbol{\lambda}(G) = (3, 2, 2)$ by reflecting along the main diagonal. In the language of Young diagrams, $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\lambda}(G)$ are *conjugates* of each other.

Having established a terminology for the existence of a (k, l)-colouring of a graph we proceed to prove a few more basic results for (k, l)-colourings of graphs. First, consider the disjoint union $G + H$ of two graphs G and H. We know that the clique covering number is additive for disjoint unions, that is, $\theta(G + H) = \theta(G) + \theta(H)$. We can obtain a similar result for (k, l)-colourings.

Proposition 2.1.7. *Let G, H be graphs and k a natural number. Then*

$$\lambda_k(G + H) = \lambda_k(G) + \lambda_k(H).$$

Proof. It suffices to show that $G + H$ admits a $(k, \lambda_k(G) + \lambda_k(H))$-colouring but not a $(k, \lambda_k(G) + \lambda_k(H) - 1)$-colouring. For the first condition, let

$$S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_{\lambda_k(G)}$$

be a $(k, \lambda_k(G))$-colouring of G and

$$S'_1, S'_2, \ldots, S'_k, C'_1, C'_2, \ldots, C'_{\lambda_k(H)}$$

be a $(k, \lambda_k(H))$-colouring of H. Since $G + H$ contains no edges between G and H, the sets $S_i \cup S'_i$ are independent for all i. Therefore

$$S_1 \cup S'_1, \ldots, S_k \cup S'_k, C_1, C_2, \ldots, C_{\lambda_k(G)}, C'_1, C'_2, \ldots, C'_{\lambda_k(H)}$$

is a $(k, \lambda_k(G) + \lambda_k(H))$-colouring of $G + H$.

Now suppose, $G+H$ admits a $(k, \lambda_k(G)+\lambda_k(H)-1)$-colouring. Again, as $G+H$ has no edges between G and H, every clique is completely contained in either G or H. By the definition of λ_k, at least $\lambda_k(G)$ of the cliques are contained in G, while at least $\lambda_k(H)$ of the cliques are contained in H, implying that there are at least $\lambda_k(G) + \lambda_k(H)$ cliques in total, a contradiction. Therefore $G + H$ does not admit a $(k, \lambda_k(G) + \lambda_k(H) - 1)$-colouring. □

Corollary 2.1.8. *For any graphs G, H,*

$$\boldsymbol{\lambda}(G + H) = \boldsymbol{\lambda}(G) + \boldsymbol{\lambda}(H),$$

where the addition is performed entrywise. □

We remark that if $\boldsymbol{\lambda}(G)$ and $\boldsymbol{\lambda}(H)$ in Corollary 2.1.8 have different length, we append zeros to the shorter one to make them the same length.

Using Proposition 2.1.7, we can show that if G and H are critical in a sense, then so is $G + H$.

Proposition 2.1.9. *Let G, H be graphs, $T = G + H$ and k be a natural number. If $\lambda_k(G') < \lambda_k(G)$ for all induced proper subgraphs G' of G and $\lambda_k(H') < \lambda_k(H)$ for all induced proper subgraphs H' of H, then $\lambda_k(T') < \lambda_k(T)$ for all induced subgraphs T' of T.*

Proof. Let T' be an induced proper subgraph of T. Since T is a disjoint union of G and H, we can write T' as the disjoint union of G' and H', where G' is an induced subgraph of G, H' is an induced subgraph of H, and at least one of them is a proper subgraph. Then by the hypothesis and Proposition 2.1.7, we obtain

$$\lambda_k(T') = \lambda_k(G') + \lambda_k(H') < \lambda_k(G) + \lambda_k(H) = \lambda_k(G + H) = \lambda_k(T).$$ □

By applying Proposition 2.1.7 and the fact that $\kappa_l(G) = \lambda_k(\overline{G})$, we can obtain the following equivalent statements for the join of two graphs.

Proposition 2.1.10. *Let G, H be graphs and l a natural number. Then*

$$\kappa_l(G \vee H) = \kappa_l(G) + \kappa_l(H).$$

Proof. We have

$$\kappa_l(G \vee H) = \lambda_l(\overline{G \vee H}) = \lambda_l(\overline{G} + \overline{H}) = \lambda_l(\overline{G}) + \lambda_l(\overline{H}) = \kappa_l(G) + \kappa_l(H). \qquad \square$$

Similarly, we obtain the following two results.

Corollary 2.1.11. *For any graphs G, H,*

$$\boldsymbol{\kappa}(G \vee H) = \boldsymbol{\kappa}(G) + \boldsymbol{\kappa}(H),$$

where the addition is performed entrywise. $\square$

Again we append zeros to the shorter of $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\kappa}(H)$, as necessary.

Proposition 2.1.12. *Let G, H be graphs, $T = G \vee H$, and l be a natural number. If $\kappa_l(G') < \kappa_l(G)$ for all induced proper subgraphs G' of G and $\kappa_l(H') < \kappa_l(H)$ for all induced proper subgraphs H' of H, then $\kappa_l(T') < \kappa_l(T)$ for all induced proper subgraphs T' of T.* $\square$

Using the Young diagrams, we can also compute $\boldsymbol{\lambda}(G \vee H)$ and $\boldsymbol{\kappa}(G+H)$. Consider $G \vee H$. By Corollary 2.1.11, $\boldsymbol{\kappa}(G \vee H)$ is obtained by adding $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\kappa}(H)$ entrywise. In terms of the Young diagram, we can picture this as writing the Young diagrams of $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\kappa}(H)$ beside each other (with the rows lining up) and moving all dots to the beginning of the row. This is equivalent to sorting the columns from largest to smallest. An example is given in Figure 2.4. We see that $\boldsymbol{\lambda}(G \vee H)$, being the conjugate of $\boldsymbol{\kappa}(G \vee H)$, is obtained by concatenating the sequence $\boldsymbol{\lambda}(G)$ with $\boldsymbol{\lambda}(H)$ and sorting the resulting sequence from largest to smallest. As we will use this operation later, we introduce the following notation.

Definition 2.1.13. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-increasing finite sequences of natural numbers. Then $\boldsymbol{a} * \boldsymbol{b}$ is the sequence obtained from concatenating $\boldsymbol{a}$ and $\boldsymbol{b}$ and sorting its entries from largest to smallest.

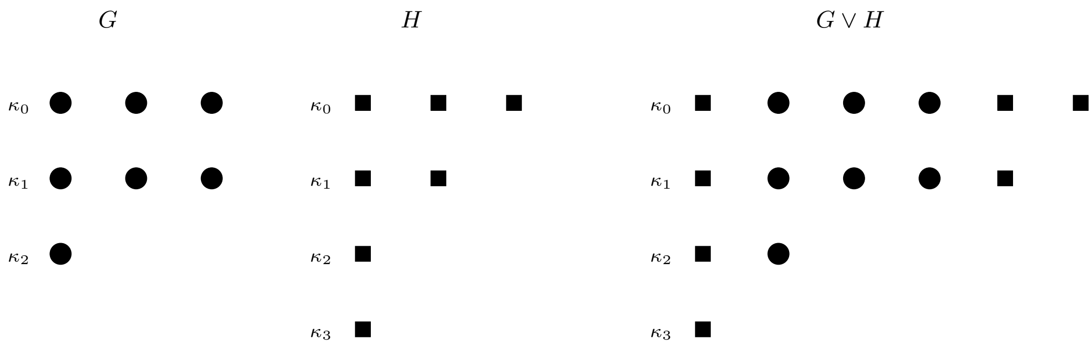

Figure 2.4: Young diagrams of $\boldsymbol{\kappa}(G) = (3, 3, 1)$, $\boldsymbol{\kappa}(H) = (3, 2, 1, 1)$ and $\boldsymbol{\kappa}(G \vee H) = (6, 5, 2, 1)$.

Corollary 2.1.14. *For any graphs G, H,*

$$\boldsymbol{\lambda}(G \vee H) = \boldsymbol{\lambda}(G) * \boldsymbol{\lambda}(H). \qquad \square$$

Corollary 2.1.15. *For any graphs G, H,*

$$\boldsymbol{\kappa}(G + H) = \boldsymbol{\kappa}(G) * \boldsymbol{\kappa}(H). \qquad \square$$

2.1.2 Small graphs that do not have a (k, l)-colouring

Using the colouring sequences κ and λ, we note that a graph G is not (k, l)-colourable if and only if $\lambda_k(G) \geq l + 1$ or, equivalently, $\kappa_l(G) \geq k + 1$. Such a graph can be constructed by taking a disjoint union of cliques.

Proposition 2.1.16. *For all natural numbers k, l,*

$$\lambda_k((l + 1)K_{k+1}) = l + 1.$$

Furthermore, every proper induced subgraph of $(l + 1)K_{k+1}$ is (k, l)-colourable.

Proof. Clearly K_{k+1} is not k-colourable, but can be covered by a single clique, therefore $\lambda_k(K_{k+1}) = 1$. By Proposition 2.1.7, we obtain $\lambda_k((l + 1)K_{k+1}) = l + 1$. Furthermore, it follows from Proposition 2.1.9 that every proper induced subgraph of $(l + 1)K_{k+1}$ is (k, l)-colourable. $\square$

We remark that the fact that $(l+1)K_{k+1}$ is not (k,l)-colourable, is well known. In fact, Lesniak and Straight [45], while calculating the cochromatic number of complete multipartite graphs (which are complements of disjoint unions of cliques), implicitly showed that $\overline{(l+1)K_{k+1}}$ is not (l,k)-colourable.

We observe that the graph $(l+1)K_{k+1}$ has $(k+1)(l+1)$ vertices. The question arises whether there exist graphs on less than $(k+1)(l+1)$ vertices which are not (k,l)-colourable. When $l=0$, the smallest example is K_{k+1}, which is incidentally the only graph on $k+1$ vertices that is not $(k,0)$-colourable. Similarly, $(l+1)K_1 = \overline{K_{k+1}}$ is the smallest graph that is not $(0,l)$-colourable.

Since $(l+1)K_{k+1}$ is a natural generalization of a clique and independent set, we might expect that this is the smallest graph that is not (k,l)-colourable. It turns out that if we restrict ourselves to perfect graphs, then this is indeed true.

Theorem 2.1.17. *Let G be a perfect graph that is not (k,l)-colourable. Then G has at least $(k+1)(l+1)$ vertices.*

Proof. We use induction on $k+l$. For $k+l=1$ the result holds. Let $k+l>1$ and without loss of generality $l>0$ and suppose that G is a perfect graph on less than $(k+1)(l+1)$ vertices which is not (k,l)-colourable. Then G is not $(k,0)$-colourable as well, and thus, as a perfect graph, contains a K_{k+1}. Removing that K_{k+1} we get:

$$|V(G-K_{k+1})| = |V(G)| - (k+1) < (k+1)(l+1) - (k+1) = (k+1)l.$$

By our induction hypothesis, $G-K_{k+1}$ is therefore $(k,l-1)$-colourable and G thus (k,l)-colourable, a contradiction to our assumption. □

Theorem 2.1.17 states that any perfect graph that is a minimal obstruction to (k,l)-colourability (that is, a graph that is not (k,l)-colourable, but whose every induced subgraph is) has at least $(k+1)(l+1)$ vertices. Feder, Hell and Hochstättler [28] showed in 2007 that every cograph that is a minimal obstruction has exactly $(k+1)(l+1)$ vertices and characterized those cographs. Later on, in Theorem 3.1.23, we will show that these are the only perfect graphs on $(k+1)(l+1)$ vertices that are not (k,l)-colourable.

As a companion to Theorem 2.1.17, it has been shown (first by Kézdy, Snevily and Wang [43] in 1996 and later by Feder and Hell [27] in 2006) that the order of a minimal obstruction for (k,l)-colourability is bounded for perfect graphs.

Interestingly, Theorem 2.1.17 becomes false for sufficiently large k and l if we drop the restriction to perfect graphs. While the Theorem remains true if $k, l \leq 3$ or if one of k and l is at most 1, as was proven in [20], in the same paper it was shown that for fixed $l \geq 2$ the minimum order of a graph that is not (k,l)-colourable is asymptotic to $2k + O(l\sqrt{k \log k})$.

Here we provide explicit examples for k and l using known lower bounds on Ramsey numbers.

Definition 2.1.18. Let r, s be two positive integers. The *Ramsey number* $R(r,s)$ is the smallest integer N such that for every graph G on N vertices either $\alpha(G) \geq r$ or $\omega(G) \geq s$.

Proposition 2.1.19. *Let k, l, r, s, n be positive integers such that $k(r-1)+l(s-1) < n < R(r,s)$. Then there exists a graph on n vertices that is not (k,l)-colourable.*

Proof. By the definition of the Ramsey number there exists a graph G on n vertices such that $\alpha(G) < r$ and $\omega(G) < s$. Therefore any collection of k independent sets and l cliques of G can contain at most $k(r-1)+l(s-1)$ vertices. Thus G cannot be (k,l)-colourable. □

We can find values for k, l, r, s, n satisfying the bounds in Proposition 2.1.19 in the survey by Radziszowski about small Ramsey numbers [52]. As an example for Proposition 2.1.19, consider $R(7,5) \geq 80$ [14]. Then with $k = 7$, $l = 9$, $r = 7$, $s = 5$ and $n = 79$, we have found an example of a graph on less than $(k+1)(l+1)$ vertices that is not (k,l)-colourable. This example can be extended to provide us with examples for any k, l with $k \geq 7$ and $l \geq 9$, as we see from the following proposition.

Proposition 2.1.20. *Suppose k, l are positive integers such that there exists a graph on less than $(k+1)(l+1)$ vertices that is not (k,l)-colourable. Then for any integers k', l' with $k' \geq k$ and $l' \geq l$ there exists a graph on less than $(k'+1)(l'+1)$ vertices that is not (k', l')-colourable.*

Proof. Let G be a graph that is not (k,l)-colourable. Then $\lambda_k(G) \geq l+1$. Consider the graph $G' = G + (l'-l)K_{k+1}$. By Propositions 2.1.7 and 2.1.16, we obtain

$$\lambda_k(G') \geq (l+1) + (l'-l) = l'+1.$$

Thus G' is not (k,l')-colourable, i.e., $\kappa_{l'}(G') \geq k+1$. Considering the graph $H = G' \vee \overline{(k'-k)K_{l'+1}}$, we find

$$\begin{aligned} \kappa_{l'}(H) &\geq (k+1) + \kappa_{l'}(\overline{(k'-k)K_{l'+1}}) \\ &= (k+1) + \lambda_{l'}((k'-k)K_{l'+1}) \\ &= (k+1) + (k'-k) \\ &= k'+1. \end{aligned}$$

Therefore H is not (k', l')-colourable. By the construction of H the number of vertices of H is less than

$$\begin{aligned} (k+1)(l+1) + (l'-l)(k+1) + (k'-k)(l'+1) &= (k+1)(l'+1) + (k'-k)(l'+1) \\ &= (k'+1)(l'+1). \end{aligned}$$

□

We can find more examples from the bounds on small Ramsey numbers. Table 2.1 gives for every k the minimum l for which the current bounds on Ramsey numbers provide us with values satisfying the inequalities in Proposition 2.1.19. The corresponding r and s are given as well as the smallest possible n under those constraints. The cases with $k > l$ are omitted as they follow by swapping k with l and r with s.

Using Table 2.1 together with Propositions 2.1.19 and 2.1.20, we see for example that for $k \geq 7$ and $l \geq 9$, there exists a graph on less than $(k+1)(l+1)$ vertices that is not (k,l)-colourable. However, there remain values of k and l for which the existence of such graphs remains an open problem.

k	l	r	s	n
3	31	12	4	127
4	17	10	4	88
5	14	10	4	88
6	13	8	5	95
7	9	7	5	79

Table 2.1: Values satisfying the inequalities in Proposition 2.1.19.

2.1.3 Complexity

The complexity of (k, l)-colouring for general graphs has first been classified by Brandstädt in [7, 8], where it was shown that the problem of deciding whether a graph is (k, l)-colourable is NP-complete for fixed k, l with $k \geq 3$ or $l \geq 3$ and polynomial time solvable otherwise. Furthermore, polynomial time algorithms for the recognition of $(1, 2)$-colourable, $(2, 1)$-colourable graphs were provided (the results can also be found in [10]).

Feder, Hell, Klein and Motwani [29] studied the (k, l)-colouring problem for some special classes of graphs. In particular, they showed that the (k, l)-colouring problem is polynomial time solvable for a graph class $\mathscr{G}$ if both the k-colouring problem and the l-clique-covering problem are polynomial time solvable for $\mathscr{G}$. On the other hand, the (k, l)-colouring problem is NP-complete for $\mathscr{G}$ if either the k-colouring problem or the l-clique-covering problem is NP-complete for $\mathscr{G}$. In particular, this proves that (k, l)-colourable perfect graphs can be recognized in polynomial time for the class of perfect graphs (which can also be deduced from the fact that there are only finitely many minimal obstructions for (k, l)-colourability that are perfect [43]).

2.2 The cochromatic number

We now turn to the concept of cocoloring and the cochromatic number, first introduced by Lesniak and Straight in [45]. An r-cocolouring of a graph is a partition of its vertex set into r partite sets such that each partite set is either an independent set or a clique. In contrast to (k, l)-colouring we make no restriction on the number of independent sets and cliques, we are only interested in the total number r, especially the minimum possible such r. We can describe this concept using (k, l)-colourings as follows.

Definition 2.2.1. The *cochromatic number* of a graph G is defined by

$$\chi^c(G) = \min \{r \mid \exists k, l \geq 0, k + l = r : G \text{ is } (k, l)\text{-colourable}\}.$$

With this definition, any (k, l)-colouring of G is an r-cocolouring of G, where $r = k + l$. The cochromatic number of G can be seen as the minimal r such that G admits an r-cocolouring.

Lesniak and Straight [45] calculated the cochromatic number of complete n-partite graphs and showed that for example graphs not containing an induced K_3 (excepting K_2) have the same cochromatic number and chromatic number.

Expanding on this result, it was shown in [24] that the maximum difference between the chromatic number and the cochromatic number for bounded clique number n grows exponentially in n.

Further research has been conducted about a variety of topics related to the cochromatic number [55, 12, 23, 49].

The cochromatic number is commonly denoted by $z(G)$. The choice to use $\chi^c(G)$ instead was made for readability and as an analogue to the closely related bichromatic number $\chi^b(G)$, defined in the next section. We start by stating a few basic results about the cochromatic number.

2.2.1 Basic properties

Proposition 2.2.2. [45] *Let G be a graph. Then*

$$\chi^c(G) \leq \min\left\{\chi(G), \theta(G)\right\}.$$

Proof. Let G be a graph with chromatic number $\chi(G) = r$. Then G is $(k, 0)$-colourable and by the definition of the cochromatic number,

$$\chi^c(G) \leq r + 0 = \chi(G).$$

Similarly, $\chi^c(G) \leq \theta(G)$. □

While the cochromatic number is bounded above by both the chromatic number and the clique covering number, graphs can have arbitrarily large chromatic number and clique covering number even if their cochromatic number is two. For example, the graph $G = K_n + (n-1)K_1$ has $\chi(G) = \theta(G) = n$ and $\chi^c(G) = 2$.

Proposition 2.2.3. [45] *Let G be a graph. Then*

$$\chi^c(\overline{G}) = \chi^c(G).$$

Proof. Let G be a graph with cochromatic number $\chi^c(G) = r$. Then there exists a (k, l)-colouring of G with $k + l = r$. But since a (k, l)-colouring of G is equivalent to a (l, k)-colouring of $\overline{G}$, we obtain

$$\chi^c(\overline{G}) \leq l + k = r = \chi^c(G).$$

Applying the same argument to $\overline{G}$, we obtain $\chi^c(G) \leq \chi^c(\overline{G})$ and therefore

$$\chi^c(G) = \chi^c(\overline{G}).$$

□

For later use, we give a reformulation of the cochromatic number in terms of the parameters $\kappa_l(G)$ and $\lambda_k(G)$.

Proposition 2.2.4. *Let G be a graph. Then*

$$\chi^c(G) = \min\{k + \lambda_k(G) \mid 0 \le k \le \chi(G)\} = \min\{l + \kappa_l(G) \mid 0 \le l \le \theta(G)\}.$$

Proof. Let G be a graph. Then

$$\begin{aligned}\chi^c(G) &= \min\{k + l \mid G \text{ is } (k,l)\text{-colourable}\}\\ &= \min\{k + \min\{l \mid G \text{ is } (k,l)\text{-colourable}\} \mid k \ge 0\}\\ &= \min\{k + \lambda_k(G) \mid k \ge 0\}\\ &= \min\{k + \lambda_k(G) \mid 0 \le k \le \chi(G)\}.\end{aligned}$$

The last equality follows from $\lambda_{\chi(G)}(G) = 0$, as the minimum will therefore not be obtained at $k > \chi(G)$.

The second equality in the proposition can be shown similarly. □

2.2.2 Complexity

The complexity of calculating the cochromatic number of a graph has been determined for a wide variety of graph classes. Gimbel, Kratsch and Stewart showed in [35] that the problem of determining the cochromatic number is NP-hard for any graph class that is closed under taking disjoint unions and on which the problem of determining the chromatic number is NP-hard. Thus, in particular, the problem is NP-hard for line graphs and therefore for general graphs. They further showed that even determining whether a graph has cochromatic number at most 3 is NP-hard. Since a graph has cochromatic number at most 2 if and only if the graph is $(0,2)$-, $(1,1)$- or $(2,0)$-colourable, the problem of determining whether a graph has cochromatic number at most 2 is polynomial.

Wagner showed in [60] that determining the cochromatic number is NP-hard for permutation graphs, a subclass of perfect graphs. Therefore the problem is NP-hard for perfect graphs.

However, it was shown in [35] that the cochromatic number of cographs and chordal graphs (see Sections 4.1 and 4.2) can be calculated in polynomial time.

2.3 The bichromatic number

The cochromatic number of a graph (see Section 2.2) is the minimum r such that there exists a (k, l)-colouring with $k + l = r$. It is natural to ask whether we can find some r such that there is a (k, l)-colouring for all $k + l = r$. This concept will be the main focus of this thesis.

Definition 2.3.1. The *bichromatic number* of a graph G is defined as

$$\chi^b(G) = \min \{r \mid \forall k, l \geq 0, k + l = r : G \text{ is } (k, l)\text{-colourable}\}.$$

Consider as an example the graph from Figure 2.1. The k, l for which the graph admits a (k, l)-colouring are shown in Figure 2.2. The pairs (k, l) with constant sum $r = k+l$ lie along the offdiagonals. From this we can see that the cochromatic number of the graph is 3, while the bichromatic number is 4 (see Figure 2.5).

We can also observe a structural difference between the two parameters. To determine that the cochromatic number is at most r, we only need to provide one r-cocolouring (a (k, l)-colouring with $k + l = r$). To prove that the bichromatic number is at most r, we need to find a $(k, r - k)$-colouring for all $r + 1$ numbers $k = 0, 1, \ldots, r$.

We now establish some basic properties of the bichromatic number.

2.3.1 Basic properties

Proposition 2.3.2. [3] *Let G be a graph. Then*

$$\chi^b(\overline{G}) = \chi^b(G).$$

Proof. Let G be a graph with bichromatic number $\chi^b(G) = r$. Then G is (k, l)-colourable for all k, l with $r = k + l$. Since every (k, l)-colouring of G is equivalent to an (l, k)-colouring of $\overline{G}$, it follows that $\overline{G}$ is (k, l)-colourable for all k, l with $k + l = r$. Hence

$$\chi^b(\overline{G}) \leq \chi^b(G).$$

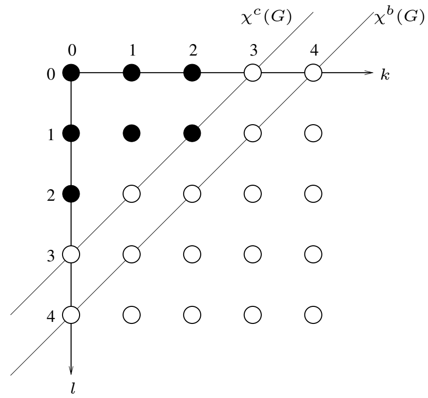

Figure 2.5: The bichromatic and cochromatic numbers of the graph from Figure 2.1.

Applying the same argument to $\overline{G}$, we obtain $\chi^b(G) \leq \chi^b(\overline{G})$ and therefore,

$$\chi^b(G) = \chi^b(\overline{G}).$$

□

Proposition 2.3.3. [51] *Let G be a graph. Then*

$$\chi^b(G) \geq \max\{\chi(G), \theta(G)\}.$$

Proof. The inequality follows from the fact that G is neither $(\chi(G) - 1, 0)$- nor $(0, \theta(G) - 1)$-colourable. □

We can also find an upper bound for the chromatic number in terms of the chromatic number and clique covering number.

Proposition 2.3.4. [51] *Let G be a graph. Then*

$$\chi^b(G) \leq \chi(G) + \theta(G) - 1.$$

Proof. Let $\chi(G) = r$ and $\theta(G) = s$. We need to show that G is (k, l)-colourable for all k, l with $k + l = r + s - 1$. So assume $k + l = r + s - 1$. Then either $k \geq r$ or $l \geq s$, and in either case, G is (k, l)-colourable. □

The following proposition summarizes the relationships among the chromatic, clique covering, cochromatic and bichromatic numbers.

Proposition 2.3.5. *Let G be a graph. Then*

$$\chi^c(G) \leq \min\{\chi(G), \theta(G)\} \leq \max\{\chi(G), \theta(G)\} \leq \chi^b(G) \leq \chi(G) + \theta(G) - 1.$$

Proof. The proposition follows directly from Propositions 2.2.2, 2.3.3 and 2.3.4. $\square$

By observing the proof of Proposition 2.3.4 we can give the following simple characterization of graphs obtaining the upper bound in Proposition 2.3.4.

Proposition 2.3.6. *Let G be a graph. Then $\chi^b(G) = \chi(G) + \theta(G) - 1$ if and only if G is not $(\chi(G) - 1, \theta(G) - 1)$-colourable.*

Proof. Let $\chi(G) = r$ and $\theta(G) = s$. Suppose that $\chi^b(G) = r + s - 1$. Then G is not (k, l)-colourable for some k, l with $k + l = r + s - 2$. However, if $k \geq r$ or $l \geq s$, then G is (k, l)-colourable. The only possibility for $k \leq r - 1$ and $l \leq s - 1$ satisfying $k + l = r + s - 2$ is $k = r - 1$ and $l = s - 1$. Therefore G is not $(r - 1, s - 1)$-colourable. On the other hand assume that G is not $(r - 1, s - 1)$-colourable. Then, by the definition of the bichromatic number,

$$\chi^b(G) \geq (r - 1) + (s - 1) + 1 = r + s - 1.$$

By Proposition 2.3.4, we obtain $\chi^b(G) = r + s - 1$. $\square$

In Section 3.1 we will establish a complete characterization of all graphs that satisfy the upper bound in Proposition 2.3.4 with equality and thereby establish that the bound is indeed sharp. It turns out that all these graphs have the property that $\chi(G) = \omega(G)$ and $\theta(G) = \alpha(G)$. However, an attempt to improve the bound by replacing $\chi(G)$ and $\theta(G)$ by $\omega(G)$ and $\alpha(G)$ fails. For example, let G be the Grötzsch graph. Then $\omega(G) = 2$ and $\alpha(G) = 4$, therefore $\omega(G) + \alpha(G) - 1 = 5$. The bichromatic number, on the other hand, is bounded from below by the clique covering number (see Proposition 2.3.3). Since G has eleven vertices and the clique number is 2, we need at least six cliques to cover the graph. Therefore $\chi^b(G) \geq \theta(G) \geq 6$.

We now turn to showing that there are only finitely many graphs with a given bichromatic number. To that end, we need a lemma about the chromatic and clique covering numbers.

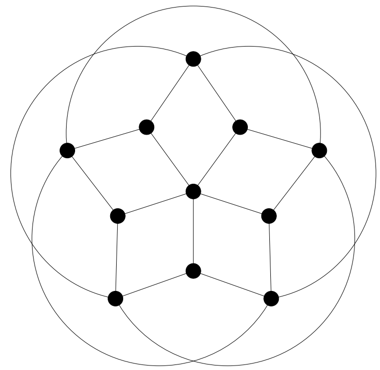

Figure 2.6: The Grötzsch graph.

Lemma 2.3.7. *Let G be a graph. Then*

$$|V(G)| \leq \chi(G)\theta(G).$$

Proof. Let $\chi(G) = r$ and $\theta(G) = s$. Then there exists an r-colouring of G with independent sets $S_1, S_2, \ldots, S_r$ and an s-clique-covering of G with cliques $C_1, C_2, \ldots, C_s$. Since both are partitions of $V(G)$, the family $\{S_i \cap C_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ forms a partition of $V(G)$ as well (where some of the sets might be empty). However, each set $S_i \cap C_j$ is both an independent set and a clique, therefore contains at most one vertex. Since there are rs sets in the family, we obtain

$$|V(G)| \leq rs = \chi(G)\theta(G).$$ □

Proposition 2.3.8. [3] *Let G be a graph. Then*

$$\chi^b(G) \geq \sqrt{|V(G)|}.$$

Proof. By Proposition 2.3.3 and Lemma 2.3.7, we have

$$\chi^b(G)\chi^b(G) \geq \chi(G)\theta(G) \geq |V(G)|.$$

Taking the square root on both sides yields the result. □

This bound has been shown to be sharp for all prime values of the bichromatic number [3]. In Section 3.2 we will prove that this bound is indeed sharp for all values of bichromatic numbers and give constructions for infinite families of graphs satisfying this bound.

Corollary 2.3.9. *Let r be a positive integer. Then the number of graphs G with $\chi^b(G) = r$ is finite.*

Proof. By Proposition 2.3.8 every graph on more than r^2 vertices has bichromatic number at least $r + 1$. Since there are only finitely many graphs on at most r^2 vertices, there are only finitely many graphs with bichromatic r. □

As in the case of the cochromatic number, we give a reformulation of the bichromatic number in terms of the parameters κ_l and λ_k.

Proposition 2.3.10. *Let G be a graph. Then*

$$\chi^b(G) = \max\{k + \lambda_k(G) \mid 0 \leq k \leq \chi(G)\} = \max\{l + \kappa_l(G) \mid 0 \leq l \leq \theta(G)\}.$$

Proof. We show the first equality, the proof for the second is identical. Let $\chi^b(G) = r$. Then G is (k, l)-colourable for all k, l with $k+l = r$ or equivalently, $l = r-k$. Therefore

$$\lambda_k(G) \leq r - k, \qquad 0 \leq k \leq r.$$

Since $r = \chi^b(G) \geq \chi(G)$, we have

$$k + \lambda_k(G) \leq r, \qquad 0 \leq k \leq \chi(G)$$

and therefore

$$\chi^b(G) \geq \max\{k + \lambda_k(G) \mid 0 \leq k \leq \chi(G)\}.$$

Furthermore, as the bichromatic number of G is r, there must exist k', l' with $k'+l' = r - 1$ (or $l' = r - k' - 1$) such that G is not (k', l')-colourable. Necessarily $k' \leq \chi(G)$, since G is $(\chi(G), 0)$-colourable. Therefore

$$\lambda_{k'}(G) \geq r - k'$$

and we obtain

$$\chi^b(G) = r \leq k' + \lambda_{k'}(G) \leq \max\{k + \lambda_k(G) \mid 0 \leq \chi(G)\},$$

completing the proof. □

2.3.2 Complexity

We start by determining the complexity of calculating the bichromatic number for general graphs.

Proposition 2.3.11. *For any fixed r, determining whether a graph G has bichromatic number at most r can be done in constant time.*

Proof. By Proposition 2.3.8, any graph on more than r^2 vertices has bichromatic number at least $r + 1$. So the problem can be reduced to determining whether a graph with at most r^2 vertices has bichromatic number at most r, which can be done in constant time as r is fixed. □

Despite this result, computing the bichromatic number is NP-hard in general.

Proposition 2.3.12. *The problem of determining the bichromatic number of a graph G is NP-hard.*

Proof. We establish the NP-hardness by exhibiting a reduction from the problem of computing the clique covering number, which is NP-hard [33]. Let G be any graph on n vertices. Consider the graph $G + (n-1)K_1$ on $2n-1$ vertices. This graph has an independent set S of order at least n and is therefore $(k, n-k)$-colourable for all $k \geq 1$, since $G - S$ has at most $n-1$ vertices and can thus be $(k-1, n-k)$-coloured trivially (each vertex being one colour class). On the other hand, the clique covering number (the case $k = 0$) of $G + (n-1)K_1$ is at least as big as the independence number, therefore at least n. Thus

$$\chi^b(G + (n-1)K_1) = \theta(G + (n-1)K_1) = n - 1 + \theta(G).$$

Determining the bichromatic number of $G + (n-1)K_1$ is therefore equivalent to determining the clique covering number of G. $\square$

Chapter 3

The bichromatic number for general graphs

In this chapter, we will investigate general bounds for the bichromatic number and establish classes of graphs satisfying the various bounds with equality.

In Section 3.1, the bound from Proposition 2.3.4 ($\chi^b(G) \leq \chi(G) + \theta(G) - 1$) is investigated with the goal of characterizing the graphs that attain equality. To this purpose, the class of box cographs is introduced in Subsection 3.1.2 with two alternative definitions, one structural and one recursive (shown to be equivalent in Proposition 3.1.9). Theorem 3.1.12 proves that box cographs are precisely the graphs attaining equality in Proposition 2.3.4, while Proposition 3.1.21 shows that box cographs are uniquely determined by their colouring sequences. Theorem 3.1.22 provides a summary of the characterization of box cographs. Finally, Theorem 3.1.23 shows that box cographs are the only perfect graphs on $(k+1)(l+1)$ vertices that are not (k,l)-colourable.

Section 3.2 gives constructions of families of graphs on r^2 vertices with bichromatic number r, that is, graphs attaining equality in Proposition 2.3.8. In particular, it is shown that such graphs exist for all r. Theorem 3.2.8 gives an exact formula for the minimum bichromatic number over all graphs on a given number of vertices.

Section 3.3 is concerned with graphs with bichromatic number equal to their cochromatic number. Proposition 3.3.6 shows a nonexhaustive range of numbers n for which there exist graphs on n vertices with bichromatic number and cochromatic number equal to r. Theorem 3.3.7 proves that there are graphs with bichromatic number equal to their cochromatic number on any number of vertices with one exception.

In Section 3.4, an analogue of Brooks' Theorem bounding the chromatic number in terms of the maximum degree is given for the bichromatic number. Theorem 3.4.4 provides a tight bound for the bichromatic number of a graph in terms of the maximum degrees of the graph and its complement and gives the complete list of graphs attaining the bound with equality.

3.1 χ^b in terms of χ and θ

In this section, we will establish a complete characterization of graphs G with $\chi^b(G) = \chi(G) + \theta(G) - 1$ (the upper bound in Proposition 2.3.4). The characterization (Theorem 3.1.12) with its proof has been published in [21]. It turns out that all graphs satisfying the above equality are cographs. We therefore start with a brief introduction to cographs and properties of cographs that we will be using.

3.1.1 Cographs

Cographs originally appeared under a variety of names in several different areas (see [15]). They can be defined in many ways. However, the following is the definition to which the name cograph was first attached [15].

Definition 3.1.1. The class of *cographs* is recursively defined as follows:

(i) K_1 is a cograph;

(ii) if G is a cograph, then $\overline{G}$ is a cograph;

(iii) if G, H are cographs, then $G + H$ is a cograph.

Theorem 3.1.2. [15] *Let G be a graph. Then the following are equivalent:*

(i) G is a cograph;

(ii) for every induced subgraph $H \neq K_1$ of G, either H or $\overline{H}$ is disconnected;

(iii) G does not contain P_4 as an induced subgraph. □

An important property of cographs is that they are perfect graphs.

Proposition 3.1.3. [54] *Let G be a cograph. Then G is perfect.* □

3.1.2 Box cographs

We now define a subclass of cographs, which will turn out to be precisely the graphs satisfying $\chi^b(G) = \chi(G) + \theta(G) - 1$.

Definition 3.1.4. Let r, s be positive integers. A graph G is a *box cograph* of dimension r times s if the following conditions are satisfied.

(i) G is a cograph;

(ii) $\chi(G) = r$;

(iii) $\theta(G) = s$;

(iv) $|V(G)| = rs$.

The class of box cographs of dimension r times s is denoted by $\mathscr{B}(r, s)$.

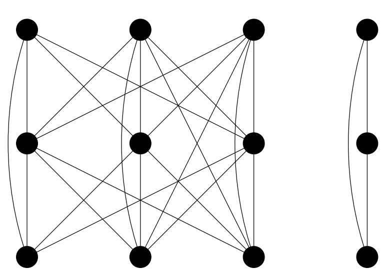

Figure 3.1: A box cograph of dimension 3 times 4.

An example of a box cograph of dimension 3 times 4 is given in Figure 3.1. We can easily calculate the various colouring parameters for box cographs.

Proposition 3.1.5. *Let* $G \in \mathscr{B}(r, s)$. *Then*

(i) $\chi(G) = \omega(G) = r$*;*

(ii) $\theta(G) = \alpha(G) = s$*;*

(iii) $\chi^c(G) = \min\{r, s\}$.

Proof. (i) By definition, $\chi(G) = r$. Since cographs are perfect, $\chi(G) = \omega(G)$.

(ii) By definition, $\theta(G) = s$. Since cographs are perfect, $\theta(G) = \alpha(G)$.

(iii) Let $\chi^c(G) = t$. Then the vertex set of G can be covered by t sets, each inducing either a clique or an independent set. Therefore each of the t sets has order at most $\max\{\alpha(G), \omega(G)\}$. Thus

$$\chi^c(G) \geq \frac{|V(G)|}{\max\{\alpha(G), \omega(G)\}} = \frac{rs}{\max\{r, s\}} = \min\{r, s\}.$$

On the other hand, by Proposition 2.2.2,

$$\chi^c(G) \leq \min\{\chi(G), \theta(G)\} = \min\{r, s\}.$$

Hence, $\chi^c(G) = \min\{r, s\}$. □

Observe from the definition of box cographs that if G is a box cograph of dimension r times s then $\overline{G}$ is a box cograph of dimension s times r; thus the class of box cographs is closed under taking complements. On the other hand, contrary to general cographs, the class of box cographs is neither closed under taking induced subgraphs ($C_4 \in \mathscr{B}(2,2)$ but P_3 is not a box cograph) nor under taking arbitrary unions - consider a box cograph of dimension 2 times 2 and one of dimension 1 times 1, for example. However, any component of a disconnected box cograph is a box cograph (note that a connected box cograph on more than one vertex has a disconnected complement by Theorem 3.1.2).

Proposition 3.1.6. *Let $G \in \mathscr{B}(r,s)$ with $G = G_1 + G_2$. Then there exist positive integers s_1, s_2 with $s_1 + s_2 = s$ such that $G_1 \in \mathscr{B}(r, s_1)$, $G_2 \in \mathscr{B}(r, s_2)$.*

Proof. Since cographs are perfect graphs, we have $\omega(G) = \chi(G) = r$. Combined with $\theta(G) = s$ and $|V(G)| = rs$ this implies that G contains s disjoint cliques of order r. Each of those cliques lies completely in either G_1 or G_2. Suppose G_1 contains s_1 of those cliques (thus $\theta(G_1) = s_1$ and $|V(G_1)| = rs_1$) and G_2 contains $s_2 = s - s_1$ of those cliques (thus $\theta(G_2) = s_2$ and $|V(G_2)| = rs_2$). The chromatic number of both G_1 and G_2 is bounded between their clique number and the chromatic number of G, therefore $\chi(G_1) = \chi(G_2) = r$. Thus we have shown that $G_1 \in \mathscr{B}(r, s_1)$ and $G_2 \in \mathscr{B}(r, s_2)$. □

We are now able to calculate the bichromatic number for any box cograph.

Proposition 3.1.7. *Let* $G \in \mathscr{B}(r, s)$. *Then* $\chi^b(G) = r + s - 1$.

Proof. We proceed by induction on $r+s$. If $r = s = 1$, then $|V(G)| = 1$, thus $G = K_1$, and we have

$$\chi^b(K_1) = 1 = r + s - 1.$$

Now assume that we have proved the statement for all pairs (r', s') with $r'+s' < r+s$. Consider $G \in \mathscr{B}(r, s)$. By Theorem 3.1.2, either G or $\overline{G}$ is disconnected. Without loss of generality let G be disconnected (otherwise consider $\overline{G}$, which is a box cograph in $\mathscr{B}(s, r)$). Suppose $G = G_1 + G_2$. By Proposition 3.1.6, $G_1 \in \mathscr{B}(r, s_1)$, $G_2 \in \mathscr{B}(r, s_2)$ for some s_1, s_2 with $s_1 + s_2 = s$. By the induction hypothesis we have for $i = 1, 2$ that

$$\chi^b(G_i) = r + s_i - 1.$$

According to Proposition 2.3.6, G_i is not $(r - 1, s_i - 1)$-colourable. It follows that $\lambda_{r-1}(G_i) = s_i$. Using Proposition 2.1.7, we obtain

$$\lambda_{r-1}(G) = \lambda_{r-1}(G_1) + \lambda_{r-1}(G_2) = s_1 + s_2 = s.$$

Thus G is not $(r - 1, s - 1)$-colourable and therefore, again by Proposition 2.3.6,

$$\chi^b(G) = \chi(G) + \theta(G) - 1 = r + s - 1.$$

□

Considering Proposition 3.1.6, we observe that we can construct any box cograph from a set of K_1's via a series of disjoint unions and complementations (using Property (ii) from Theorem 3.1.2). This motivates the following recursive definition reminiscent of the recursive definition of cographs.

Definition 3.1.8. Let the graph class $\mathscr{B}$ be defined by:

(i) $K_1 \in \mathscr{B}$;

(ii) if $G \in \mathscr{B}$ then $\overline{G} \in \mathscr{B}$;

(iii) if $G, H \in \mathscr{B}$ and $\chi(G) = \chi(H)$ then $G + H \in \mathscr{B}$.

The class $\mathscr{B}$ is precisely the class of box cographs, as the following proposition shows.

Proposition 3.1.9.

$$\mathscr{B} = \bigcup_{r,s} \mathscr{B}(r, s).$$

Proof. Suppose that $G \in \mathscr{B}(r, s)$. We will show by induction on the number of vertices that G is also contained in $\mathscr{B}$. If $G = K_1$ then $G \in \mathscr{B}$. Otherwise either G or $\overline{G}$ is disconnected, since G is a cograph. Assume that G is disconnected, say $G = G_1 + G_2$ (otherwise consider $\overline{G}$, which lies in $\mathscr{B}(s, r)$). By Proposition 3.1.6, there exist s_1, s_2 such that $G_1 \in \mathscr{B}(r, s_1)$ and $G_2 \in \mathscr{B}(r, s_2)$. By the induction hypothesis, both graphs are contained in $\mathscr{B}$, thus, by Property (iii) of the definition of $\mathscr{B}$, $G_1 + G_2 = G$ is in $\mathscr{B}$.

We now show that $\mathscr{B} \subseteq \bigcup_{r,s} \mathscr{B}(r, s)$ by proving that applying the recursion for $\mathscr{B}$ to graphs in $\bigcup_{r,s} \mathscr{B}(r, s)$ yields a graph that is again in $\bigcup_{r,s} \mathscr{B}(r, s)$.

We have $K_1 \in \mathscr{B}(1, 1)$. If $G \in \mathscr{B}(r, s)$, then $\overline{G} \in \mathscr{B}(s, r)$.

Last, suppose that $G, H \in \bigcup_{i,j} \mathscr{B}(r, s)$ with $\chi(G) = \chi(H)$. Since G and H have the same chromatic number, suppose that $G \in \mathscr{B}(r, s_1)$ and $H \in \mathscr{B}(r, s_2)$. Consider $G + H$. Since both G and H are cographs, so is $G + H$. We have

$$\chi(G + H) = \max\{\chi(G), \chi(H)\} = r$$

and

$$\theta(G + H) = \theta(G) + \theta(H) = s_1 + s_2.$$

Finally,

$$|V(G + H)| = |V(G)| + |V(H)| = rs_1 + rs_2 = r(s_1 + s_2).$$

Hence $G + H \in \mathscr{B}(r, s_1 + s_2)$. □

In the proof of Proposition 3.1.7, we calculated $\lambda_{r-1}(G) = s$ for any box cograph $G \in \mathscr{B}(r,s)$. This provides us with the complete knowledge of the colouring sequence $\boldsymbol{\lambda}(G)$ as defined in Subsection 2.1.1, since $\boldsymbol{\lambda}(G)$ is a nonincreasing sequence and $\lambda_0(G) = \theta(G) = s$. Therefore $\boldsymbol{\lambda}(G)$ is a sequence of r entries, each of which equals s. As we will use this particular sequence again later, we introduce a notation for it.

Definition 3.1.10. For positive integers r, s, define

$$[s]^r = (\underbrace{s, s, \ldots, s}_{r}).$$

With this notation, we obtain the following proposition.

Proposition 3.1.11. *For any box cograph $G \in \mathscr{B}(r,s)$,*

$$\boldsymbol{\lambda}(G) = [s]^r \text{ and } \boldsymbol{\kappa}(G) = [r]^s.$$

Proof. We just showed $\boldsymbol{\lambda}(G) = [s]^r$. The result for $\boldsymbol{\kappa}(G)$ is a direct consequence of $\boldsymbol{\lambda}(G)$ and $\boldsymbol{\kappa}(G)$ being conjugates. □

3.1.3 Characterization of graphs with $\chi^b = \chi + \theta - 1$

We will now proceed to establish that the box cographs are precisely the graphs G satisfying $\chi^b(G) = \chi(G) + \theta(G) - 1$.

Theorem 3.1.12. [21] *Let G be a graph. Then $\chi^b(G) = \chi(G) + \theta(G) - 1$ if and only if $G \in \mathscr{B}$.*

Proof. By Proposition 3.1.7, if $G \in \mathscr{B}$ then $\chi^b(G) = \chi(G) + \theta(G) - 1$. We will show the other direction via a series of lemmas. So let G be an arbitrary graph with

$$\chi(G) = r, \qquad \theta(G) = s$$

and

$$\chi^b(G) = r + s - 1.$$

Furthermore, let $S_1, S_2, \ldots, S_r$ be a fixed r-colouring and $C_1, C_2, \ldots, C_s$ be a fixed s-clique covering of G.

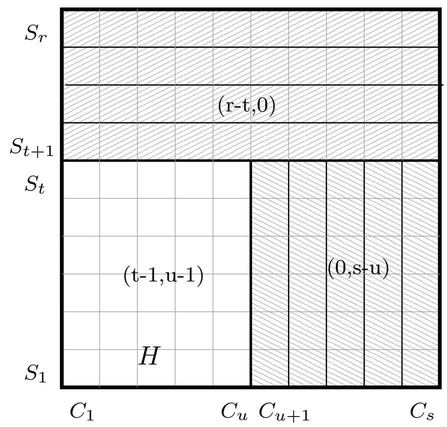

Figure 3.2: $(r-1, s-1)$-colouring G by parts.

Lemma 3.1.13. *Let t, u be positive integers. For any $1 \leq i_1 < \cdots < i_t \leq r$ and $1 \leq j_1 < \cdots < j_u \leq s$, the subgraph H of G induced by $(S_{i_1} \cup \cdots \cup S_{i_t}) \cap (C_{j_1} \cup \cdots \cup C_{j_u})$ is not $(t-1, u-1)$-colourable.*

Proof. We assume without loss of generality that H is the subgraph of G induced by $(S_1 \cup \cdots \cup S_t) \cap (C_1 \cup \cdots \cup C_u)$. Suppose H is $(t-1, u-1)$-colourable. Consider $G - V(H)$. This graph is the subgraph of G induced by $(S_{t+1} \cup \cdots \cup S_r) \cup (C_{u+1} \cup \cdots \cup C_s)$. Set

$$T = S_{t+1} \cup \cdots \cup S_r.$$

Then $\{S_{t+1}, \ldots, S_r, C_{u+1} \setminus T, \ldots, C_s \setminus T\}$ is an $(r-t, s-u)$-colouring of $G - V(H)$. Combining this with the $(t-1, u-1)$-colouring of H, we obtain an $(r-1, s-1)$-colouring of G (see Figure 3.2). However, by Proposition 2.3.6, this contradicts $\chi^b(G) = r + s - 1$. □

Lemma 3.1.14. *For all i, j, $|S_i \cap C_j| = 1$.*

Proof. Since S_i is an independent set and C_j is a clique, $S_i \cap C_j$ is both an independent set and a clique and therefore can contain at most one vertex. By Lemma 3.1.13, the graph induced by $S_i \cap C_j$ is not $(0, 0)$-colourable, thus must contain at least one vertex. □

Lemma 3.1.15. *The graph G has rs vertices.*

Proof. Both $\{S_1, S_2, \dots, S_r\}$ and $\{C_1, C_2, \dots, C_s\}$ are partitions of the vertex set of G. Therefore

$$\{S_i \cap C_j \mid 1 \le i \le r, 1 \le j \le s\}$$

is a partition of the vertex set (into rs sets) as well. By Lemma 3.1.14, each of those sets contains one vertex, therefore G has rs vertices. $\square$

Definition 3.1.16. Define $v_{i,j}$ to be the unique vertex in $S_i \cap C_j$. For fixed (but not necessarily distinct) $a, b \in \{1, 2, \dots, r\}$, define the relation $\sim_{a,b}$ as follows. For all $x, y \in \{1, \dots, s\}$,

$$x \sim_{a,b} y \Leftrightarrow v_{a,x}v_{b,y} \in E(G).$$

We note that the relation $\sim_{a,a}$ is the empty relation, since the vertices $v_{a,x}$ and $v_{a,y}$ both lie in the independent set S_a, thus cannot be adjacent.

Lemma 3.1.17. *For any $a, b \in \{1, 2, \dots, r\}$ the relation $\sim_{a,b}$ is equal to the relation $\sim_{b,a}$.*

Proof. Let $x, y \in \{1, 2, \dots, s\}$. Suppose that $x \sim_{a,b} y$ and $x \not\sim_{b,a} y$, that is, $v_{a,x}v_{b,y}$ is an edge but $v_{b,x}v_{a,y}$ is not an edge of G. Then the subgraph induced by those four vertices (which is the same as the subgraph induced by $(S_a \cup S_b) \cap (C_x \cup C_y)$) is P_4, thus $(1, 1)$-colourable, contradicting Lemma 3.1.13. $\square$

Lemma 3.1.18. *For any $a, b \in \{1, 2, \dots, r\}$, the relation $\sim_{a,b}$ is an equivalence relation.*

Proof. For any $x \in \{1, 2, \dots, s\}$ the vertices $v_{a,x}$ and $v_{b,x}$ lie in the clique C_x, therefore $v_{a,x}v_{b,x}$ is an edge and thus $x \sim_{a,b} x$, which proves reflexivity of the relation. Suppose $x \sim_{a,b} y$, i.e., $v_{a,x}v_{b,y}$ is an edge in G. Then we also have $y \sim_{b,a} x$, and by Lemma 3.1.17, $y \sim_{a,b} x$, which proves the symmetry of the relation. Finally, assume that $x \sim_{a,b} y \sim_{a,b} z$, but $x \not\sim_{a,b} z$. Then $v_{a,x}v_{b,y}$ and $v_{a,y}v_{b,z}$ are edges of G and (by Lemma 3.1.17) $v_{a,z}v_{b,x}$ is not an edge of G. This provides us with a $(1, 2)$-colouring of the subgraph induced by $(S_a \cup S_b) \cap (C_x \cup C_y \cup C_z)$ which is a contradiction to Lemma 3.1.13. This proves the transitivity of the relation. $\square$

Lemma 3.1.19. *For any (not necessarily distinct) $a, b, c \in \{1, 2, \ldots, r\}$ and (not necessarily distinct) $x, y, z \in \{1, \ldots, s\}$ the following statements hold:*

(i) if $x \sim_{a,b} y$ and $x \not\sim_{b,c} y$, then $x \sim_{a,c} y$;

(ii) if $x \not\sim_{a,b} y$ and $x \not\sim_{b,c} y$, then $x \not\sim_{a,c} y$;

(iii) if $x \not\sim_{a,b} y$, $x \sim_{a,c} z$ and $y \sim_{b,c} z$, then $x \sim_{a,c} y$ and $x \sim_{b,c} y$;

(iv) if $x \sim_{a,b} y$, $x \not\sim_{a,c} z$ and $y \not\sim_{b,c} z$, then $x \not\sim_{a,b} z$;

(v) if $x \not\sim_{a,b} y$, $x \sim_{b,c} y$ and $x \sim_{a,c} y \sim_{a,c} z$, then $x \sim_{b,c} z$.

Proof. (i) Assume that (i) is false, i.e.

$$x \sim_{a,b} y,\ x \not\sim_{b,c} y,\ x \not\sim_{a,c} y.$$

Then by the definition of the relations and their symmetry, the sets $\{v_{b,x}, v_{c,y}\}$ and $\{v_{c,x}, v_{a,y}\}$ are independent sets while the set $\{v_{a,x}, v_{b,y}\}$ is a clique. Together these three sets form a $(2,1)$-colouring of the graph induced by $(S_a \cup S_b \cup S_c) \cap (C_x \cup C_y)$, a contradiction to Lemma 3.1.13.

(ii) In the same fashion as for (i), the negation of the statement leads to a $(2,1)$-colouring of the graph induced by $(S_a \cup S_b \cup S_c) \cap (C_x \cup C_y)$.

(iii) Suppose that $x \not\sim_{a,b} y$, $x \sim_{a,c} z$, $y \sim_{b,c} z$, but $x \not\sim_{a,c} y$. The symmetry and transitivity of $\sim_{a,c}$ implies $y \sim_{a,c} z$ while part (ii) yields $x \not\sim_{b,c} y$.

(iv) Suppose that $x \sim_{a,b} y$, $x \not\sim_{a,c} z$ and $y \not\sim_{b,c} z$, but $x \sim_{a,b} z$. Then, by the transitivity of $\sim_{a,b}$, $y \sim_{a,b} z$. By (i), we obtain $x \sim_{b,c} z$ and $y \sim_{a,c} z$. The transitivity of $\sim_{b,c}$ and $\sim_{a,c}$ implies $x \not\sim_{b,c} y$ and $x \not\sim_{a,c} y$ respectively. But this contradicts (i).

(v) Suppose that $x \not\sim_{a,b} y$, $x \sim_{b,c} y$ and $x \sim_{a,c} y \sim_{a,c} z$, but $x \not\sim_{b,c} z$. The transitivity of $\sim_{b,c}$ implies $y \not\sim_{b,c} z$. By (i), we obtain $x \sim_{a,b} z$ and $y \sim_{a,b} z$, contradicting the assumption $x \not\sim_{a,b} y$ due to the transitivity of $\sim_{a,b}$. □

Lemma 3.1.20. *The graph G is in $\mathscr{B}(r,s)$.*

Proof. By Lemma 3.1.15, G has rs vertices. It remains to show that G is a cograph. Suppose not. By Theorem 3.1.2, G then contains a P_4, say $v_{a,x}v_{b,y}v_{c,z}v_{d,w}$ (neither a, b, c, d nor x, y, z, w necessarily distinct). Then we have the following relations:

$$x \sim_{a,b} y,\ y \sim_{b,c} z,\ z \sim_{c,d} w,\ x \not\sim_{a,c} z,\ x \not\sim_{a,d} w,\ y \not\sim_{b,d} w.$$

Using (iii) from Lemma 3.1.19 for y, w and z we conclude $y \sim_{b,c} w$ and $y \sim_{c,d} w$. Similarly, by (iii) of the same lemma for x, z and y, we get $x \sim_{b,c} z$. This together with $y \sim_{b,c} z$ implies $x \sim_{b,c} y$. Recapitulating, we have

$$y \not\sim_{b,d} w,\ y \sim_{c,d} w,\ y \sim_{b,c} w \sim_{b,c} x,$$

and thus, by (v) from Lemma 3.1.19, we have $x \sim_{c,d} y$. Hence we have $z \sim_{c,d} w$, $y \sim_{c,d} w$ and $x \sim_{c,d} y$. The transitivity of $\sim_{b,d}$ implies $w \sim_{b,d} x \sim_{b,d} z$. This together with $x \not\sim_{a,d} w$ and $x \not\sim_{a,c} z$ contradicts (iv) from Lemma 3.1.19. □

This completes the proof of Theorem 3.1.12. □

We present an interesting consequence of Theorem 3.1.12 regarding the colouring sequences $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\lambda}(G)$ defined in Subsection 2.1.1. We showed in Proposition 3.1.11 that $\boldsymbol{\lambda}(G) = [s]^r$ for any box cograph $G \in \mathscr{B}(r,s)$. We can now prove that box cographs in $\mathscr{B}(r,s)$ are in fact the only graphs with colouring sequence $\boldsymbol{\lambda}(G) = [s]^r$.

Proposition 3.1.21. *For any graph G and positive integers r, s,*

$$G \in \mathscr{B}(r,s) \Leftrightarrow \boldsymbol{\lambda}(G) = [s]^r \Leftrightarrow \boldsymbol{\kappa}(G) = [r]^s.$$

Proof. The second equivalence is a direct consequence of $\boldsymbol{\lambda}(G)$ and $\boldsymbol{\kappa}(G)$ being conjugates, so we only need to prove the first equivalence. If $G \in \mathscr{B}(r,s)$ then $\boldsymbol{\lambda}(G) = [s]^r$

by Proposition 3.1.11. If on the other hand G is a graph with $\boldsymbol{\lambda}(G) = [s]^r$, then we obtain from the definition of $\boldsymbol{\lambda}(G)$ that $\chi(G) = r$, $\theta(G) = s$ and $\lambda_{r-1}(G) = s$. Therefore G is not $(\chi(G)-1, \theta(G)-1)$-colourable and thus $\chi^b(G) = \chi(G)+\theta(G)-1$. By Theorem 3.1.12, G is a box cograph. □

The following theorem summarizes the characterization of graphs G with $\chi^b(G) = \chi(G) + \theta(G) - 1$.

Theorem 3.1.22. *Let G be a graph with $r = \chi(G)$, $s = \theta(G)$. Then the following are equivalent:*

(i) $\chi^b(G) = r + s - 1$;

(ii) G is not $(r-1, s-1)$-colourable;

(iii) G has rs vertices and is a cograph;

(iv) $G \in \mathscr{B}$.

(v) $\boldsymbol{\kappa}(G) = [r]^s$

Proof. The equivalence of (i) and (ii) is Proposition 2.3.6, the equivalence of (i) and (iv) is Theorem 3.1.12, the equivalence between (iii) and (iv) is Proposition 3.1.9 and the equivalence between (iii) and (iv) is Proposition 3.1.21. □

Also, we can now finish the discussion from Theorem 2.1.17.

Theorem 3.1.23. *Let G be a perfect graph on $(k+1)(l+1)$ vertices. Then G is not (k,l)-colourable if and only if $G \in \mathscr{B}(k+1, l+1)$.*

Proof. If $G \in \mathscr{B}(k,l)$, then G is not (k,l)-colourable by Proposition 3.1.7 and Proposition 2.3.6. Now assume that G is not (k,l)-colourable. Then the chromatic number of G is at least $k+1$. Suppose $\chi(G) \geq k+2$. Then $\omega(G) \geq k+2$, since G is a perfect graph. Let C be a clique of order $k+2$ and consider the subgraph of G induced by $V(G) \setminus C$. This graph has $(k+1)l - 1$ vertices. By Theorem 2.1.17 it is

$(k, l-1)$-colourable. But then G has to be (k, l)-colourable, a contradiction, proving $\chi(G) = k+1$. Similarly we can establish $\theta(G) = l+1$. Proposition 2.3.6 then shows that $\chi^b(G) = \chi(G) + \theta(G) - 1$, which in turn implies $G \in \mathscr{B}(k+1, l+1)$ by Theorem 3.1.12. $\square$

3.2 Square graphs

We now consider the lower bound on the bichromatic number given in Proposition 2.3.8, that is

$$\chi^b(G) \geq \sqrt{|V(G)|}.$$

In this section we will show that this bound is sharp and provide constructions of infinite families of graphs that achieve the bound with equality.

Definition 3.2.1. A graph G on r^2 vertices with $\chi^b(G) = r$ is called a *square graph.*

A construction for square graphs on p^2 vertices, where p is a prime number, was given by Axenovich, Kézdy and Martin [3] in 2008. Using this construction, the following proposition was proven.

Proposition 3.2.2. [3] *Let n be a positive integer. Then*

$$\sqrt{n} \leq \min_{|V(G)|=n} \chi^b(G) \leq \sqrt{n} + (1 + o(1))n^{0.2625}. \qquad \square$$

We will construct square graphs on r^2 vertices for every integer r, implying a precise formula for $\min\limits_{|V(G)|=n} \chi^b(G)$.

A necessary condition for square graphs is that both the chromatic number and clique covering number must equal r (see the proof of Proposition 2.3.8). Suppose G is a graph on r^2 vertices with $\chi(G) = \theta(G) = r$. Then Proposition 2.3.5 gives us

$$r \leq \chi^b(G) \leq 2r - 1.$$

In Section 3.1 it was shown that G satisfies the upper bound if and only if G is a cograph (see Theorem 3.1.22). This suggests that, to find classes of graphs satisfying the lower bound, it might be advantageous to consider graphs that are far from being cographs.

As the simplest example (apart from K_1), consider square graphs on four vertices. Of all the graphs on four vertices, all but three contain either a clique or an independent set of order 3, thus their bichromatic number is at least 3. The three remaining

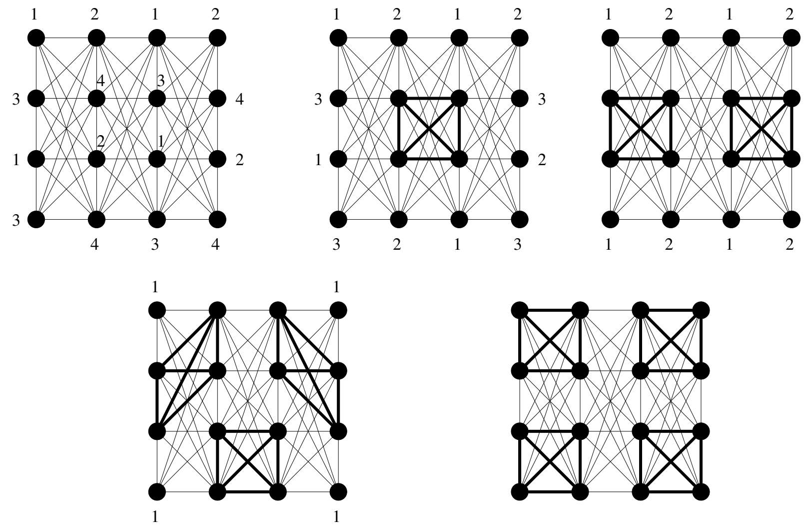

Figure 3.3: (k, l)-colourings with $k + l = 4$.

graphs are C_4, $2K_2$ and P_4. The first two are not $(1, 1)$-colourable, which leaves P_4 as the only square graph on four vertices, incidentally also the only graph on four vertices that is not a cograph.

Now consider the graph arising from P_4 by replacing every vertex by another P_4 and each edge by a complete bipartite graph. As we can see from Figure 3.3, this graph is a square graph as well. This graph can be described as the lexicographic product of P_4 with itself. We will investigate lexicographic products in detail in Section 5.2. In particular, we will calculate the bichromatic number of lexicographic powers of P_4 in Corollary 5.2.18, which also shows that all these graphs are square graphs, providing us with a square graph for every order that is a power of 2.

We proceed to give a construction of a class of square graphs that is inspired by finite geometry. The construction can be seen as a generalization of the one found in [3].

Definition 3.2.3. An *affine plane $\mathscr{A}$ of order r* consists of a point set

$$\mathscr{P} = \{1, 2, \ldots, r\} \times \{1, 2, \ldots, r\}$$

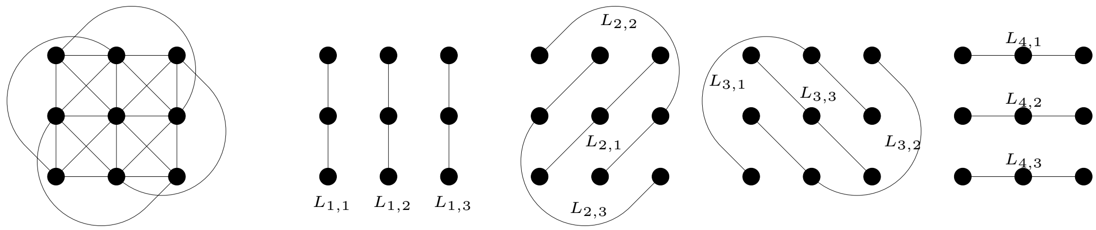

Figure 3.4: An affine plane of order 3 with its parallel classes.

and a line set

$$\mathscr{L} = \{L_{i,j} \mid 1 \leq i \leq r+1,\, 1 \leq j \leq r\}\,,$$

such that each line contains r points, any pair of points is contained in exactly one common line and the set $\{L_{i,1}, L_{i,2}, \ldots, L_{i,r}\}$ (called a *parallel class*) forms a partition of the point set for each $1 \leq i \leq r+1$.

An example of an affine plane of order 3 is given in Figure 3.4. It is known (see for example [59]) that an affine plane of order r exists whenever r is a prime power. It is conjectured that there are no affine planes of an order different from a prime power.

We will use the $r+1$ parallel classes of an affine plane to construct a graph on r^2 vertices that is $(k, r-k)$-colourable for the $r+1$ values of $k \in \{0, 1, \ldots, r\}$.

Definition 3.2.4. Let $\mathscr{A}$ be an affine plane of order r. We define the *affine plane graph* G to be the graph with vertex set $\mathscr{P}$ and edge set defined by

$$uv \in E(G) \Leftrightarrow i \leq j \text{ for the unique line } L_{i,j} \text{ with } u, v \in L_{i,j}.$$

We remark that a different labelling of the lines of an affine plane might result in a different affine plane graph.

Proposition 3.2.5. *Any affine plane graph is a square graph.*

Proof. Let G be an affine plane graph on r^2 vertices. We will show that G is $(k, r-k)$-colourable for all $k \in \{0, 1, \ldots, r\}$. Let k be fixed. Consider the parallel class

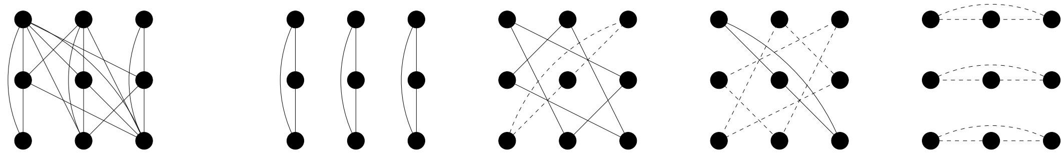

Figure 3.5: An affine plane graph with the various $(k, 3-k)$-colourings indicated.

$\{L_{k+1,1}, L_{k+1,2}, \ldots, L_{k+1,r}\}$. By the definition of the affine plane graph the points lying in a line $L_{k+1,j}$ form a clique if and only if $k+1 \leq j$. Therefore we obtain k independent sets and $r-k$ cliques from this parallel class. Since every point lies on some line from the parallel class, these independent sets and cliques give a $(k, r-k)$-colouring of G. □

As an example, the affine plane graph arising from the affine plane in Figure 3.4 is given in Figure 3.5 together with the independent sets and cliques showing each $(k, 3-k)$-colouring.

While the affine plane graphs provide us with a richer collection of graphs satisfying $\chi^b(G) = \sqrt{|V(G)|}$, we still do not have such graphs for each integer value of $\sqrt{|V(G)|}$. The next construction will fill that gap.

We once again consider P_4, this time drawn such that the cliques providing the 2-clique-covering appear vertically and the independent sets providing the 2-colouring appear horizontally.

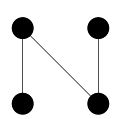

Figure 3.6: The graph P_4.

Drawn this way, the diagonal edge lies on a line with negative slope, while the diagonal nonedge lies on a line with positive slope. We will now define a graph class where the edge sets are based on this slope idea. This graph class can be seen as a generalization of P_4 and we will show that each of those graphs satisfies

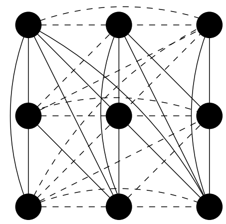

Figure 3.7: Slope graph of order 9.

$\chi^b(G) = \sqrt{|V(G)|}$.

Definition 3.2.6. Let r be a positive integer. A *slope graph* of order r^2 is a graph G with vertex set

$$V(G) = \{1, 2, \ldots, r\} \times \{1, 2, \ldots, r\}$$

and an edge set satisfying the following properties for any $1 \leq x_1, x_2, y_1, y_2 \leq n$ and $(x_1, y_1) \neq (x_2, y_2)$:

$$y_1 - y_2 \geq x_2 - x_1 \geq 0 \implies (x_1, y_1)(x_2, y_2) \in E(G),$$
$$x_2 - x_1 \geq y_2 - y_1 \geq 0 \implies (x_1, y_1)(x_2, y_2) \notin E(G).$$

It is important to note that the two properties given in the definition of slope graphs do not characterize the edge set. For example, the two properties do not determine whether, say, $(1, 1)(2, 3)$ is an edge. While we could give more restrictive conditions, the given definition turns out to be the most general that is still sufficient for our purpose.

The only slope graph of order 4 is P_4. Figure 3.7 shows the slope graphs of order 9. The forced edges according to the definition are shown as solid lines, while the forced nonedges are shown as dashed lines. Any missing line between two vertices indicates that the corresponding edge may or may not be in the edge set of the slope graph.

We note that for large r there are very many slope graphs of order r^2, as the forced edges and nonedges only make up about half of all possible $\binom{r^2}{2}$ edges.

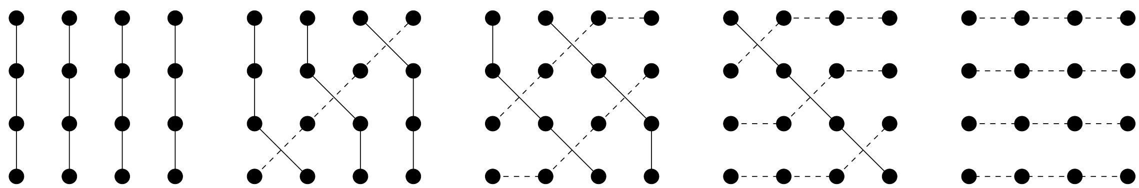

Figure 3.8: Various $(k, 4-k)$-colourings of a slope graph of order 16.

Proposition 3.2.7. *Any slope graph is a square graph.*

Proof. Let G be a slope graph of order r^2. We will show that G is $(k, r-k)$-colourable for all $k \in \{0, 1, \dots, r\}$. Let k be fixed. Define the sets S_i $(1 \le i \le k)$ and C_j $(1 \le j \le r-k)$ as

$$S_i = \{(1,i), \dots, (k-i+1, i), \dots, (r-i+1, r-k+i), \dots, (r, r-k+i)\}$$

and

$$C_j = \{(j,r), \dots, (j, k+j), \dots, (j+k, j), \dots, (j+k, 1)\}.$$

Checking the definition of slope graphs we see that each S_i forms an independent set and every C_j forms a clique. Furthermore, the sets partition the vertex set of G. Thus $S_1, S_2, \dots, S_k, C_1, C_2, \dots, C_{r-k}$ is a $(k, r-k)$-colouring of G. □

As an example, Figure 3.8 indicates the $(k, 4-k)$-colourings ($k \in \{0, \dots, 4\}$) of slope graphs of order 16, where, for the sake of simplicity, each clique is shown as a solid path and each independent set as a dashed path.

Proposition 3.2.7 shows that there is a graph with $\chi^b(G) = \sqrt{|V(G)|}$ whenever $|V(G)|$ is a square number. We can now show our main result about the minimum size of the bichromatic number of graphs with a fixed number of vertices.

Theorem 3.2.8. *Let n be a positive integer. Then*

$$\min_{|V(G)|=n} \chi^b(G) = \lceil \sqrt{n}\, \rceil.$$

Proof. Proposition 2.3.8 shows that the left hand side is bounded from below by the right hand side. To show equality, it is therefore sufficient to give an example for every n of a graph G with $\chi^b(G) = \lceil \sqrt{n}\, \rceil$. For $n = r^2$, any slope graph of order r^2 achieves equality. If n is not a square let G be a graph on n vertices that is an induced subgraph of a slope graph H on r^2 vertices, where $(r-1)^2 < n < r^2$, thus $r = \lceil \sqrt{n}\, \rceil$. Then, using Propositions 2.3.8 and 3.2.7, we get

$$r - 1 < \sqrt{n} \leq \chi^b(G) \leq \chi^b(H) = r.$$

Therefore $\chi^b(G) = \lceil \sqrt{n}\, \rceil$. □

3.3 Graphs with $\chi^b = \chi^c$

In this section we will investigate graphs whose bichromatic number equals their cochromatic number. We will show that for every positive integer $n \neq 2$ there exists a graph on n vertices with $\chi^b = \chi^c$. We also will establish that every graph is an induced subgraph of a graph with bichromatic number equal to its cochromatic number.

First, we provide a sufficient condition for the bichromatic number equalling the cochromatic number that ties into the last section.

Lemma 3.3.1. *Let G be a graph with $\chi^b(G) = r$. If*

$$r(r-1) < |V(G)|,$$

then $\chi^c(G) = \chi^b(G)$.

Proof. Since $\max\{\chi(G), \theta(G)\} \leq \chi^b(G)$ by Proposition 2.3.3 and

$$r(r-1) < |V(G)| \leq \chi(G)\theta(G) \leq r^2,$$

we have $\chi(G) = \theta(G) = r$. Therefore both $\alpha(G)$ and $\omega(G)$ are at most r, since the two parameters are bounded from above by $\theta(G)$ and $\chi(G)$, respectively. Thus we need at least r cliques and independent sets to cover the vertex set of G, implying $\chi^c(G) \geq r$. As $\chi^c(G) \leq \chi^b(G) = r$, we obtain $\chi^c(G) = \chi^b(G)$. □

Lemma 3.3.1 establishes that the square graphs encountered in Section 3.2 (e.g. affine plane graphs, slope graphs) have bichromatic number equal to their cochromatic number. In Theorem 3.2.8, the existence of graphs on n vertices with bichromatic number $\lceil \sqrt{n}\, \rceil$ was proven. In light of lemma 3.3.1, we thus can find a graph G on n vertices with $\chi^b(G) = \chi^c(G) = r$, whenever $r(r-1) < n \leq r^2$ (for example slope graphs on r^2 vertices with up to $r-1$ vertices removed). We will now endeavour to show that we can also find such graphs if $(r-1)^2 < n \leq r(r-1)$ (for $r > 2$). For that purpose we first make a few observations about the parameters λ_k and κ_l defined in

Section 2.1. The reason is that we can reformulate the property of $\chi^b = \chi^c$ in terms of λ_k or κ_l.

Proposition 3.3.2. *Let G be a graph with $\chi^b(G) = r$. Then the following statements are equivalent:*

(i) $\chi^b(G) = \chi^c(G)$*;*

(ii) $\lambda_0(G) = r$ *and* $\lambda_k(G) = \lambda_{k-1}(G) - 1$ *for all* $1 \leq k \leq r$*;*

(iii) $\kappa_0(G) = r$ *and* $\kappa_l(G) = \kappa_{l-1}(G) - 1$ *for all* $1 \leq l \leq r$*.*

Proof. Propositions 2.2.4 and 2.3.10 state that

$$\chi^c(G) = \min \{k + \lambda_k(G) \mid 0 \leq k \leq \chi(G)\}$$

and

$$\chi^b(G) = \max \{k + \lambda_k(G) \mid 0 \leq k \leq \chi(G)\}.$$

Therefore $\chi^b(G) = \chi^c(G)$ if and only if $k + \lambda_k(G) = r$ for all $0 \leq k \leq \chi(G)$ or, equivalently, $\lambda_k(G) = r - k$, which in turn is equivalent to property (ii). The equivalence of (i) and (iii) is shown in the same way. □

Lemma 3.3.3. *Let G be a graph. Then*

(i) $$\lambda_k(G + K_m) = \begin{cases} \lambda_k(G) + 1 & k < m \\ \lambda_k(G) & k \geq m; \end{cases}$$

(ii) $$\lambda_k(G \vee mK_1) = \begin{cases} \lambda_k(G) & \lambda_k(G) \geq m \\ \min\{m, \lambda_{k-1}(G)\} & \lambda_k(G) < m. \end{cases}$$

Proof. To prove (i), we note that $\lambda_k(K_m) = 1$ if $k < m$ and $\lambda_k(K_m) = 0$ for $k \geq m$. The statement now follows from Proposition 2.1.7.

For (ii), suppose that $\lambda_k(G) \geq m$. We extend a $(k, \lambda_k(G))$-colouring of G to a $(k, \lambda_k(G))$-colouring of $G \vee mK_1$ by adding one vertex of mK_1 each to m of the cliques of the colouring. Therefore $\lambda_k(G \vee mK_1) \leq \lambda_k(G)$ and since G is an induced subgraph of $G \vee mK_1$, equality follows.

Now suppose that $\lambda_k(G) < m$. Then $G \vee mK_1$ is (k, m)-colourable by the same technique of adding a vertex of mK_1 each to every clique in a (k, m)-colouring of G. Furthermore, a $(k, \lambda_{k-1}(G))$-colouring of $G \vee mK_1$ can be obtained from a $(k-1, \lambda_{k-1})$-colouring of G by adding an independent set containing all the vertices of mK_1. Thus $\lambda_k(G \vee mK_1) \leq \min\{m, \lambda_{k-1}(G)\}$. To show equality, consider a (k, l)-colouring of $G \vee mK_1$. Suppose none of the vertices of mK_1 appear in an independent set. Then we need to have at least m cliques, thus $l \geq m$. On the other hand if some independent set in the colouring contains a vertex from mK_1, then this independent set cannot contain any vertex from G. Restricting the colouring to G, we obtain a $(k-1, l)$-colouring of G, implying that $l \geq \lambda_{k-1}(G)$. □

Corollary 3.3.4. *Let G be a graph. Then*

(i) $$\kappa_l(G \vee mK_1) = \begin{cases} \kappa_l(G) + 1 & l < m \\ \kappa_l(G) & l \geq m; \end{cases}$$

(ii) $$\kappa_l(G + K_m) = \begin{cases} \kappa_l(G) & \kappa_l(G) \geq m \\ \min\{m, \kappa_{l-1}(G)\} & \kappa_l(G) < m. \end{cases}$$

Proof. The corollary is a direct consequence of Lemma 3.3.3 by using the property that $\kappa_l(H) = \lambda_l(\overline{H})$ for any graph H. □

In particular, Lemma 3.3.3 allows us to compute $\boldsymbol{\lambda}$ for complete multipartite graphs.

Corollary 3.3.5. *Let $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$. Then*

$$\lambda_k(K_{m_1, m_2, \ldots, m_s}) = \begin{cases} m_{k+1} & k < s \\ 0 & k \geq s. \end{cases}$$

Proof. We use induction on s. For $s = 1$ we have an edgeless graph $\overline{K_{m_1}}$ with $\lambda_0(\overline{K_{m_1}}) = m_1$ and $\lambda_k(\overline{K_{m_1}}) = 0$ for $k \geq 1$. Suppose the statement holds for $G = K_{m_1,m_2,\ldots,m_{s-1}}$ Then $\lambda_k(G) \geq m_s$ for all $k < s-1$ and $\lambda_{s-1}(G) = 0$. Applying Lemma 3.3.3 for $G \vee m_s K_1$ finishes the proof. □

We remark that Corollary 3.3.5 can also be obtained from a characterization of (k,l)-colourable chordal graphs (see Theorem 4.2.5, first proven in [41]), as complete multipartite graphs are the complements of disjoint union of cliques which are chordal. Also, the cochromatic number of complete multipartite graphs has been computed by Lesniak and Straight in [45].

In view of Proposition 3.3.2, Corollary 3.3.5 implies that $K_{r,r-1,\ldots,1}$ is the only complete multipartite graph whose bichromatic and cochromatic numbers both equal r. This graph has $\binom{r+1}{2}$ vertices, fewer than the graphs from Lemma 3.3.1 for $r > 2$.

Proposition 3.3.6. *For any positive integers n, r with $\binom{r+1}{2} \leq n \leq r^2$ there exists a graph G on n vertices with $\chi^b(G) = \chi^c(G) = r$.*

Proof. We will use induction on r. For $r = 1$, the graph K_1 satisfies all requirements. Suppose the statement holds for $r-1$. If $r(r-1) < n \leq r^2$ then, by Theorem 3.2.8, there exists a graph G on n vertices with $\chi^b(G) = r$, and by Lemma 3.3.1, G must have $\chi^b(G) = \chi^c(G)$.

Now suppose that $\binom{r+1}{2} \leq n \leq r(r-1)$. Then

$$\binom{r}{2} \leq n - r < (r-1)^2.$$

Thus by the induction hypothesis there exists a graph H on $n-r$ vertices with $\chi^b(H) = \chi^c(H) = r-1$. By Proposition 3.3.2, we have $\lambda_0(H) = r-1$ and $\lambda_k(H) = \lambda_{k-1}(H) - 1$ for all $1 \leq k \leq r-1$. Set $G = H \vee rK_1$. Since $\lambda_k(H) < r$ for all values of k, Proposition 3.3.3 implies $\lambda_0(G) = r$ and $\lambda_k(G) = \lambda_{k-1}(H)$ for all $k \geq 1$. Thus $\lambda_1(G) = \lambda_0(H) = r-1 = \lambda_0(G) - 1$ and for $2 \leq k \leq r$,

$$\lambda_k(G) = \lambda_{k-1}(H) = \lambda_{k-2}(H) - 1 = \lambda_{k-1}(G) - 1.$$

Thus, by Proposition 3.3.2, $\chi^b(G) = \chi^c(G)$. □

We remark that while the upper bound $n \leq r^2$ in Proposition 3.3.6 is tight (as any graph on more than r^2 vertices has bichromatic number more than r), the lower bound can be improved. For example, C_5 satisfies $\chi^b(C_5) = \chi^c(C_5) = 3$ but has less than $\binom{4}{2}$ vertices.

Theorem 3.3.7. *For any positive integer $n \neq 2$ there exists a graph on n vertices with $\chi^b(G) = \chi^c(G)$.*

Proof. For a given n, set $r = \lceil \sqrt{n}\, \rceil$. Then

$$(r-1)^2 < n \leq r^2.$$

Since $\binom{r+1}{2} \leq (r-1)^2 + 1$ for $r \geq 4$, we have

$$\binom{r+1}{2} \leq n \leq r^2,$$

unless $n = 2$ or $n = 5$. By Proposition 3.3.6 there then exists a graph on n vertices with $\chi^b(G) = \chi^c(G)$. If $n = 5$, the graph C_5 satisfies $\chi^b(C_5) = \chi^c(C_5)$. For $n = 2$, the only two graphs on two vertices are K_2 and $\overline{K_2}$. Both have bichromatic number 2 and cochromatic number 1. □

We finish by showing that graphs with bichromatic number equal to their cochromatic number are "general" in the sense that any graph is an induced subgraph of such a graph.

Proposition 3.3.8. *For any graph G, there exists a graph H with $\chi^b(H) = \chi^c(H)$ such that G is an induced subgraph of H.*

Proof. We will use Proposition 3.3.2 to construct the graph H in two steps. First, we will construct a graph G' from G by taking disjoint unions with complete graphs in such a way that $\lambda_{k-1}(G') > \lambda_k(G')$ for all $0 < k \leq \chi(G')$. To that purpose, consider G. By the definition of λ_k, we have

$$\lambda_0(G) \geq \lambda_1(G) \geq \cdots \geq \lambda_{\chi(G)}(G) = 0.$$

Let I be the set of indices $0 < k \leq \chi(G)$ such that $\lambda_{k-1}(G) = \lambda_k(G)$. Consider $G+K_i$ for some $i \in I$. By Proposition 3.3.3, we obtain that $\lambda_{k-1}(G+K_i) = \lambda_k(G+K_i)$ if and only if $k \in I \setminus \{i\}$. Therefore, by setting

$$G' = G + \sum_{i \in I} K_i,$$

we obtain $\lambda_{k-1}(G') > \lambda_k(G')$ for $0 < k \leq \chi(G) = \chi(G')$.

We will now construct the graph H from G' by taking joins with edgeless graphs. Let J be the set of values between 0 and $\lambda_0(G')$ that are not taken by $\lambda_k(G')$ for $k \in \{0, 1, \ldots, \chi(G')\}$. Consider $G \vee jK_1$ for some $j \in J$ and let k_j be the smallest index such that $\lambda_{k_j}(G') < j$ (thus $\lambda_{k_j - 1} > j$ by the definition of J). By Proposition 3.3.3, we have for $0 \leq k \leq \chi(G') + 1 = \chi(G' \vee jK_1)$,

$$\lambda_k(G' \vee jK_1) = \begin{cases} \lambda_k(G') & k < k_j \\ j & k = k_j \\ \lambda_{k-1}(G') & k > k_j. \end{cases}$$

Thus $\lambda_0(G' \vee jK_1) = \lambda_0(G')$ and $\lambda_k(G' \vee jK_1) > \lambda_{k+1}(G' \vee jK_1)$ for $0 \leq k < \chi(G') + 1 = \chi(G' \vee jK_1)$. Furthermore, the values between 0 and $\lambda_0(G' \vee jK_1)$ that are not taken by $\lambda_k(G' \vee jK_1)$ are precisely the ones in $J \setminus \{j\}$. Setting

$$H = G' \vee \bigvee_{j \in J} jK_1,$$

we obtain that $\lambda_k(H) > \lambda_{k+1}(H)$ for $0 \leq k < \chi(H)$ and that $\lambda_k(H)$ takes on all values between 0 and $\lambda_0(H)$. It follows that $\lambda_k(H) = \lambda_{k-1}(H) - 1$ for all $1 \leq k \leq \chi(H)$ and thus, by Proposition 3.3.3, $\chi^b(G) = \chi^c(G)$. Furthermore, G is an induced subgraph of H by the construction of H. □

Corollary 3.3.9. *Let G be a perfect graph. Then there exists a perfect graph H with $\chi^b(H) = \chi^c(H)$ such that G is an induced subgraph of H.*

Proof. If G is a perfect graph then all the graphs used in the construction in the proof of Proposition 3.3.8 are perfect as well. □

3.4 A Brooks-type theorem for χ^b

A well-known theorem of Brooks about graph colourings states:

Theorem 3.4.1 (Brooks' Theorem [13])**.** *For any graph G,*

$$\chi(G) \leq \Delta(G) + 1.$$

Furthermore, equality holds if and only if one of the components of G containing a vertex of maximum degree is either a complete graph or an odd cycle. □

The importance of Brooks' Theorem lies in the fact that, while it is an NP-complete problem to compute the chromatic number of a graph, the maximum degree can be found in linear time in terms of the number of vertices plus the number of edges. Thus Brooks' Theorem provides an easy yet sharp upper bound for the chromatic number. Furthermore, it gives a complete characterization of the graphs that achieve this upper bound, which implies a polynomial time algorithm to check whether a graph G has chromatic number $\Delta(G) + 1$.

We are going to establish a theorem, similar to Brooks', for the bichromatic number. Since the bichromatic number is invariant under complementation, we cannot expect to find an upper bound in terms of just the maximum degree. Indeed, we have $\chi^b(\overline{K_n}) = n$, while $\Delta(\overline{K_n}) = 0$. If we want a bound using the maximum degree, we have to take the maximum degree of the complement into account as well. This justifies the following definition.

Definition 3.4.2. The *bidegree* of a graph G is defined as

$$\Delta^b(G) = \max\left\{\Delta(G), \Delta(\overline{G})\right\}.$$

The following lemma states a few simple properties of the bidegree.

Lemma 3.4.3. *Let G be a graph. Then*

(i) $\Delta^b(\overline{G}) = \Delta^b(G)$;

(ii) $\Delta^b(G) \geq \Delta(G)$*;*

(iii) $\Delta^b(G) \geq \Delta(G) + 1$ *if* G *is disconnected;*

(iv) $\Delta^b(G) = \max\{\Delta(G), n-1-\delta(G)\}$*;*

(v) $\Delta^b(G) \geq \frac{n-1}{2}$*, where equality holds if and only if* G *is* $\frac{n-1}{2}$*-regular.*

Proof. From the definition of the bidegree, (i) and (ii) are immediately apparent. For (iii), let G be disconnected and let r be the order of the largest component of G. Then $\Delta(G) \leq r-1$. However, a vertex in a different component is not adjacent to any of the vertices of the largest component, therefore its degree in $\overline{G}$ is at least r, thus

$$\Delta^b(G) \geq \Delta(\overline{G}) \geq r \geq \Delta(G) + 1.$$

The equality (iv) follows from the fact that $\Delta(\overline{G}) = n-1-\delta(G)$. For (v), we use part (iv) to calculate

$$\Delta^b(G) = \max\{\Delta(G), n-1-\delta(G)\} \geq \max\{\Delta(G), n-1-\Delta(G)\} \geq \frac{n-1}{2}.$$

If $\Delta^b(G) = \frac{n-1}{2}$, then equality must hold in each step of this calculation. The last inequality then yields $\Delta(G) = \frac{n-1}{2}$ and hence $\delta(G) = \frac{n-1}{2}$, as $\delta(G) < \frac{n-1}{2}$ implies $\Delta^b(G) > \frac{n-1}{2}$ by part (iv). Therefore G is $\frac{n-1}{2}$-regular. □

The graph Q in Figure 3.9 is a 4-regular graph on nine vertices having bidegree four. The graph is obtained from $K_{4,4}$ by removing two disjoint edges and joining the four involved vertices to a ninth vertex.

We can now state an analogue of Brooks' Theorem for the bichromatic number.

Theorem 3.4.4. *For any graph* G*,*

$$\chi^b(G) \leq \Delta^b(G) + 1.$$

Furthermore, equality holds if and only if G *is one of* K_n*,* $K_{m,m}$*,* C_5*,* Q *or their complements.*

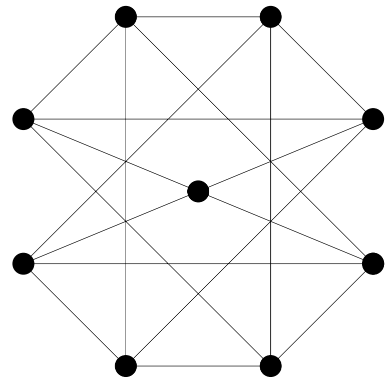

Figure 3.9: The graph Q.

The proof of Theorem 3.4.4 will be conducted in a series of lemmas. First we are going to need a couple of technical lemmas that provide us with simple sufficient conditions for the existence of a (k, l)-colouring of a graph in terms of the number of vertices and the bidegree, provided neither k nor l is zero.

Lemma 3.4.5. *Let G be a graph on n vertices and $k, l \geq 1$ with $n \leq 2(k + l) - 1$. Then G is (k, l)-colourable.*

Proof. We use induction on $k + l$. If $k = l = 1$, then G has at most three vertices and it is easy to see that G is $(1, 1)$-colourable. Otherwise we either have $k \geq 2$ or $l \geq 2$. Suppose $l \geq 2$ (the proof for $k \geq 2$ is similar). If G is an edgeless graph, then it is $(1, 0)$-colourable, thus (k, l)-colourable. Otherwise G contains a clique C of order at least two. Therefore

$$|V(G - C)| \leq n - 2 \leq 2(k + l) - 3 = 2(k + (l - 1)) - 1.$$

By the induction hypothesis, $G - C$ admits a $(k, l - 1)$-colouring. Adding C to this colouring provides a (k, l)-colouring of G. $\square$

We note that Lemma 3.4.5 may fail if $k = 0$ or $l = 0$, as for example K_n has less than $2(n - 1) - 1$ vertices, but is not $(n - 1, 0)$-colourable.

Solving for $k + l$ in Lemma 3.4.5 provides us with the following lemma, which we state so as to simplify subsequent proofs in this section.

Lemma 3.4.6. *Let G be a graph on n vertices. If $k, l \geq 1$ with*

$$k + l > \left\lceil \frac{n-1}{2} \right\rceil,$$

then G is (k, l)-colourable. □

We can now prove that the upper bound in Theorem 3.4.4 holds for all graphs.

Proposition 3.4.7. *Let G be a graph. Then*

$$\chi^b(G) \leq \Delta^b(G) + 1.$$

Proof. We need to show that G is (k, l)-colourable for all k, l with

$$k + l = \Delta^b(G) + 1.$$

Suppose $l = 0$. Then

$$k = \Delta^b(G) + 1 = \max\left\{\Delta(G), \Delta(\overline{G}\right\} + 1 \geq \Delta(G) + 1.$$

Therefore G is $(k, 0)$-colourable by Brooks' Theorem. For $k = 0$, we use the same proof to show that $\overline{G}$ is $(l, 0)$-colourable and hence G is $(0, l)$-colourable.

Now suppose $k, l \geq 1$. Then

$$\begin{aligned} 2(k+l) - 1 &= 2\Delta^b(G) + 1 \\ &\geq 2 \cdot \frac{n-1}{2} + 1 \\ &= n. \end{aligned}$$

Therefore, by Lemma 3.4.5, G is (k, l)-colourable. □

As a next step, we show that the bichromatic numbers of the graphs mentioned in Theorem 3.4.4 all meet the upper bound with equality. Since both bichromatic number and bidegree are invariant under complementation, we only need to consider K_n, $K_{m,m}$, C_5 and Q.

Lemma 3.4.8. *Let $G \in \{K_n, K_{m,m}, C_5, Q\}$. Then*

$$\chi^b(G) = \Delta^b(G) + 1.$$

Proof. By Proposition 3.4.7, it suffices to show that $\chi^b(G) \geq \Delta^b(G) + 1$. For K_n, we have

$$\chi^b(K_n) = n, \qquad \Delta^b(K_n) = n - 1.$$

The bichromatic number of $K_{m,m}$ has been calculated in Proposition 3.1.7, as $K_{m,m}$ is a box cograph of dimensions 2 times m. Thus

$$\chi^b(K_{m,m}) = m + 1, \qquad \Delta^b(K_{m,m}) = m.$$

The bichromatic number and the bidegree of C_5 are:

$$\chi^b(C_5) = 3, \qquad \Delta^b(C_5) = 2.$$

Finally, Q is a 4-regular graph, thus $\Delta^b(Q) = 4$. It remains to show that $\chi^b(Q) \geq 5$. We will prove this by showing that Q is not $(1, 3)$-colourable. Suppose to the contrary that Q is $(1, 3)$-colourable. Let v be the center vertex in Figure 3.9. If we remove any maximal clique containing v, we obtain $K_{3,3}$, which is not $(1, 2)$-colourable. Therefore v cannot be in a clique of a $(1, 3)$-colouring of Q. However, the removal of any maximal independent set containing v leaves behind a graph with an independent set of order 4, which is therefore not $(0, 3)$-colourable. Thus v cannot be in an independent set of a $(1, 3)$-colouring of Q, either. Hence Q cannot be $(1, 3)$-colourable. $\square$

We will show that K_n, $K_{m,m}$, C_5, Q and their complements are the only graphs that achieve equality in the upper bound. To establish this fact, we will show that any other graph G can be (k, l)-coloured for any k, l with $k + l = \Delta^b(G)$, implying $\chi^b(G) \leq \Delta^b(G)$.

Lemma 3.4.9. *Let G be a graph on n vertices other than C_5 or K_n. Then G is $(\Delta^b(G), 0)$-colourable.*

Proof. If G is a connected graph other than C_5 or K_n then G can be coloured with $\Delta(G)$ colours by Brooks' Theorem. Since $\Delta^b(G) \geq \Delta(G)$, the statement follows. If G is not connected, then the maximum degree of $\overline{G}$ is at least one larger than the maximum degree of G. Thus

$$\Delta^b(G) \geq \Delta(G) + 1.$$

By Brooks' Theorem, G is colourable with $\Delta(G)+1$ colours, and therefore $(\Delta^b(G), 0)$-colourable. □

Lemma 3.4.10. *Let G be a graph on n vertices other than C_5 or $\overline{K_n}$. Then G is $(0, \Delta^b(G))$-colourable.*

Proof. The statement follows directly from Lemma 3.4.9 by considering $\overline{G}$. □

Lemma 3.4.11. *Let G be a graph on n vertices, n odd, with $\omega(G) = 2$ other than C_5. Then G is $(1, \Delta^b(G) - 1)$-colourable.*

Proof. If $\Delta^b(G) > \frac{n-1}{2}$ (thus $\Delta^b(G) \geq 2$), then G is $(1, \Delta^b(G) - 1)$-colourable by Lemma 3.4.6. The remaining case is $\Delta^b(G) = \frac{n-1}{2}$ by Lemma 3.4.3, i.e., when G is a $\frac{n-1}{2}$-regular graph. We will show that the only $\frac{n-1}{2}$-regular graph with clique number 2 is C_5. There is no 1-regular graph on three vertices and the only 2-regular graph on five vertices is C_5. We will show that there is no $\frac{n-1}{2}$-regular graph on more than five vertices with clique number 2.

Suppose G is such a graph. Let v be an arbitrary vertex of G and set

$$S = N(v), \qquad T = V(G) - N[v].$$

Since G is $\frac{n-1}{2}$-regular, we obtain $|S| = |T| = \frac{n-1}{2}$. We now count the number of edges. There are $\frac{n-1}{2}$ edges between v and S. As G does not contain a 3-clique, there are no edges between vertices within S. Hence each vertex of S must be adjacent to all but one of the vertices of T. Thus the number of edges between S and T is

$$\frac{n-1}{2}\left(\frac{n-1}{2} - 1\right).$$

Since G is $\frac{n-1}{2}$ regular, it has $\frac{n(n-1)}{4}$ edges overall. We obtain that the number of edges between vertices of T is

$$\frac{n(n-1)}{4} - \frac{n-1}{2} - \frac{(n-1)(n-3)}{4} = \frac{n-1}{4}.$$

Since $n > 5$, this number is at least 2. Let e, f be two edges between vertices of T. If e and f share no common vertex then consider a vertex w of S. As w is adjacent to all but one of the vertices of T, it is adjacent to both endpoints of either e or f, thus creating a clique of order 3. If on the other hand e and f share a vertex, then this vertex must have a neighbour in S, since every vertex has degree $\frac{n-1}{2}$, but there are only $\frac{n-1}{4}$ edges in T. This neighbour is adjacent to at least one of the other two endpoints of e and f, thus again creating a clique of order 3. Hence there is no $\frac{n-1}{2}$-regular graph on more than five vertices with clique number 2. □

Lemma 3.4.12. *Let G be a bipartite graph on n vertices, n even, that is not a box cograph. Then G is $(1, \frac{n}{2} - 1)$-colourable.*

Proof. Consider the independence number of G. The independence number is at least as big as the order of the larger partite set, i.e., at least $\frac{n}{2}$. Suppose that the independence number of G is greater than $\frac{n}{2}$. Then we obtain a $(1, \frac{n}{2} - 1)$-colouring of G by having the independent set be a maximum independent set and the cliques each consist of one of the remaining vertices.

Now suppose that the independence number of G is $\frac{n}{2}$. Since bipartite graphs are perfect graphs, the clique covering number of G is also $\frac{n}{2}$. By Theorem 3.1.22, G is either a box cograph or $(1, \frac{n}{2} - 1)$-colourable. □

Lemma 3.4.13. *Let G be a graph on n vertices, n even, with $\omega(G) = 2$ and G not bipartite. Then G is $(1, \frac{n}{2} - 1)$-colourable.*

Proof. Assume the statement is false and let G be a counterexample on the smallest number of vertices. Consider an odd cycle C (of length at least 5, since $\omega(G) = 2$). Take a K_2 with at least one vertex outside of C (if such a K_2 does not exist then G is a disjoint union of an odd cycle with an independent set, which is $(1, \frac{n}{2} - 1)$-colourable).

If $G - K_2$ still contains an odd cycle then it must have a $(1, \frac{n}{2} - 2)$-colouring by the assumption of G being a smallest counterexample. Extending this colouring by adding the K_2 yields a $(1, \frac{n}{2} - 1)$-colouring of G, a contradiction. Therefore $G - K_2$ is bipartite. By Lemma 3.4.12, $G - K_2$ must then be a box cograph, and cannot contain a P_4. This is a contradiction since the remaining vertices of C form a path of order at least 4. □

Lemma 3.4.14. *Let G be a graph with $\omega(G) = 2$ other than C_5, $2K_2$ and $K_{m,m}$. Then G is $(1, \Delta^b(G) - 1)$-colourable.*

Proof. Let n be the number of vertices of G. If n is odd, then by Lemma 3.4.11, G is $(1, \Delta^b(G) - 1)$-colourable. If n is even and G is not a bipartite box cograph, then by Lemmas 3.4.12 and 3.4.13, G is $(1, \frac{n}{2} - 1)$-colourable. By Lemma 3.4.3, $\Delta^b(G) \geq \frac{n}{2}$ if n is even, thus G is also $(1, \Delta^b(G) - 1)$-colourable. The only case that remains to be considered is if G is a bipartite box cograph other than $2K_2$ and $K_{m,m}$.

The only bipartite box cographs on less than six vertices are $2K_1$ (with clique number 1), $K_{1,1}$, $2K_2$ and $K_{2,2}$, all of which have been excluded. We thus may assume that $n \geq 6$. As a cograph, either G or $\overline{G}$ is disconnected. The graph $\overline{G}$ consists of two cliques (the partite sets in G) and at least one edge connecting them, since G is not a complete bipartite graph by assumption. Therefore G must be disconnected. By considering a vertex from the largest partite set of the smallest component of G, we find that $\delta(G) \leq \frac{n}{4}$ and thus

$$\Delta^b(G) \geq n - 1 - \delta(G) \geq \left\lceil \frac{3n}{4} \right\rceil - 1 \geq \frac{n}{2} + 1.$$

By Lemma 3.4.6, G is then $(1, \Delta^b(G) - 1)$-colourable. □

Lemma 3.4.15. *Let G be a graph on n vertices, n even, with $\omega(G) = 3$. Then G is $(1, \frac{n}{2} - 1)$-colourable.*

Proof. Any graph on four vertices with clique number 3 is $(1, 1)$-colourable, thus suppose $n \geq 6$. Take a K_3 in G and consider $G - K_3$. We obtain

$$|V(G - K_3)| = n - 3 = 2\left(1 + \left(\frac{n}{2} - 2\right)\right) - 1.$$

By Lemma 3.4.5, $G - K_3$ thus admits a $(1, \frac{n}{2} - 2)$-colouring. Extending this colouring by adding the K_3 yields a $(1, \frac{n}{2} - 1)$-colouring of G. □

Lemma 3.4.16. *Let G be a graph on n vertices, n odd, with $\omega(G) = 3$ other than Q. Then G is $(1, \Delta^b(G) - 1)$-colourable.*

Proof. Since $\omega(G) = 3$, the bidegree of G is at least 2. By Lemma 3.4.6, G is $(1, \Delta^b(G) - 1)$-colourable if $\Delta^b(G) > \frac{n-1}{2}$. Therefore we only need to consider $\frac{n-1}{2}$-regular graphs (in particular $n \geq 9$, since there are no $\frac{n-1}{2}$-regular graphs on less vertices with clique number 3). Take a K_3 in G and consider $G - K_3$. We want to show that $G - K_3$ admits a $(1, \frac{n-1}{2} - 2)$-colouring, as we can (by including the K_3) extend such a colouring to a $(1, \frac{n-1}{2} - 1)$-colouring of G. By Lemmas 3.4.12, 3.4.13 and 3.4.15, $G - K_3$ admits such a colouring unless it is a bipartite box cograph. Thus suppose $G - K_3$ is a bipartite box cograph (independent of the choice of K_3).

As a cograph, either $G - K_3$ or $\overline{G - K_3}$ is disconnected. We will show that $\overline{G - K_3}$ must be disconnected. Suppose to the contrary that $G - K_3$ is disconnected. By taking a vertex from the largest partite set of the smallest component, we find that

$$\delta(G - K_3) \leq \frac{n-3}{4}.$$

On the other hand every vertex of G had degree $\frac{n-1}{2}$ and was adjacent to at most two of the vertices from K_3 (otherwise the clique number would be at least 4). Therefore

$$\delta(G - K_3) \geq \frac{n-1}{2} - 2.$$

The only way to satisfy both inequalities is by having $n \leq 7$. However, we already established $n \geq 9$. Therefore $G - K_3$ has to be connected and thus $\overline{G - K_3}$ disconnected. Since $G - K_3$ is bipartite, $\overline{G - K_3}$ must be a disjoint union of two cliques, and as a box cograph therefore must be $2K_m$, where $m = \frac{n-3}{2}$. Thus $G - K_3 = K_{m,m}$. Using this fact, we will count the number of edges of G in two ways. Since G is a $\frac{n-1}{2}$-regular graph,

$$|E(G)| = \frac{n(n-1)}{4} = \frac{1}{4}(n^2 - n).$$

On the other hand, using $G - K_3 = K_{\frac{n-3}{2},\frac{n-3}{2}}$, we see that there are $\left(\frac{n-3}{2}\right)^2$ edges in $G - K_3$, while there are $3\left(\frac{n-1}{2} - 2\right)$ edges between K_3 and $G - K_3$, and 3 edges in K_3. So we obtain

$$\begin{aligned} |E(G)| &= \left(\frac{n-3}{2}\right)^2 + 3\left(\frac{n-1}{2} - 2\right) + 3 \\ &= \frac{1}{4}(n^2 - 9). \end{aligned}$$

Thus, the only possible value for n is 9. We are going to show that G is isomorphic to the graph Q.

To recapitulate, G is a 4-regular graph on nine vertices and the removal of any K_3 leaves behind a $K_{3,3}$. Consider a K_3. Since G is regular, any vertex in K_3 is adjacent to two vertices in $G - K_3$ and all these vertices must be distinct, since $K_{3,3}$ is 3-regular. Then one of the vertices must be adjacent to vertices from both partite sets of $K_{3,3}$ (this is going to be the center vertex in Figure 3.9). If we remove the K_3 formed by those three vertices, the reminder must be again $K_{3,3}$, and the constructed graph is precisely Q. □

Lemma 3.4.17. *Let G be a graph on n vertices with $\omega(G) \geq 4$. Then G is $(1, \Delta^b(G) - 1)$-colourable.*

Proof. Since $\omega(G) \geq 4$, the bidegree of G is at least 3. Take a K_4 in G and consider $G - K_4$. Since $\Delta^b(G) \geq \frac{n-1}{2}$, we obtain

$$|V(G - K_4)| = n - 4 \leq 2\Delta^b(G) - 3 = 2(1 + (\Delta^b(G) - 2)) - 1.$$

By Lemma 3.4.5, $G - K_4$ admits a $(1, \Delta^b(G) - 2)$-colouring, which we can extend to a $(1, \Delta^b(G) - 1)$-colouring of G by adding the K_4 as a clique. □

Lemma 3.4.18. *Let G be a graph on n vertices, where n is even. Then G is (k, l)-colourable for any $k, l \geq 2$ with $k + l = \Delta^b(G)$.*

Proof. If $\Delta^b(G) > \frac{n}{2}$, then G is (k, l)-colourable by Lemma 3.4.6. Thus we may assume that $\Delta^b(G) = \frac{n}{2}$. Then $n \geq 8$, since $\Delta^b(G) = k + l \geq 4$. Since the Ramsey number $R(3,3)$ equals 6, we know that G either contains a clique or an independent set of order 3. Suppose G contains a K_3 (the other case is similar). Then

$$|V(G-K_3)| = n-3 = 2\Delta^b(G)-3 = 2(k+(l-1))-1.$$

By Lemma 3.4.5, $G-K_3$ then admits a $(k,l-1)$-colouring, which we can extend to a (k,l)-colouring of G by including K_3 as a clique. □

Lemma 3.4.19. *Let G be a graph on n vertices, where n is odd. Then G is (k,l)-colourable for any $k,l \geq 2$ with $k+l = \Delta^b(G)$.*

Proof. If $\Delta^b(G) > \frac{n-1}{2}$, then G is (k,l)-colourable by Lemma 3.4.6. Thus we may assume that $\Delta^b(G) = \frac{n-1}{2}$. Then $n \geq 9$, since $\Delta^b(G) = k+l \geq 4$. Suppose G contains a K_4. Then

$$|V(G-K_4)| \leq 2\Delta^b(G)-3 = 2(k+(l-1))-1.$$

By Lemma 3.4.5, $G-K_4$ admits a $(k,l-1)$-colouring, which we can extend to a (k,l)-colouring of G by including the K_4 as a clique. In the same manner, we can find a (k,l)-colouring of G, if G contains $\overline{K_4}$. It remains to show that G is (k,l)-colourable if both clique number and independence number of G are at most 3. Since the Ramsey number $R(3,4)$ equals 9, any graph contains either a clique of order 4 or an independent set of order 3. Thus we can conclude that G contains an independent set of order 3. With the same argument, we can show that G contains a clique of order 3. To recapitulate, G is an $\frac{n-1}{2}$-regular graph on $n \geq 9$ vertices (where n is odd) with $\alpha(G) = \omega(G) = 3$.

Now assume that $n \geq 13$. Then either k or l must be at least three (as $k+l = \Delta^b(G) = \frac{n-1}{2}$), say without loss of generality $l \geq 3$. Let C be clique of order 3 in G and consider $G-C$. Since $|V(G-C)| \geq 10$ and $G-C$ cannot contain an independent set of order at least 4, we again (by the same argument involving $R(3,4)$ as above) have $\omega(G-C) = 3$. Let C' be a clique of order 3 in $G-C$. We have

$$|V(G-C-C')| = n-6 = 2\Delta^b(G)-5 = 2(k+(l-2))-1.$$

Thus, by Lemma 3.4.5, $G - C - C'$ admits a $(k, l-2)$-colouring, which we can extend to a (k, l)-colouring by the inclusion of C and C' as cliques.

Since there is no 5-regular graph on eleven vertices, the only case remaining is $n = 9$, which implies $k = l = 2$. Let S be an independent set in G of order 3 and consider $G - S$. Since $R(3, 3) = 6$, $G - S$ either contains an independent set or a clique of order 3. Suppose C is a clique of order 3 in $G - S$. Then $G - S - C$ has three vertices and is therefore $(1, 1)$-colourable. By including C as a clique and S as an independent set, we can extend such a colouring to a $(2, 2)$-colouring of G. Thus we may assume that $\omega(G - S) = 2$ (the case of $\omega(G - S) = 1$ is not possible as this would imply that $V(G - S)$ is an independent set of order 6). By Lemmas 3.4.12 and 3.4.13, $G - S$ is $(1, 2)$-colourable (and thus G would be $(2, 2)$-colourable by the inclusion of S), unless $G - S$ is a bipartite box cograph. We can compute the number of edges of $G - S$, since we know that G, as a 4-regular graph, has 18 edges and each vertex of S has 4 incident edges, all of which must be distinct, since S is an independent set. Therefore $G - S$ has six edges. This is a contradiction, since no bipartite box cograph on six vertices has exactly six edges (the only bipartite box cographs on six vertices are $K_{3,3}$, $C_4 + K_2$ and $3K_2$). Hence all 4-regular graphs on nine vertices are $(2, 2)$-colourable. □

We now have all the ingredients to prove the analogue of Brooks' theorem for the bichromatic number.

Proof of Theorem 3.4.4. By Proposition 3.4.7, the upper bound holds and by Lemma 3.4.8 the graphs mentioned achieve equality in the upper bound. It remains to show that any other graph G satisfies $\chi^b(G) \leq \Delta^b(G)$, i.e., is (k, l)-colourable for all k, l with $k + l = \Delta^b(G)$. Furthermore, we can restrict ourselves to $k \leq l$, since for $k > l$, we can equivalently show that $\overline{G}$ is (l, k)-colourable.

If $k = 0$, then G is (k, l)-colourable by Lemma 3.4.10. For $k = 1$ (thus $l = \Delta^b(G) - 1$), consider the clique number of G. If $\omega(G) = 1$, then G is an edgeless graph which is trivially (k, l)-colourable. For $\omega(G) = 2$, Lemma 3.4.14 shows that

G is (k,l)-colourable, while the case of $\omega(G) = 3$ is handled by Lemmas 3.4.15 and 3.4.16 and the case of $\omega(G) \geq 4$ by Lemma 3.4.17. Finally, if $k \geq 2$, then G is (k,l)-colourable by Lemmas 3.4.18 and 3.4.19. $\square$

Chapter 4

Special graph classes

In this chapter, we investigate (k, l)-colourings of a few specialized graph classes (cographs, chordal graphs and proper circular arc graphs) and establish some algorithms determining the (k, l)-colourability for such graphs.

Section 4.1 is concerned with (k, l)-colourings of cographs. Proposition 4.1.1 shows a connection between the colouring sequence of a cograph and its number of vertices. Following this, it is shown in Proposition 4.1.2 that cographs have a particularly nice representation, called a Young diagram representation, relating to (k, l)-colourings. Proposition 4.1.3 and Corollary 4.1.4 show that the box cographs are precisely the obstructions regarding (k, l)-colourings and the bichromatic number of cographs. In Subsection 4.1.1, we give a summary of how a cograph can be represented as a tree, called the cotree and introduce the related, more general, representation of pseudocotrees. Subsection 4.1.2 introduces a terminology for algorithms on pseudocotrees and demonstrates it using two simple algorithms for the chromatic number and for finding a maximum clique in a cograph. In Subsection 4.1.3 algorithms for the (k, l)-colourability of cographs, for finding induced box cographs and for obtaining the Young diagram representation of cographs are established.

In Section 4.2 we consider (k, l)-colourings of chordal graphs. Theorem 4.2.5 characterizes the obstructions for (k, l)-colouring chordal graphs. Though the theorem has been known before, the proof presented here is new, relying on the perfect elimination property of chordal graphs. Subsection 4.2.2 investigates the class of k-trees, a subclass of chordal graphs, with particular interest in relating the bichromatic number to the clique covering number of k-trees.

The subject of Section 4.3 is (k, l)-colourings of proper circular arc graphs. For convenience of terminology they are investigated in their oriented form as round digraphs. Subsection 4.3.1 looks at (k, l)-colourings with $l \geq 1$. Some structural results are obtained and Proposition 4.3.16 presents a method for checking (k, l)-colourability in some cases. Subsection 4.3.2 considers (k, l)-colourings with $l = 0$, that is, k-colourings. The object of this part is characterizing k-colourable round digraphs using an algebraic approach involving permutations, leading to Subsection 4.3.3, in which this approach is further transformed to obtain a characterization of all maximally k-colourable round digraphs in Theorem 4.3.32. Finally, Subsection 4.3.4 provides a partial list of obstructions for the (k, l)-colourability of round digraphs.

4.1 Cographs

We will now consider the class of cographs in more detail. In Theorem 3.1.22 we showed that all graphs G satisfying

$$\chi^b(G) = \chi(G) + \theta(G) - 1$$

are cographs and more precisely box cographs as defined in Subsection 3.1.2. We will show that the box cographs in $\mathscr{B}(k,l)$ are precisely the forbidden subgraphs for the (k,l)-colourability of a cograph. We will then proceed to use this result to derive algorithms that determine whether a cograph is (k,l)-colourable and find its cochromatic and bichromatic numbers as well as an algorithm that finds an induced box cograph of dimensions k times l, given that the cograph is not $(k-1,l-1)$-colourable. The algorithms will rely on the fact that every cograph can be represented as a tree, called the cotree. This concept was first introduced in [15]. In addition, we will give a new representation of cographs using Young diagrams which allows us to answer several questions regarding (k,l)-colourability of a cograph directly from that representation.

The first time that (k,l)-colouring of cographs appeared was in a paper by Gimbel, Kratsch and Stewart in 1994 [35]. In the paper, an algorithm for the computation of the cochromatic number of a cograph using its cotree was presented. The algorithm, which runs in time $O(n^2)$, implicitly uses (k,l)-colourings and can easily be adapted to check the (k,l)-colourability or to compute the bichromatic number of a cograph. Demange, Ekim and de Werra [17] gave two different algorithms concerning the (k,l)-colouring of cographs. The first, which also uses cotrees, calculates a maximum (k,l)-colourable induced subgraph of a cograph (thereby also checking the (k,l)-colourability of the cograph itself) in time $O((k^3l + kl^3)n)$. For the purpose of the second algorithm, it is shown that if G is a (k,l)-colourable cograph (with $l \geq 1$) and C a maximum clique of G, then $G - C$ is $(k, l-1)$-colourable. Using this, the algorithm finds a (k,l)-colouring of a cograph (if one exists) by successively removing l maximum cliques and finding a k-colouring of the remaining graph. The algorithm runs in time $O(n(m+n))$ where m is the number of edges of the graph. An adaptation of this idea is presented which calculates the cochromatic number in time $O(n^{3/2})$.

A characterization of (k,l)-colourable cographs via forbidden subgraphs was first stated by de Souza Francisco, Klein and Nogueira [16], omitting the proof for $k \geq 2$ (the two cases for $k = 1$ and $l = 1, 2$ had also appeared in [17]). A full proof was given by Feder, Hell and Hochstättler [28].

To start, we recall one of the characterizations for cographs from Theorem 3.1.2 that will turn out to be very useful in this section.

A graph G is a cograph if and only if for any induced subgraph $H \neq K_1$ of G either H or $\overline{H}$ is disconnected.

This characterization implies that any induced subgraph of a cograph is a cograph itself. Since G is an induced subgraph of itself, we can obtain from the above characterization the following recursive property of cographs.

- *For every disconnected cograph G, there exist cographs G_1, G_2 such that $G = G_1 + G_2$.*
- *For every connected cograph $G \neq K_1$, there exist cographs G_1, G_2 such that $G = G_1 \vee G_2$.*

Now recall the parameters $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ concerning (k,l)-colourings as defined in Subsection 2.1.1. Corollaries 2.1.8, 2.1.11, 2.1.14 and 2.1.15 provide us with formulas for the disjoint union and the join of graphs.

$$\boldsymbol{\kappa}(G_1 + G_2) = \boldsymbol{\kappa}(G_1) * \boldsymbol{\kappa}(G_2) \qquad \boldsymbol{\kappa}(G_1 \vee G_2) = \boldsymbol{\kappa}(G_1) + \boldsymbol{\kappa}(G_2)$$
$$\boldsymbol{\lambda}(G_1 + G_2) = \boldsymbol{\lambda}(G_1) + \boldsymbol{\lambda}(G_2) \qquad \boldsymbol{\lambda}(G_1 \vee G_2) = \boldsymbol{\lambda}(G_1) * \boldsymbol{\lambda}(G_2)$$

Thus we can calculate $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ recursively for cographs, only using the two operations $+$ and $*$.

An interesting consequence of these formulas is the following result.

Proposition 4.1.1. *For any cograph G,*

$$\sum_{i\geq 0}\kappa_i(G)=\sum_{i\geq 0}\lambda_i(G)=|V(G)|\,.$$

Proof. The first equality is a direct consequence of $\boldsymbol{\kappa}(G)$ and $\boldsymbol{\lambda}(G)$ being conjugates (as established in Subsection 2.1.1). We will show

$$\sum_{i\geq 0}\kappa_i(G)=|V(G)|$$

by induction on the number of vertices of G. If $G=K_1$, then $\boldsymbol{\kappa}(G)=(1)$, thus the statement holds. Assume that $G\neq K_1$ and the statement holds for all cographs with fewer vertices than G. Suppose G is disconnected. Then there exist cographs G_1, G_2 such that $G=G_1+G_2$ and we have $\boldsymbol{\kappa}(G)=\boldsymbol{\kappa}(G_1)*\boldsymbol{\kappa}(G_2)$. By the definition of the operation $*$ and the induction hypothesis,

$$\begin{aligned}\sum_{i\geq 0}\kappa_i(G)&=\sum_{i\geq 0}\kappa_i(G_1)+\sum_{i\geq 0}\kappa_i(G_2)\\&=|V(G_1)|+|V(G_2)|\\&=|V(G_1)\cup V(G_2)|\\&=|V(G)|\,.\end{aligned}$$

If on the other hand G is connected, there exist cographs G_1, G_2 with $G=G_1\vee G_2$ and we obtain as above

$$\begin{aligned}\sum_{i\geq 0}\kappa_i(G)&=\sum_{i\geq 0}(\kappa_i(G_1)+\kappa_i(G_2))\\&=\sum_{i\geq 0}\kappa_i(G_1)+\sum_{i\geq 0}\kappa_i(G_2)\\&=|V(G)|\,.\end{aligned}$$

□

We remark that neither does this result hold for general graphs, nor could we replace the equality by an inequality, as we can see from the examples of P_4 and C_5:

$$\sum_{i\geq 0}\kappa_i(P_4)=2+1=3<|V(P_4)|\,,$$

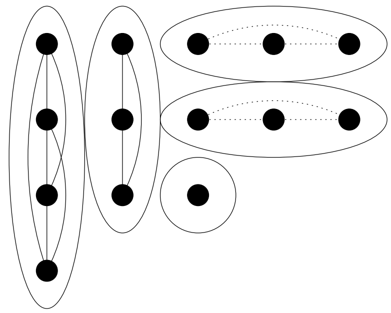

Figure 4.1: A $(3,2)$-colouring of a cograph given Young diagram representation.

$$\sum_{i \geq 0} \kappa_i(C_5) = 3 + 2 + 1 = 6 > |V(C_5)|\,.$$

Now consider graphs G that can be represented as follows. The vertices of G correspond to the points of a Young diagram, where vertices in the same column form a clique and vertices in the same row an independent set. We will call this a *Young diagram representation* of G. Suppose we want to (k,l)-colour such a graph. One strategy would be to take the l leftmost columns as cliques and then the k topmost rows as independent sets (see Figure 4.1). We see that we can do so if k is at least the number of vertices in the $(l+1)$-st column, thus

$$\kappa_l(G) \leq \#\text{ vertices in the } (l+1)\text{-st column.}$$

If we sum the righthand sides over all l, we get the number of vertices of G. The sum over the lefthand sides is $\sum_{i \geq 0} \kappa_i(G)$. Thus graphs G with a Young diagram representation satisfy

$$\sum_{i \geq 0} \kappa_i(G) \leq |V(G)|\,.$$

If, in addition, G is a cograph, we know that the sum on the lefthand side equals $|V(G)|$. Therefore all the inequalities are equalities. So $\kappa_l(G)$ is precisely the number of vertices in the $(l+1)$-st column and $\boldsymbol{\kappa}(G)$ thus the type of the conjugate of the Young diagram. (The type of the Young diagram equals $\boldsymbol{\lambda}(G)$, as $\boldsymbol{\lambda}(G)$ is the conjugate of $\boldsymbol{\kappa}(G)$).

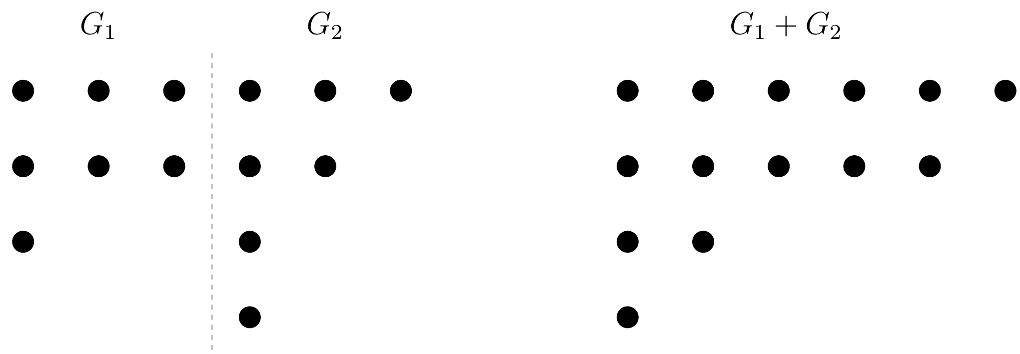

Figure 4.2: Young diagram representation of G_1, G_2 and $G_1 + G_2$.

We have already seen in Subsection 3.1 that box cographs can be represented in this fashion. Clearly we can also give a Young diagram representation of disjoint unions of cliques and their complements, complete multipartite graphs, thereby providing us with an alternative proof of Corollary 3.3.5. However, we can show much more.

Proposition 4.1.2. *Every cograph G has a Young diagram representation.*

Proof. The proof is by induction on the number of vertices of G. If $G = K_1$, then the Young diagram consists of a single point. Assume that G has more than one vertex and every cograph on fewer vertices than G has a Young diagram representation. Suppose, G is disconnected. Then there exist cographs G_1, G_2 with $G = G_1 + G_2$. By the induction hypothesis, G_1 and G_2 have a Young diagram representation. Consider what happens if we write the two diagrams side by side (see the left side of Figure 4.2 for an example). Each column completely belongs to either G_1 or G_2, therefore forms a clique. Each row may have vertices from both G_1 and G_2, but the vertices in each of the two graphs form an independent set, and therefore the whole row must form an independent set in $G_1 + G_2$. The diagram we have might not be a Young diagram, though. However, this can be remedied by permuting the columns. As this does not change the sets of rows and columns, we obtain a Young diagram representation of G (see the right side of Figure 4.2 for the example).

If G is connected, there exist cographs G_1, G_2 with $G = G_1 \vee G_2$ and we obtain a Young diagram representation of G by writing the two Young diagram representations of G_1 and G_2 on top of each other and permuting the rows instead. □

As an easy corollary of Proposition 4.1.2, we obtain a result that has first been stated in [16] and proven in [28].

Proposition 4.1.3. [28] *A cograph G is (k,l)-colourable if and only if it contains no induced subgraph $H \in \mathscr{B}(k+1, l+1)$.*

Proof. Suppose G contains an induced subgraph $H \in \mathscr{B}(k+1, l+1)$. As H is not (k,l)-colourable (Theorem 3.1.22), neither is G. On the other hand, suppose G is not (k,l)-colourable. Then $\kappa_l(G) \geq k+1$. Therefore each of the $l+1$ leftmost columns of the Young diagram representation of G contain at least $k+1$ vertices. These $(k+1)(l+1)$ vertices induce a box cograph in $\mathscr{B}(k+1, l+1)$. □

Corollary 4.1.4. *Every cograph contains an induced box cograph of the same bichromatic number.*

Proof. Let G be a cograph and let k, l be such that $k + l = \chi^b(G) - 1$ and that G is not (k,l)-colourable. Then G contains an induced subgraph $H \in \mathscr{B}(k+1, l+1)$ by Proposition 4.1.3. The bichromatic number of H is $(k+1) + (l+1) - 1$, which is equal to the bichromatic number of G. □

In light of Section 3.3, we define one more subclass of cographs.

Definition 4.1.5. A cograph G is called a *triangular cograph* of dimension n if $\boldsymbol{\kappa}(G) = (n, n-1, \ldots, 1)$.

The term triangular cograph is chosen from the shape of the Young diagram representation (similar to the rectangular shape of the Young diagram representation of Box cographs). The definition immediately implies the following result.

Proposition 4.1.6. *For any triangular cograph G,*

$$\chi^b(G) = \chi^c(G) = n.$$ □

4.1.1 Cotrees

In the beginning of this section, we stated that any cograph G on at least two vertices can be either written as $G = G_1 + G_2$ or $G = G_1 \vee G_2$ for some cographs G_1, G_2. These cographs G_1, G_2 need not be unique, though, as G or $\overline{G}$ might have more than one component. However, if we let $G_1, G_2, \ldots, G_t$ be the components of G or $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_t$ be the components of $\overline{G}$ (depending on whether G is disconnected or not), we get the following property of cographs.

- Every disconnected cograph G can be written uniquely (up to order) as

$$G = G_1 + G_2 + \cdots + G_t,$$

  where each G_i is a connected component of G and a cograph.

- Every connected cograph $G \neq K_1$ can be written uniquely (up to order) as

$$G = G_1 \vee G_2 \vee \cdots \vee G_t,$$

  where each $\overline{G_i}$ is a connected component of $\overline{G}$ and each G_i is a cograph.

This allows us to represent a cograph G as a tree, called the *cotree* of G (first introduced in [15]). The cotree of G, denoted by T_G, is a rooted tree where every internal node is labelled with either 0 or 1. We recursively construct the cotree as follows.

- If $G = K_1$, then we define T_G to be the rooted tree on a single vertex.
- If G is disconnected, let $G_1, G_2, \ldots, G_t$ be the connected components of G. We take the cotrees of $G_1, G_2, \ldots, G_r$ and add an edge from each of the roots to a new vertex, which we label with a 0. The tree thus constructed, with the root at the new vertex, is the cotree T_G.
- If $G \neq K_1$ is connected, let $\overline{G_1}, \overline{G_2}, \ldots, \overline{G_t}$ be the connected components of $\overline{G}$. We take the cotrees of $G_1, \ldots, G_r$ and add an edge from each of the roots to a

new vertex, which we label with a 1. The tree thus constructed, with the roots at the new vertex, is the cotree T_G.

We remark that the construction implies that the cotree is unique. Every leaf of the cotree T_G represents a vertex of G and every internal node represents the subgraph of G induced by the vertices that are descendents of that node. Every 0-node represents a disconnected subgraph, every 1-node a connected subgraph. By the construction, all children of a 0-node represent connected cographs, thus are either 1-nodes or leaves. Similarly the children of a 1-node are either 0-nodes or leaves. Also, we note that two vertices of the cograph are adjacent if and only if the lowest common ancestor of the corresponding leaves is a 1-node. An example of a cotree is given in Figure 4.3. The corresponding cograph is shown in Figure 4.4, where the thick edges stand for complete adjacency.

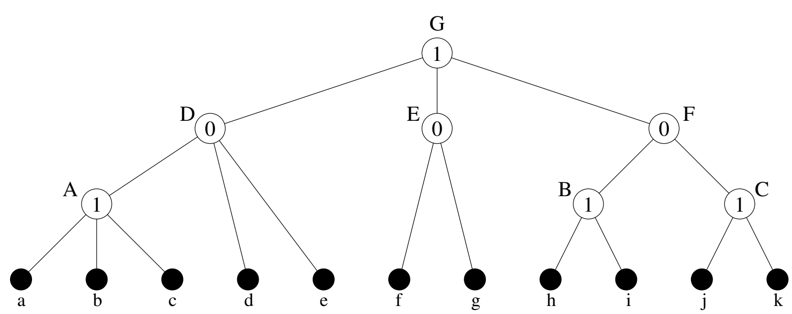

Figure 4.3: A cotree.

For the purpose of later proofs, it will be advantageous to consider a more general variant of cotrees. Note that every cotree satisfies the following three properties.

- Each internal node is labeled either 0 or 1;
- Each internal node has at least two children;
- The 0-nodes and 1-nodes alternate on each path from a leaf to the root.

If we want to construct the cograph to a given cotree, we associate each leaf with a vertex and interpret the 0-nodes as disjoint unions and 1-nodes as joins. We note

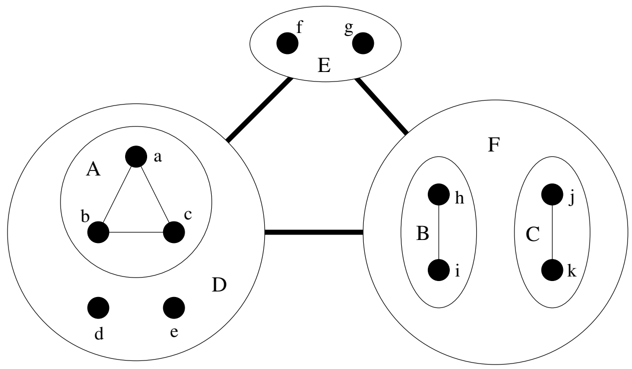

Figure 4.4: The cograph corresponding to the cotree from Figure 4.3.

that we only use the first property of the tree. We will call a tree satisfying the first property a *pseudocotree* of the cograph obtained by the procedure described. An example of a pseudocotree of the graph in Figure 4.4 is given in 4.5.

In general, a cograph G can be represented by different pseudocotrees. The following lemma shows how to obtain the unique cotree T_G of a cograph G from any pseudocotree.

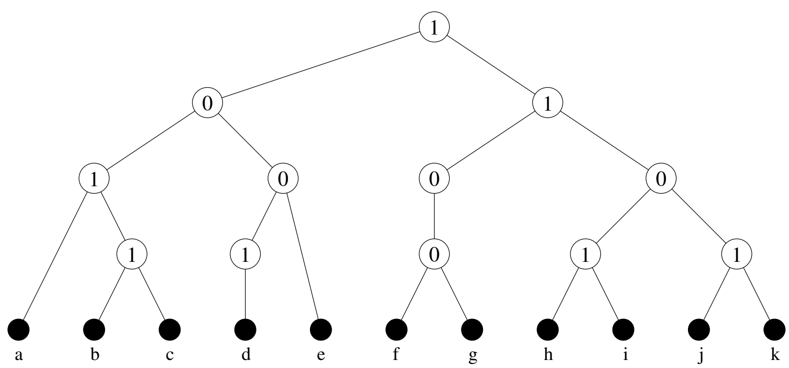

Figure 4.5: A pseudocotree of the graph from Figure 4.4.

Lemma 4.1.7. *Let T be a pseudocotree of a cograph G. Then the cotree T_G can be obtained from T by repeatedly applying the following steps:*

(1) if the root B has only one child C, delete B and make C the new root;

(2) if an internal node B with parent A has only one child C, delete B and make C the child of A;

(3) if an internal node B with children $C_1, C_2, \ldots, C_t$ has the same label as its parent A, delete that node and make $C_1, C_2, \ldots, C_t$ the children of A.

Proof. Since each of the three steps removes a node, the process of repeatedly applying the three steps will eventually terminate in a rooted tree with internal nodes labelled 0 or 1, where every internal node has at least two children (otherwise we could apply step (1) or step (2)) and where 0-nodes and 1-nodes alternate (otherwise we could apply step (3)). These properties define a cotree, hence it remains to show that this cotree is indeed the cotree of the graph G. We will prove this by showing that the application of each of the three steps preserves the associated cograph.

Consider steps (1) and (2). Since both the disjoint union and join of a single graph return the graph itself, the graph induced by the node B is the same as the one induced by C. Thus applying one of the two steps does not change the associated cograph.

For step (3), assume without loss of generality that A and B are 0-nodes (the proof for 1-nodes is almost identical) and that A has children $A_1, A_2, \ldots, A_s$ apart from B. By the associativity and commutativity of the disjoint union, we can therefore write for the graphs induced by A and B

$$A = B + A_1 + \cdots + A_s,$$

$$B = C_1 + \cdots + C_t.$$

Combining these two equations yields

$$A = C_1 + \cdots + C_t + A_1 + \cdots + A_s,$$

which is precisely the graph induced by A after applying step (3). Thus the associated cograph does not change by applying either of the three steps. □

4.1.2 Basic algorithms on cotrees

There are two standard types of algorithms that can be implemented on cotrees (respectively pseudocotrees). The first type is the bottom-up algorithm, which traverses the cotree from the leaves to the root. A bottom-up algorithm can be described as follows.

Algorithm. (generic bottom-up algorithm)

- INPUT: Pseudocotree T.
- INITIALIZATION: Argument assigned to the leaves of T.
- 0-NODE OPERATOR: Formula for calculating an argument for a 0-node A depending on the arguments of its children $A_1, A_2, \ldots, A_t$.
- 1-NODE OPERATOR: Formula for calculating an argument for a 1-node A depending on the arguments of its children $A_1, A_2, \ldots, A_t$.
- OUTPUT: Argument of the root of T.

A typical bottom-up algorithm calculates a graph invariant, such as for example the chromatic number, the number of cliques (see [15] for both) and the cochromatic number (see [35]). As an example, we present an algorithm that calculates the chromatic number of a cograph (from [15], but represented in the form of a bottom-up algorithm as given here).

Algorithm. (CHROMATIC NUMBER)

- INPUT: Pseudocotree T of a cograph G.
- INITIALIZATION: Assign $\chi = 1$ to each leaf of T.
- 0-NODE OPERATOR: $\chi(A) = \max_{1 \leq i \leq t} \chi(A_i)$.
- 1-NODE OPERATOR: $\chi(A) = \sum_{1 \leq i \leq t} \chi(A_i)$.

- OUTPUT: $\chi(G)$.

Proof of correctness. We observe that the argument assigned to the leaves is $\chi(K_1)$. As the two operators correspond to the formulas for the chromatic number of disjoint unions and joins of graphs, the argument of every node is the chromatic number of the cograph induced by that node. □

An example of the output of CHROMATIC NUMBER is given in Figure 4.6, where the arguments of the leaves (all being 1) have been omitted.

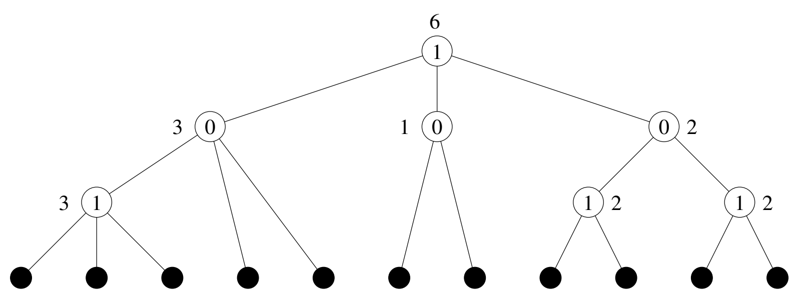

Figure 4.6: CHROMATIC NUMBER on the cotree from Figure 4.3.

We can see that CHROMATIC NUMBER has the same output regardless of which pseudocotree T of G has been chosen. A particularly nice class of algorithms that satisfy this property is the following.

Definition 4.1.8. A bottom-up algorithm is called *natural* if

(i) both operators are associative and commutative;

(ii) both operators are the identity operators on nodes with only one child;

(iii) there exists a *neutral* argument that has no impact on either of the operators.

With this definition, CHROMATIC NUMBER is a natural algorithm with neutral argument 0.

Lemma 4.1.9. *The output of a natural algorithm is the same for any pseudocotree of G.*

Proof. We will show that the output of a natural algorithm for any pseudocotree T of G is the same as for the cotree T_G. It suffices to show that the output does not change, when applying one of the three steps from Lemma 4.1.7 to T. This follows directly from property (ii) of natural algorithms for steps (1) and (2) and from property (i) for step (3). □

The effect of the neutral argument in the definition of natural algorithms becomes evident when we consider induced subgraphs of a cograph. Suppose we have a graph G and an induced subgraph H and we are given the cotree T_G. The easiest pseudocotree of H to find is not T_H but the one arising from T_G by pruning all the branches not containing a vertex of H. In Figure 4.7, we can see such a pseudocotree. We observe that running CHROMATIC NUMBER on that pseudocotree has the same result as when we run CHROMATIC NUMBER on T_G with the modification of assigning a 0 (the neutral argument of CHROMATIC NUMBER) to the leafs not in H. This property is the key reason for considering natural algorithms.

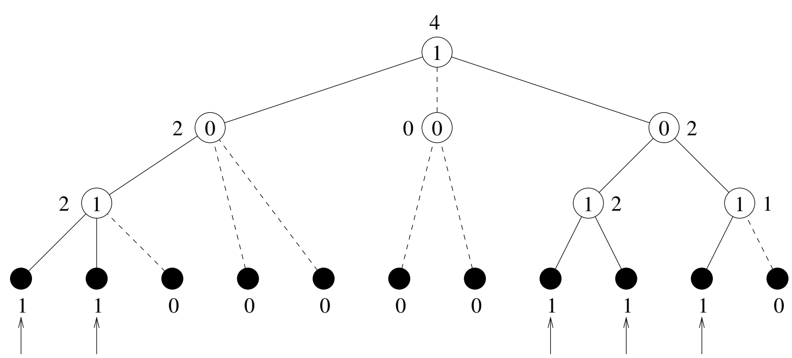

Figure 4.7: CHROMATIC NUMBER on an induced subgraph containing the vertices indicated by the arrows.

Lemma 4.1.10. *Let H be an induced subgraph of a cograph G. Then the output of a natural algorithm for T_H is the same as for T_G if we replace the initial argument of every leaf in T_G that is not in H by the neutral argument.*

Proof. Consider the natural algorithm on T_G, where the leaves not belonging to H have been assigned the neutral argument. By property (iii) of natural algorithms, we can remove all nodes that do not have a leaf of H as a descendent without changing the output of the algorithm. The resulting pseudocotree is a pseudocotree of H. The statement follows from Lemma 4.1.9 □

We now turn to the second type of standard algorithm, the top-down algorithm that traverses the cotree from the root to the leaves. A top-down algorithm can be described as follows.

Algorithm. (generic top-down algorithm)

- INPUT: Pseudocotree T, parameter assigned to each node (for example computed by a bottom-up algorithm).
- INITIALIZATION: Argument assigned to the root of T, depending on the parameter assigned to the root.
- 0-NODE OPERATOR: Formula for calculating an argument for each of the children $A_1, A_2, \ldots, A_t$ of a 0-node A depending on the argument of A and the parameters of $A_1, A_2, \ldots, A_t$.
- 1-NODE OPERATOR: Formula for calculating an argument for each of the children $A_1, A_2, \ldots, A_t$ of a 1-node A depending on the argument of A and the parameters of $A_1, A_2, \ldots, A_t$.
- OUTPUT: Argument of the leaves of T.

Top-down algorithms are suited for example for finding an induced subgraph with certain properties, such as a maximum clique. The following algorithm performs just that.

Algorithm. (MAXIMUM CLIQUE)

- INPUT: Pseudocotree T of a cograph G, χ for each node of T.

- INITIALIZATION: Assign $c(G) = \chi(G)$ to the root of T.
- 0-NODE OPERATOR: Set $c(A_i) \in \mathbb{N}$ such that

$$\begin{aligned} &c(A_j) = c(A) \text{ for some } j \text{ with } \chi(A_j) = \chi(A), \\ &c(A_i) = 0 \text{ for } i \neq j. \end{aligned}$$

- 1-NODE OPERATOR: Set $c(A_i) \in \mathbb{N}$ such that

$$\begin{aligned} \sum_i c(A_i) &= 0, \\ c(A_i) &\leq \chi(A_i). \end{aligned}$$

- OUTPUT: Leaves with $c = 1$ inducing a maximum clique.

Before proving the correctness of this algorithm, we remark that contrary to CHROMATIC NUMBER the equations for the operators of MAXIMUM CLIQUE allow for more than one solution, and different solutions will lead to different maximum cliques.

Proof of Correctness. We need to show that the two operators are well-defined and that the output is indeed a maximum clique. We first observe that by the definition of the operators we have $c(A) \leq \chi(A)$ for all nodes A (in particular $c(A) \leq 1$ for any leaf A). To prove that the operator for 0-nodes always has a feasible solution, we remark that

$$\chi(A) = \max_i \chi(A_i),$$

hence there exists a j such that $\chi(A) = \chi(A_j)$. Similarly for 1-nodes, we have

$$\chi(A) = \sum_i \chi(A_i),$$

which implies that we can always find a feasible solution for the operator for 1-nodes. Since both operators satisfy

$$c(A) = \sum_i c(A_i),$$

we obtain that the sum of the arguments of the leaves (each being 0 or 1) equals the argument of the root $\chi(G)$. Thus the output is a set of $\chi(G)$ leaves. It remains to be shown that they form a clique. This however can easily be seen from the 0-node operator, as all but one child of a 0-node receive an argument of 0. Hence the lowest common ancestor of any two leaves with an argument of 1 cannot be a 0-node, thus the corresponding vertices must be pairwise adjacent. □

Figure 4.8 shows the output of MAXIMUM CLIQUE. The parameter χ is shown to the left of each node (omitted for the leaves) and the argument c to the right of each node in brackets.

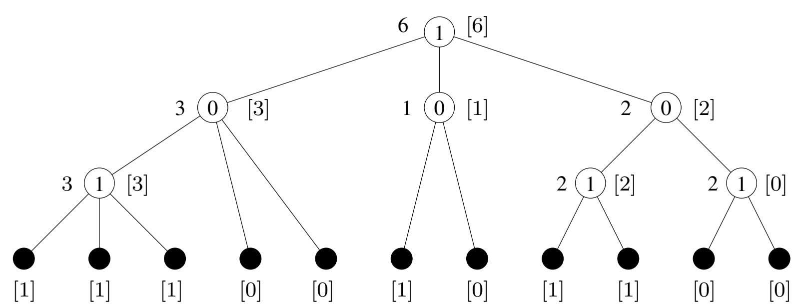

Figure 4.8: MAXIMUM CLIQUE on the cotree from Figure 4.3.

There is an alternative argument showing that this algorithm has the correct output. If we compare MAXIMUM CLIQUE with CHROMATIC NUMBER, we see that the operators of CHROMATIC NUMBER are inverses to those of MAXIMUM CLIQUE. That means that if we initialize CHROMATIC NUMBER with the arguments of the leaves obtained by MAXIMUM CLIQUE, the output will be $c(G) = \chi(G)$. If we let H be the subgraph of G induced by the leaves of T that got assigned a 1, then Lemma 4.1.10 together with the fact that CHROMATIC NUMBER is a natural algorithm with neutral argument 0 implies

$$\chi(H) = \chi(G).$$

Since H has $\chi(G)$ vertices, H must be a clique. While this argument might seem slightly more complicated than the one given in the proof, we will use it later to our benefit.

4.1.3 Cotrees and (k, l)-colouring

We now turn to algorithms concerning (k, l)-colouring of cographs. The first algorithm, similar to CHROMATIC NUMBER, is a bottom-up algorithm that calculates the parameter $\boldsymbol{\kappa}$ for a cograph, relying on the formulas for $\boldsymbol{\kappa}$ given in Corollaries 2.1.11 and 2.1.15. The algorithm is similar in nature to the one presented in [35] for the cochromatic number. However, the presentation is much simpler due to the formulas for $\boldsymbol{\kappa}$ established here.

Algorithm. (KAPPA)

- INPUT: Pseudocotree T of a cograph G.
- INITIALIZATION: Assign $\boldsymbol{\kappa} = (1)$ to each leaf of T.
- 0-NODE OPERATOR: $\boldsymbol{\kappa}(A) = \boldsymbol{\kappa}(A_1) * \boldsymbol{\kappa}(A_2) * \cdots * \boldsymbol{\kappa}(A_t)$.
- 1-NODE OPERATOR: $\boldsymbol{\kappa}(A) = \boldsymbol{\kappa}(A_1) + \boldsymbol{\kappa}(A_2) + \cdots + \boldsymbol{\kappa}(A_t)$.
- OUTPUT: $\boldsymbol{\kappa}(G)$.

Proof of Correctness. The correctness follows directly from the fact that $\boldsymbol{\kappa}(K_1) = K_1$ and from the formulas for $\boldsymbol{\kappa}$ for the disjoint union and join of graphs. □

An example of the output of KAPPA is shown in Figure 4.9, where the arguments of the leaves (all being (1)) have been omitted.

As both operations $*$ and $+$ are associative and commutative, KAPPA is a natural algorithm with the neutral argument being the empty sequence (). As we can easily calculate the cochromatic number and bichromatic number from $\boldsymbol{\kappa}$ (Propositions 2.2.4 and 2.3.10), KAPPA can be seen as an algorithm for the (k, l)-colourability, the cochromatic number and the bichromatic number of a cograph.

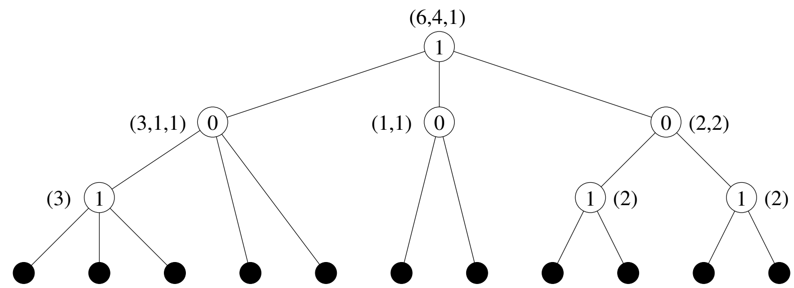

Figure 4.9: KAPPA for the cotree from Figure 4.3.

We will briefly discuss the complexity of KAPPA. Since pseudocotrees can have arbitrarily many vertices, we only consider KAPPA on cotrees. Let n be the number of vertices of G. Then the number of operations ($*$ or $+$) performed by the algorithm is exactly $n-1$. We will show that each operation only needs time $O(n)$. Note that each sequence $\boldsymbol{\kappa}(A)$ has length at most n. To calculate $\boldsymbol{\kappa}(A_1) * \boldsymbol{\kappa}(A_2)$, say, we need to sort the concatenated sequence of $\boldsymbol{\kappa}(A_1)$ and $\boldsymbol{\kappa}(A_2)$. Since both sequences are already sorted, we only need to scan each sequence once. As both sequences have length at most n, this can be done in $O(n)$. To calculate $\boldsymbol{\kappa}(A_1) + \boldsymbol{\kappa}(A_2)$, we need to perform at most n additions, which is also $O(n)$. Thus the algorithm can be implemented in time $O(n^2)$, which matches the complexity of the algorithm from [35]. However, with a bit more care, it is possible to implement KAPPA in such a way that $\boldsymbol{\kappa}(A_1) * \boldsymbol{\kappa}(A_2)$ and $\boldsymbol{\kappa}(A_1) + \boldsymbol{\kappa}(A_2)$ can be calculated in time $O(\min\{|A_1|, |A_2|\})$, which in turn is sufficient to show that the algorithm runs in time $O(n \log n)$, which is faster than the algorithms from [35] and [17] for the cochromatic number and the (k,l)-colourability.

In the same way as the algorithm for finding a maximum clique arose out of the algorithm for the chromatic number, we can use KAPPA to establish a top-down algorithm that finds a certain box cograph. As a reminder, the obstructions for $(k-1, l-1)$-colourability of cographs are precisely the box cographs in $\mathscr{B}(k,l)$, similar to k-cliques being the obstructions for $(k-1)$-colourability. Also recall that $[r]^s$ is the finite sequence consisting of s entries of r.

Algorithm. (BOX COGRAPH)

- INPUT: Pseudocotree T of a cograph G with $\kappa_{l-1}(G) \geq k$ and $\boldsymbol{\kappa}$ for each node of T.
- INITIALIZATION: Assign $c(G) = [k]^l$ to the root of T.
- 0-NODE OPERATOR: For $c(A) = [r]^s$, set $c(A_i)$ such that

$$\begin{aligned} c(A_i) &= [r]^{s_i}, \\ \kappa_{s_i-1}(G_i) &\geq r, \\ s_1 + s_2 + \cdots + s_t &= s. \end{aligned}$$

- 1-NODE OPERATOR: For $c(A) = [r]^s$, set $c(A_i)$ such that

$$\begin{aligned} c(A_i) &= [r_i]^{s}, \\ \kappa_{s-1}(G_i) &\geq r_i, \\ r_1 + r_2 + \cdots + r_t &= r. \end{aligned}$$

- OUTPUT: Leaves with $c = (1)$ inducing a box cograph $H \in \mathscr{B}(k,l)$.

Proof of Correctness. We start by showing that the two operators are well-defined. Let A be a node that got $c(A) = [r]^s$ assigned. By the initialization and the definition of the operators, we know that

$$\kappa_{s-1}(A) \geq r.$$

Suppose A is a 0-node. Then

$$A = A_1 + A_2 + \cdots + A_t$$

and therefore

$$\boldsymbol{\kappa}(A) = \boldsymbol{\kappa}(A_1) * \boldsymbol{\kappa}(A_2) * \cdots * \boldsymbol{\kappa}(A_t).$$

As $\kappa_{s-1}(A) \geq r$ implies that there are at least s entries greater than or equal to r in $\boldsymbol{\kappa}(A)$ and since $\boldsymbol{\kappa}(A)$ arises from the concatenation of $\boldsymbol{\kappa}(A_1), \boldsymbol{\kappa}(A_2), \ldots, \boldsymbol{\kappa}(A_t)$, we know that we can find values $s_1, s_2, \ldots, s_t$ such that $\boldsymbol{\kappa}(A_i)$ contains at least s_i entries greater than or equal to r and $s_1 + s_2 + \cdots + s_t = s$. Therefore the 0-operator is well-defined.

If A is a 1-node, then

$$A = A_1 \vee A_2 \vee \cdots \vee A_t$$

and we have

$$\boldsymbol{\kappa}(A) = \boldsymbol{\kappa}(A_1) + \boldsymbol{\kappa}(A_2) + \cdots + \boldsymbol{\kappa}(A_t),$$

thus

$$\kappa_{s-1}(A) = \kappa_{s-1}(A_1) + \kappa_{s-1}(A_2) + \cdots + \kappa_{s-1}(A_t).$$

Hence we can find values $r_1, r_2, \ldots, r_t$ such that $\kappa_{s-1}(A_i) \geq r_i$ and $r_1+r_2+\cdots+r_t = r$. Therefore the 1-operator is well-defined.

To show that the vertices with $c = (1)$ induce a box cograph in $\mathscr{B}(k, l)$, we first note that for any 0-node, the sum over the entries of $c(A)$ is

$$rs = (r_1 + r_2 + \cdots + r_t)s = r_1 s + r_2 s + \cdots + r_t s,$$

which is the sum over all entries of all $c(A_i)$. The same holds for 1-nodes. Hence the sum over all entries of all sequences assigned to the leaves equals the sum over the entries of the sequence assigned to the root, which is kl. As the only possible assignments to the leaves are (1) and the empty sequence (), we must have kl leaves with (1) assigned to them. By Proposition 3.1.21, it suffices to show that the graph H induced by these leaves satisfies $\boldsymbol{\kappa}(H) = [k]^l$. To do so, we apply KAPPA to T, where we initialize the leaves by the arguments assigned to them by this algorithm. The operators of KAPPA are inverses to the ones of BOX COGRAPH. Therefore the output of KAPPA will be $c(G) = [k]^l$. Since KAPPA is a natural algorithm with neutral element (), Lemma 4.1.10 implies $\boldsymbol{\kappa}(H) = [k]^l$, which proves that $H \in \mathscr{B}(k, l)$. □

An example of the output of BOX COGRAPH is shown in Figure 4.10, where $\boldsymbol{\kappa}$ is shown in round brackets to the left of each node and c in square brackets to the right of each node. For the leaves, $\boldsymbol{\kappa}$ and c have been omitted, except when $c = [1]$.

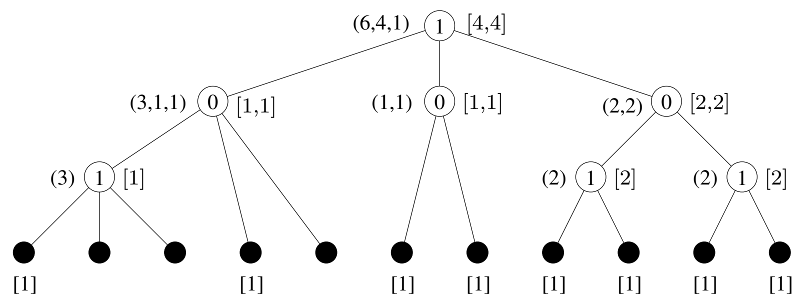

Figure 4.10: BOX COGRAPH with $k = 4$ and $l = 2$ for the cotree from Figure 4.3.

As a final algorithm, we present a bottom-up algorithm, calculating the Young diagram presentation of a cograph.

Algorithm. (YOUNG DIAGRAM)

- INPUT: Pseudocotree T of a cograph G.
- INITIALIZATION: Assign $\mathscr{Y} = \bullet$ to each leaf of T, labelled with the name of the leaf.
- 0-NODE OPERATOR: $\mathscr{Y}(A)$ is the Young diagram representation consisting of the columns of the Young diagram representations $\mathscr{Y}(A_1), \mathscr{Y}(A_2), \ldots, \mathscr{Y}(A_t)$, sorted by size.
- 1-NODE OPERATOR: $\mathscr{Y}(A)$ is the Young diagram representation consisting of the rows of the Young diagram representations $\mathscr{Y}(A_1), \mathscr{Y}(A_2), \ldots, \mathscr{Y}(A_t)$, sorted by size.
- OUTPUT: Young diagram representation of G.

Proof of Correctness. The correctness follows directly from the proof of Proposition 4.1.2. □

As sorting of columns and rows is associative and commutative, YOUNG DIAGRAM is a natural algorithm with neutral argument being the empty Young diagram. An example of an output of YOUNG DIAGRAM is given in Figure 4.11. The cotree

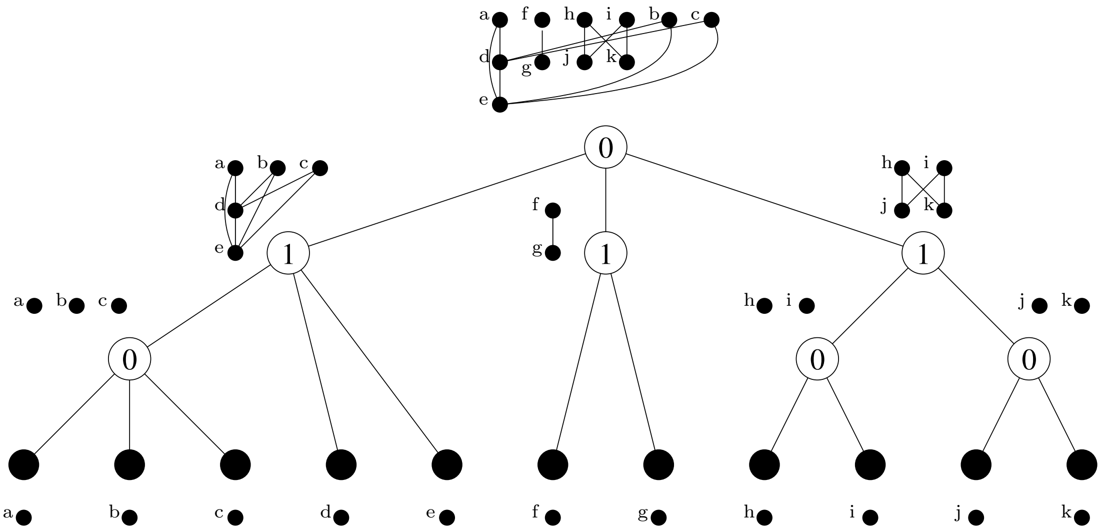

Figure 4.11: YOUNG DIAGRAM for the complement of the cotree from Figure 4.3.

is the one from Figure 4.3 except that the labels of the internal nodes have been swapped for visual reasons (the graph represented by this cotree is the complement of the one from Figure 4.4).

As mentioned, from the Young diagram representation we can easily determine, whether a graph is (k, l)-colourable and find an box cograph obstruction if not. For example, the graph in Figure 4.11 is not $(1, 3)$-colourable. A box cograph in $\mathscr{B}(2, 4)$ is induced by the vertices a, f, h, i, d, g, j, k.

4.2 Chordal graphs

We turn to another interesting graph class, the class of chordal graphs. Chordal graphs have first been introduced in 1958 (though not under that name) by Hajnal and Surányi [38], who showed that the independence number of chordal graphs is equal to their clique covering number. In 1960, Berge [4] extended this and proved that chordal graphs are perfect. Chordal graphs generalize several other types of graphs, such as trees, interval graphs and split graphs (in fact, split graphs are precisely the chordal graphs, whose complement is chordal as well). The (k, l)-colourability of chordal graphs has been investigated by Hell et. al. [41], who gave a complete characterization of (k, l)-colourable chordal graphs via forbidden subgraphs. Presented here is an alternative proof of this characterization

Definition 4.2.1. A graph is *chordal* if and only if it contains no induced cycle of order at least 4.

Theorem 4.2.2. [4] *Chordal graphs are perfect.* □

4.2.1 Perfect elimination ordering and (k, l)-colourings

Definition 4.2.3. Let G be a graph. A *perfect elimination ordering* is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that the set of vertices $N_>(v_i) = \{v_j \mid j > i,\, v_iv_j \in E(G)\}$ forms a clique for each i.

Theorem 4.2.4. [31] *A graph is chordal if and only if it admits a perfect elimination ordering.* □

We remark that the perfect elimination ordering can be used to obtain an optimal colouring and clique covering of a chordal graph. An optimal clique covering can be found using a greedy algorithm on the vertices in perfect elimination order, while an optimal colouring can be obtained by greedily colouring the vertices in reverse perfect elimination order.

The following characterization of (k,l)-colourable chordal graphs has been proven in [41] by two different methods. The first one relies on establishing properties of the adjacency graph of all $(k+1)$-cliques of a chordal graph. The second one is based on a greedy-style algorithm applied to a perfect elimination ordering of a chordal graph. We present here an alternative direct proof of the result, also based on perfect elimination orderings.

Theorem 4.2.5. [41] *Let G be a chordal graph. Then G is (k,l)-colourable if and only if G does not contain $(l+1)K_{k+1}$ as an induced subgraph.*

Proof. We will show that if G is minimally not (k,l)-colourable then G has to be a disjoint union of K_{k+1}'s. As lK_{k+1} is (k,l)-colourable, while $(l+1)K_{k+1}$ is not, the result then follows.

Suppose that G is minimally not (k,l)-colourable. Let $1,2,\ldots,n$ be a perfect elimination ordering of the vertices of G such that the vertices of each component appear consecutively while the order of the components is arbitrary (this is possible as each component is itself a chordal graph and we can combine their perfect elimination orderings). Let v be the first vertex such that the set

$$S := \{w \in V(G) \mid w \leq v\}$$

contains a K_{k+1} (such a vertex exists, as G is perfect and not $(k,0)$-colourable, thus contains K_{k+1}). Note that the graph induced by S is not $(k,0)$-colourable, while the graph induced by $S-\{v\}$ can be $(k,0)$-coloured (since it contains no K_{k+1}). Assume without loss of generality that the graph induced by S is connected, otherwise reorder the perfect elimination ordering by starting with the component of v. Using the perfect elimination ordering, we show that $N_{>}(v)$ is a clique cutset, separating S from $V(G)-S-N_{>}(v)$. Suppose x is the largest vertex in S having a neighbour x' in $V(G)-S-N_{>}(v)$. If x has a larger neighbour in S then by the perfect elimination order this neighbour is adjacent to x' which is a contradiction to the choice of x. Otherwise take the smallest vertex y on a shortest path from x to v.

Both its neighbours on said path are larger than y, thus have to be adjacent by the perfect elimination order, contradicting that the path was a shortest path. Thus $N_>(v)$ separates S from $V(G) - S - N_>(v)$.

Now suppose that either $N_>(v)$ is not empty or $S \neq K_{k+1}$. Then there exists a vertex $u \in N_>(v) \cup (S - \{v\})$ such that the graph induced by $S - \{u\}$ is not $(k, 0)$-colourable. By the minimality of G, the graph $G - u$ is (k, l)-colourable. Since the graph induced by $S - \{u\}$ is not $(k, 0)$-colourable, at least one of its vertices must be covered by a clique in the (k, l)-colouring. The clique can only contain vertices from $S \cup N_>(v) - \{u\}$ since there are no edges between S and $V(G) - S - N_>(v)$. Thus the graph induced by $V(G) - S - N_>(v)$ is $(k, l-1)$-colourable. But then we can construct a (k, l)-colouring of G by adding the clique $\{v\} \cup N_>(v)$ and k-colouring the vertices in $S - v$ (none of those vertices being adjacent to any vertices in $V(G) - S - N_>(v)$), contradicting our assumption. Therefore $S = K_{k+1}$ and S is a component of G.

Hence each component of G that is not equal to K_{k+1} does not contain a K_{k+1}. However, such a component is $(k, 0)$-colourable and can be removed without changing the property of the graph not being (k, l)-colourable, contradicting the minimality of G. □

4.2.2 Bichromatic number of k-trees

It was shown in Proposition 2.3.5 that the bichromatic number of a graph G is bounded:

$$\theta(G) \leq \chi^b(G) \leq \theta(G) + \chi(G) - 1.$$

In Theorem 3.1.22 we gave a complete characterization of the graphs achieving equality in the upper bound. The question arises whether we can find a characterization of graphs satisfying $\chi^b(G) = \theta(G) + r$ for $r < \chi(G) - 1$. We will show that a forbidden subgraph characterization of these graphs does not exist even for the class of k-trees, which are a natural generalization of trees. This class was first introduced by Harary and Palmer [40].

Definition 4.2.6. The class of *k-trees* is recursively defined as follows:

(i) K_{k+1} is a k-tree;

(ii) If T is a k-tree and C is a k-clique of T, then the graph arising from T by adding a vertex v with $N(v) = C$ is a k-tree.

It is easy to see that every k-tree is a chordal graph having chromatic number $k + 1$ (note that the converse is false).

Proposition 4.2.7. *Let T be a k-tree. Then T is chordal and $\chi(T) = k + 1$.* □

Proposition 4.2.8. *Let T be a k-tree. Then*

(i) $\theta(T) \leq \chi^b(T) \leq \theta(T) + k$*;*

(ii) $\chi^b(T) = \theta(T) + k \Leftrightarrow T = K_{k+1}$.

Proof. Since $\chi(T) = k + 1$, (i) follows directly from Proposition 2.3.5. For (ii), Proposition 3.1.22 shows that $\chi^b(T) = \theta(T) + k$ is equivalent to T being a box cograph. The only chordal box cographs are disjoint union of cliques and the only connected chordal box cographs are thus cliques, which proves the statement. □

We will need a special type of k-tree for the later proofs.

Definition 4.2.9. A *balanced k-path* of order n ($n \geq k + 1$) is a graph on vertices $v_1, v_2, \ldots, v_n$, such that v_i is adjacent to v_j if and only if $|i - j| \leq k$.

We remark that a balanced k-path or order n is equivalent to P_n^k, the k-th power of the path P_n.

An example of a balanced 2-path (equivalent to P_8^2) is given in Figure 4.12.

Lemma 4.2.10. *Let P be a balanced k-path of order n. Then*

(i) P is a k-tree;

(ii) if $n \geq (k + 1)(s + k)$ with $1 \leq s \leq k$, then P is $(s, \theta(P) - s)$-colourable.

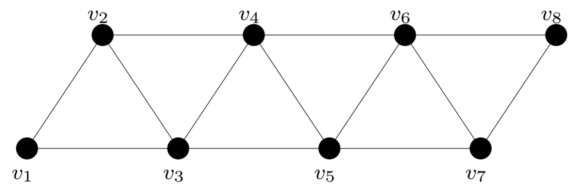

Figure 4.12: A balanced 2-path.

Proof. Let P be a balanced k-path of order n. If $n = k+1$, then P is a clique and thus a k-tree. Otherwise assume by induction that the balanced k-path of order $n-1$ on the vertices $v_1, v_2, \ldots, v_{n-1}$ is a k-tree. Since $N(v_n) = \{v_{n-k}, \ldots, v_{n-2}, v_{n-1}\}$ forms a clique, adding v_n produces a k-tree again.

Suppose $n \geq (k+1)(s+k)$. Using the greedy algorithm on the perfect elimination order $v_1, v_2, \ldots, v_n$, we find an optimal clique covering of P where the vertices $v_1, v_2, \ldots, v_{(k+1)(s+k)}$ are covered by $s+k$ cliques. We will replace these cliques by an (s,k)-colouring on those vertices, thus obtaining an $(s, \theta(P) - s)$-colouring of P. For that purpose we define independent sets $S_1, S_2, \ldots, S_s$ by

$$v_i \in S_j \Leftrightarrow i \equiv j \pmod{s+k+1}$$

and cliques $C_1, C_2, \ldots, C_k$ by

$$v_i \in C_j \Leftrightarrow j(s+k+1) - k \leq i \leq j(s+k+1).$$

Every vertex $v_1, v_2, \ldots, v_{(k+1)(s+k)}$ is contained in precisely one of the independent sets or cliques. Thus $S_1, S_2, \ldots, S_s, C_1, C_2, \ldots, C_k$ is an (s,k)-colouring of the subgraph induced by these vertices, which completes the proof. □

Proposition 4.2.11. *For every k-tree T there exists a k-tree T', such that T is an induced subgraph of T' and $\chi^b(T') = \theta(T')$.*

Proof. Let T be a k-tree and let $u_1, u_2, \ldots, u_k$ be the vertices of a k-clique in T. Let T' be the k-tree arising from T by appending vertices $v_1, v_2, \ldots, v_{2k(k+1)}$, such that $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{2k(k+1)}$ forms a balanced k-path. Using the greedy algorithm on a perfect elimination order starting with the appended vertices, we obtain an optimal clique covering containing $2k$ cliques covering precisely the appended vertices, thus yielding

$$\theta(T') = \theta(T) + 2k > k + 1.$$

By Lemma 4.2.10, the appended path is $(s, 2k - s)$-colourable for all $1 \leq s \leq k$, as $(k+1)(s+k) \leq 2k(k+1)$. Extending this colouring to T' by using the optimal clique covering of T, we find that T' is $(s, \theta(T') - s)$-colourable for all s (where the case $s \geq k + 1$ follows from the $(k + 1)$-colourability of k-trees), thus $\chi^b(T') = \theta(T')$. □

Proposition 4.2.12. *For every k-tree T there exists a k-tree T', such that T is an induced subgraph of T' and $\chi^b(T') = \theta(T') + k - 1$.*

Proof. Let T be a k-tree. Construct T' as follows. For every vertex $v \in V(T)$ take a clique of order $k + 1$ in T containing v. Let the vertices in said clique be $v_1, v_2, \ldots, v_{k+1} = v$. Now append vertices $v_{k+2}, v_{k+3}, \ldots, v_{2k+1}$, such that the vertices $v_1, v_2, \ldots, v_{2k+1}$ form a balanced k-path. Let the resulting graph be T'. The clique covering number of T' is $|V(T)|$, as the endvertices of the appended paths form an independent set of order $|V(T)|$ and T' can be covered by $|V(T)|$ cliques by choosing

$$\{v_{k+1}, v_{k+2}, \ldots, v_{2k+1}\}, \quad (v \in V(T))$$

as cliques, each clique covering one appended path and one vertex of T. However, T' is not $(k - 1, |V(T)| - 1)$-colourable, as the set

$$\{\{v_{k+2}, v_{k+3}, \ldots, v_{2k+1}\} \mid v \in V(T)\}$$

forms an induced $|V(T)|\, K_k$ (Theorem 4.2.5). Therefore we have

$$\chi^b(T') \geq |V(T)| + k - 1 = \theta(T') + k - 1.$$

Since T' is by construction not a clique, Proposition 4.2.8 implies equality in the equation. □

Proposition 4.2.13. *For every k-tree T and every integer $0 \leq r \leq k-1$, there exists a k-tree T_r, such that T is an induced subgraph of T_r and $\chi^b(T_r) = \theta(T_r) + r$.*

Proof. Setting $f(G) = \chi^b(G) - \theta(G)$ for any graph G, we need to show that there exists a k-tree T_r, containing T as an induced subgraph and with $f(T_r) = r$, for all integers $0 \leq r \leq k-1$. Consider the parameter f. For any vertex $v \in V(G)$, we obtain

$$|f(G) - f(G-v)| = \left|(\chi^b(G) - \theta(G)) - (\chi^b(G-v) - \theta(G-v))\right| \leq 1,$$

since $(\chi^b(G) - \chi^b(G-v)), (\theta(G) - \theta(G-v)) \in \{0, 1\}$. By Proposition 4.2.11, there exists a k-tree T' containing T, for which $f(T) = 0$. By successively removing vertices from T' until we get to T, we find trees T_r containing T as an induced subgraph with $f(T_r) = r$ for all values of r between 0 and $f(T)$. Similarly, using Proposition 4.2.12, we can find trees T_r containing T as an induced subgraph and with $f(T_r) = r$ for all values of r between $f(T)$ and $k-1$. □

4.3 Round digraphs (proper circular arc graphs)

In Sections 4.1 and 4.2 we have studied (k,l)-colourings of cographs and chordal graphs. Both cographs and chordal graphs are perfect graphs. In this section we will consider a class of graphs which are not perfect in general, but are in a sense close to being perfect. This class of graphs is best described using the terminology of digraphs. It should be kept in mind, however, that we are interested in the underlying undirected graph. Unless explicitly referring to properties of the digraph, any (k,l)-colourings of digraphs in this section are to be understood as (k,l)-colourings of the underlying undirected graph.

Definition 4.3.1. A *round digraph* is a simple loopless digraph D on a set of circularly enumerated vertices $v_0, v_1, \ldots, v_{n-1}$ such that for all $0 \leq i \leq n-1$

$$v_i v_{i+h} \in A(D) \implies v_i v_{i+j}, v_{i+j} v_{i+h} \in A(D) \qquad 1 \leq j < h < n,$$

where all indices are taken modulo n. The circular enumeration $v_0, v_1, \ldots, v_{n-1}$ will be referred to as a *round enumeration* of D.

Remark. For this section, any indices of vertices are understood to be taken modulo n. Furthermore, while there might be more than one round enumeration for a given round digraph, when speaking about a round digraph D, we assume that it comes with a round enumeration $v_0, v_1, \ldots, v_{n-1}$ of the vertices and we will refer to this round enumeration.

By the definition, every induced subgraph D' of a round digraph D is itself a round digraph, where the round enumeration of D' is obtained by enumerating the vertices of D' in the same order as they appear in the round enumeration of D.

An example of a round digraph is shown in Figure 4.13. The definition of round digraphs implies that if $v_i v_{i+h} \in A(D)$, then the set of vertices $\{v_i, v_{i+1}, \ldots, v_{i+h}\}$ forms a clique. In particular, the closed outneighbourhood of any vertex v_i is

$$N^+[v_i] = \left\{v_i, v_{i+1}, \ldots, v_{i+d^+(v_i)}\right\},$$

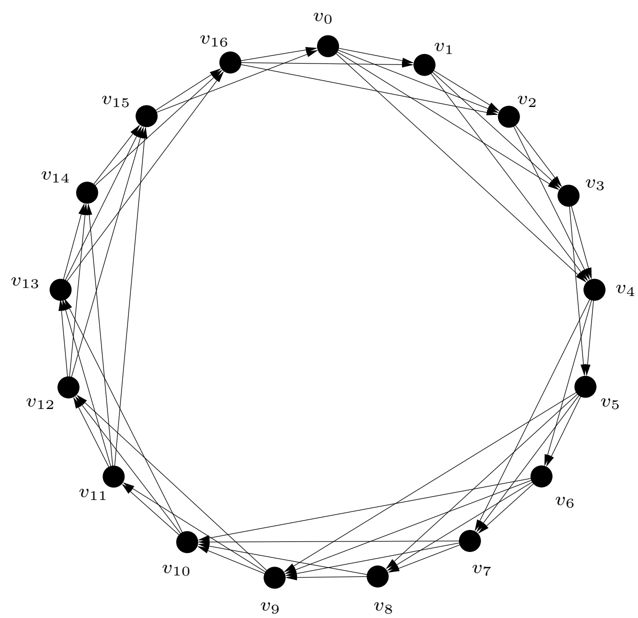

Figure 4.13: A round digraph.

where $d^+(v_i)$ is the outdegree of v_i, and forms a clique. Similarly, with $d^-(v_i)$ being the indegree of v_i, the closed inneighbourhood of v_i is

$$N^-[v_i] = \left\{v_{i-d^-(v_i)}, \ldots, v_{i-1}, v_i\right\}.$$

It has been shown in [5] that the underlying graphs of round digraphs are precisely the intersection graphs of inclusion-free families of circular arcs on a fixed circle. These graphs are also called *proper circular arc graphs* [57].

Let us briefly consider the special case of round digraphs with a vertex of outdegree 0. Suppose without loss of generality that v_{n-1} has outdegree 0. For any arc $v_iv_j \in A(D)$, we must have $i < j$, since if $j > i$, then the definition of round digraphs implies $v_jv_0 \in A(D)$, which in turn implies $v_{n-1}v_0 \in A(D)$. Therefore the linear ordering $v_0, v_1, \ldots, v_{n-1}$ is a perfect elimination ordering of the vertices (see Subsection 4.2.1) of the underlying graph of D, implying that the underlying graph of D is a chordal graph. In fact, these graphs are precisely the intersection graphs of inclusion-free families of intervals, called *proper interval graphs* [58].

For notational convenience, we make the following definition.

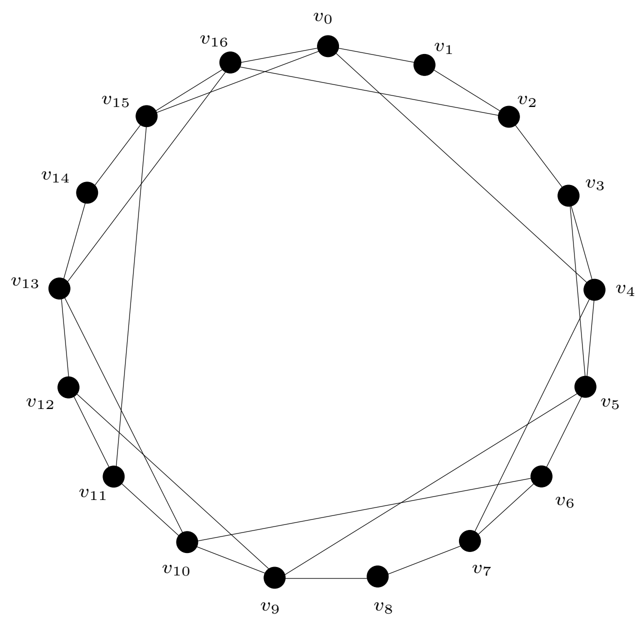

Figure 4.14: The essential arcs of the round digraph from Figure 4.13.

Definition 4.3.2. Let D be a round digraph. For any two vertices v_i, v_j, the closed interval $[v_i, v_j]$ is defined as the set of vertices $\{v_i, v_{i+1}, \ldots, v_j\}$. Open and half-open intervals are defined analogously.

With this notation, the closed outneighbourhood of a vertex v_i can be written as $N^+[v_i] = [v_i, v_{i+d^+(v_i)}]$ and the inneighbourhood as $N^-[v_i] = [v_{i-d^-(v_i)}, v_i]$.

We note that to define the arc set of a round digraph, it is sufficient to provide a subset of the arc set. For example, consider the arc v_5v_9 in Figure 4.13. Knowing that v_5v_9 is an arc, we can use the definition of round digraphs to conclude that, for example, v_5v_7 is an arc. Thus the arc v_5v_7 can be seen to be implied by the arc v_5v_9. However, we cannot find any arc that implies v_5v_9. We want to give this type of arcs a name.

Definition 4.3.3. Let D be a round digraph. An arc $v_iv_j \in A(D)$ is *essential* if $v_{i-1}v_j, v_iv_{j+1} \notin A(D)$.

We note that if v_iv_j is an essential arc, then $j = i + d^+(v_i)$ by the condition $v_iv_{j+1} \notin A(D)$. The essential arcs of the round digraph from Figure 4.13 can be seen in Figure 4.14, where the arcs between consecutive vertices, not being essential, are included for purely optical reasons. The orientation of each arc is omitted for the sake of simplicity, but assumed to be clockwise, here and in the following figures.

Clearly the set of essential arcs of a round digraph completely defines the arc set by implication (using the definition of round digraphs). We also note that any round digraph is determined by the list of the outdegrees $d^+(v_0), d^+(v_1), \ldots, d^+(v_{n-1})$, since the outneighbours of each vertex are the direct successors of that vertex in the round enumeration. We can make an observation about the outdegrees of consecutive vertices. Consider two consecutive vertices v_i and v_{i+1}. Since $N^+[v_i] = [v_i, v_{i+d^+(v_i)}]$ is a clique, we obtain that $v_{i+1}v_{i+d^+(v_i)} \in A(D)$ and thus $d^+(v_{i+1}) \geq d^+(v_i) - 1$. So the outdegree can go down by at most 1 between two consecutive vertices. For later, we give the vertices where the outdegree does not go down a name.

Definition 4.3.4. Let D be a round digraph. An *essential vertex* is a vertex v_i with $d^+(v_i) \leq d^+(v_{i+1})$.

We can relate essential vertices to essential arcs.

Proposition 4.3.5. *Let v_i be a vertex of a round digraph. Then v_i is essential if and only if the arc $v_{i+1}v_{i+1+d^+(v_{i+1})}$ is essential.*

Proof. Set $j = i+1+d^+(v_{i+1})$. Suppose $v_{i+1}v_j$ is an essential arc. Then $v_iv_j \notin A(D)$, and therefore $d^+(v_i) \leq d^+(v_{i+1})$, showing that v_i is an essential vertex. On the other hand, suppose v_i is an essential vertex. Then $i + d^+(v_i) \leq i + d^+(v_{i+1}) < j$, implying $v_iv_j \notin A(D)$. Since $v_{v+1}v_{j+1} \notin A(D)$ by definition of j, the arc $v_{i+1}v_j$ is essential. □

4.3.1 (k, l)-colourings with $l \geq 1$

Before considering (k, l)-colourings of round digraphs we briefly discuss cliques in round digraphs. There are two different types of cliques that can occur. The first type is the *transitive clique*, which consists of a subset of $N^+[v]$ for some vertex v. In particular, the maximal transitive cliques (which are also called *overlap cliques*

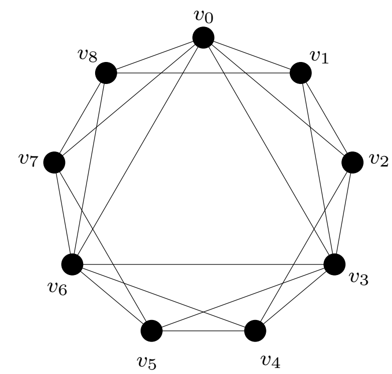

Figure 4.15: The clique consisting of v_0, v_3, v_6 is maximal but not transitive.

[48]) are all of the form $N^+[v]$. Note that $N^+[v]$ does not necessarily have to be a maximal clique. Now consider what happens if we remove the closed outneighbourhood of a vertex from a round digraph. Consider the vertex v_{i-1} in $D - N^+[v_i]$. All outneighbours of v_{i-1} in D are contained in $N^+[v_i]$, therefore the outdegree of v_{i-1} in $D - N^+[v_i]$ is 0, implying that the underlying graph of $D - N^+[v_i]$ is a proper interval graph. This property will be instrumental in finding (k, l)-colourings of round digraphs.

However, not all cliques in round digraphs need be transitive. For instance, the clique $\{v_0, v_3, v_6\}$ in Figure 4.15 is a clique that is not transitive. We note that if we remove that clique, we do not obtain a proper interval graph. Fortunately, we can neglect nontransitive cliques for (k, l)-colourings if $l \geq 2$, as the following lemma shows.

Lemma 4.3.6. *Let D be a round digraph and U be a set of vertices in D that can be covered by two cliques. Then there exist two transitive cliques in D covering U.*

Proof. Since the graph induced by U is a round digraph as well, we may assume without loss of generality that $U = V(D)$. Consider a clique covering C_1, C_2 of D. If both C_1 and C_2 are transitive cliques, we are done. Suppose at least one clique, say C_1, is not transitive. Let v_r, v_s, v_t be vertices in C_1 forming a directed 3-cycle (in that order). Now let v_i be the last vertex in $[v_r, v_{s-1}]$ such that v_i is in C_1 and $v_t v_i$ is an arc in D. Consider v_{i+1}. If $v_{i+1}v_t$ is an arc, then the two transitive cliques $[v_t, v_i]$ and $[v_{i+1}, v_{t-1}]$ cover the vertex set of D. Otherwise v_{i+1} is not adjacent to v_t and must therefore be in C_2. In this case, let v_j be the last vertex in $[v_s, v_{t-1}]$ such that $v_{i+1}v_j$ is an arc (this is well defined, since $v_{i+1}v_s$ is an arc as v_rv_s is an arc and v_{i+1} is between v_r and v_s). Consider v_{j+1}. By the definition of v_j, we know that $v_{i+1}v_{j+1}$ cannot be an arc and therefore neither can v_iv_{j+1}. Thus, if v_{j+1} is in C_1 we must have that $v_{j+1}v_i$ is an arc and if v_{j+1} is in C_2 we must have that $v_{j+1}v_{i+1}$ is an arc (and therefore $v_{j+1}v_i$ as well). In either case, the two transitive cliques $[v_{i+1}, v_j]$ and $[v_{j+1}, v_i]$ cover the vertex set of D. □

Lemma 4.3.7. *Let D be a round digraph that is (k,l)-colourable, where $l \geq 2$. Then there exists a (k,l)-colouring of D, where all cliques are transitive.*

Proof. Given a (k,l)-colouring of D that contains a clique that is not transitive, set U to be the vertex set consisting of that clique and another clique from the (k,l)-colouring. By Lemma 4.3.6, we can cover U by two transitive cliques and thus obtain a (k,l)-colouring of D with more transitive cliques. Repeating this procedure proves the claim. □

Definition 4.3.8. For any round digraph D and natural number k, let $\lambda_k^T(D)$ be the minimum l such that there exists a (k,l)-colouring of D, where all cliques are transitive.

Lemma 4.3.9. *For any round digraph D,*

(i) $\lambda_k^T(D) \neq 2 \implies \lambda_k(D) = \lambda_k^T(D)$;

(ii) $\lambda_k^T(D) = 2 \implies \lambda_k(D) \in \{1, 2\}$.

Proof. By definition $\lambda_k(D) = 0$ if and only if $\lambda_k^T(D) = 0$. If $\lambda_k(D) \geq 2$, then by Lemma 4.3.7, $\lambda_k(D) = \lambda_k^T(D)$. Finally, if $\lambda_k(D) = 1$, then D is $(k,1)$-colourable, therefore $(k,2)$-colourable. Thus Lemma 4.3.7 implies $\lambda_k^T(D) \leq 2$. □

We see that the only possible case in which $\lambda_k(D)$ and $\lambda_k^T(D)$ can differ is if $\lambda_k(D) = 1$ and $\lambda_k^T(D) = 2$. Figure 4.15 shows such a round digraph for $k = 2$. The round digraph is $(2,1)$-colourable, but the clique in a $(2,1)$-colouring has to be the nontransitive clique $\{v_0, v_3, v_6\}$.

On the basis of Lemma 4.3.9, we now present an algorithm that calculates $\lambda_k^T(D)$ for any round digraph D that is not k-colourable. We will establish the algorithm in a series of Lemmas.

Definition 4.3.10. Let D be a round digraph that is not k-colourable. For any vertex $v_i \in V(D)$, we let $F(v_i)$ be the first vertex in $D - N^+[v_i]$ with respect to the vertex ordering $v_{i+d^+(v_i)+1}, \ldots, v_{i-2}, v_{i-1}$ such that the indegree in $D - N^+[v_i]$ is at least k. If there exists no such vertex, set $F(v_i) = v_i$.

The reasoning behind this definition becomes more clear, when considering a (k,l)-colouring of D containing $N^+[v_i]$ as a clique. The digraph $D - N^+[v_i]$ is, as shown earlier, a chordal graph with perfect elimination ordering $v_{i+d^+(v_i)+1}, v_{i+d^+(v_i)+2}, \ldots, v_{i-1}$. By definition, $F(v)$ is the first vertex in this ordering such that the digraph induced by the closed interval $[v_{i+d^+(v_i)+1}, F(v)]$ contains a K_{k+1}, thus is not k-colourable. Hence every $(k, l-1)$-colouring of $D - N^+[v_i]$ must contain a vertex from this interval. We can generalize this notion.

Lemma 4.3.11. *Let D be a round digraph and $W = \{w_1, w_2, \ldots, w_l\}$ be a set of vertices, listed in the order of the round enumeration of D. Then*

$$D - \bigcup_{i=1}^{l} N^+[w_i]$$

is k-colourable if and only if $w_{i+1} \in (w_i, F(w_i)]$ for all $1 \le i \le l$.

Proof. Denote $H = D - \bigcup_{i=1}^{l} N^+[w_i]$. The l cliques $N^+[w_1], \ldots, N^+[w_l]$ separate the vertices of H into l (possibly empty) sets of consecutive vertices $[w_i, w_{i+1}] \setminus N^+[w_i]$ $(1 \le i \le l)$. Thus H is k-colourable if and only if all the graphs induced by one of those sets are k-colourable. By the definition of $F(w_i)$, the graph induced by $[w_i, w_{i+1}] \setminus N^+[w_i]$ is k-colourable if and only if $w_{i+1} \in (w_i, F(w_i)]$. $\square$

Lemma 4.3.12. *Let D be a round digraph and $v, w \in V(D)$ such that $w \in [v, F(v)]$. Then $F(w) \in [F(v), F^2(v)]$.*

Proof. We start by showing that $F(v) \in [w, F(w)]$. To do so, we need to show that the graph induced by the vertex set $[w, F(v)] \setminus N^+[w]$ is k-colourable. As the graph induced by the vertex set $[v, F(v)] \setminus N^+[v]$ is k-colourable by the definition of $F(v)$, it suffices to show

$$[w, F(v)] \setminus N^+[w] \subseteq [w, F(v)] \setminus N^+[v] \subseteq [v, F(v)] \setminus N^+[v].$$

The second containment clearly holds, since $w \in [v, F(v)]$. As for the first, suppose to the contrary that there is a vertex $u \in [w, F(v)] \setminus N^+[w]$ such that $u \notin [w, F(v)] \setminus N^+[v]$. This implies that $w \in [v, u]$ and that vu is an arc but wu is not, contradicting the definition of round digraphs. By repeating the same argument for w and $F(v)$ instead of v and w, we can show that $F(w) \in [F(v), F^2(v)]$. □

Definition 4.3.13. Let D be a round digraph. For any vertex $v \in V(D)$, define

$$l_v = \min \left\{ i \geq 1 \;\middle|\; v \in (F^{i-1}(v), F^i(v)] \right\}$$

and

$$W^v = \left\{ v, F(v), F^2(v), \ldots, F^{l_v - 1}(v) \right\}.$$

Proposition 4.3.14. *Let D be a round digraph that is not k-colourable. Then*

$$\lambda_k^T(D) = \min_{v \in V(D)} l_v.$$

Proof. Suppose there exists a (k, l)-colouring of D where all cliques are transitive. By adding vertices to the cliques, if necessary, we can then find a (k, l)-colouring of D such that all cliques are the closed outneighbourhoods of vertices. Therefore, by Lemma 4.3.11, $\lambda_k^T(D)$ is equal to the minimum size of a set $W = \{w_1, \ldots, w_l\}$ with $w_{i+1} \in (w_i, F(w_i)]$ for all $1 \leq i \leq l$. Call a set with that property *feasible*. Suppose W is a feasible set of minimum size. Consider w_2. If $w_3 \in (w_1, F(w_1)]$, then $W \setminus \{w_2\}$ is a smaller feasible set, a contradiction. If $w_3 \notin (w_1, F(w_1)]$, then let W' be the set arising from W by replacing w_2 by $F(w_1)$. To show that W' is feasible, it suffices to show that $w_3 \in (F(w_1), F^2(w_1)]$. By assumption, we have $w_3 \in (w_2, F(w_2)]$. Since

$$w_2 \in (w_1, F(w_1)], \qquad F(w_2) \in [F(w_1), F^2(w_1)]$$

by assumption and Lemma 4.3.12, we obtain $w_3 \in (w_1, F^2(w_1)]$. Combining this with $w_3 \notin (w_1, F(w_1)]$, we conclude

$$w_3 \in (F(w_1), F^2(w_1)].$$

Therefore W' is a feasible set as well. Repeating this procedure for $w_3, \dots, w_l$ and setting $v = w_1$, we obtain the feasible set W^v. Thus, to find $\lambda_k^T(D)$, we only need to find the minimum size over the sets W^v, which proves the statement. □

Proposition 4.3.14 allows us to construct a polynomial time algorithm to determine whether a round digraph is (k, l)-colourable if $l \geq 2$. All we need is to calculate l_v for every vertex v (which can be done in polynomial time, since $F(v)$ can be calculated in polynomial time). By Lemma 4.3.9, D is (k, l)-colourable if and only if $l \geq \min_{v \in V(D)} l_v$. Two remarks are in order. First, we can still use this algorithm if D is k-colourable, since in that case $F(v) = v$ for every vertex v and therefore $\min_{v \in V(D)} l_v = 1 \leq l$. Second, if we want to find out whether D is $(k, 1)$-colourable, we can attempt the algorithm. Unless the result is $\min_{v \in V(D)} l_v = 2$, the algorithm still returns the correct result. However, that one case is not resolved. As an example, consider Figure 4.15. The algorithm returns $\lambda_k^T = 2$, whereas $\lambda_k = 1$.

To calculate $\lambda_k^T(D)$ from Proposition 4.3.14, we need to calculate l_v for every vertex v. With just a bit more effort, we can improve on Proposition 4.3.14, and show that it is sufficient to calculate l_v for only one vertex.

Lemma 4.3.15. *Let D be a round digraph and $v \in V(D)$. Then $l_v \geq l_{F(v)}$.*

Proof. By the definition of l_v, we have $v \in (F^{l_v - 1}(v), F^{l_v}(v)]$. By Lemma 4.3.12, we then have $F(v) \in (F^{l_v}(v), F^{l_v+1}(v)]$. Therefore either

$$W^{F(v)} = \left\{F(v), F^2(v), \dots, F^{l_v - 1}(v)\right\}$$

or

$$W^{F(v)} = \left\{F(v), F^2(v), \ldots, F^{l_v}(v)\right\}.$$

Thus $l_{F(v)} \in \{l_v - 1, l_v\}$. □

Proposition 4.3.16. *Let D be a round digraph that is not k-colourable and $v \in V(D)$. If w is the first repeating vertex in $v, F(v), F^2(v), \ldots,$ then $\lambda_k^T(D) = l_w$.*

Proof. By Lemma 4.3.15, the sequence $l_v, l_{F(v)}, l_{F^2(v)}, \ldots$ is nonincreasing, thus becomes stationary, once l_w has been reached. By Proposition 4.3.14, there exists a $u \in V(D)$ such that $\lambda_k^T(D) = l_u$. Again by Lemma 4.3.15, the sequence $l_u, l_{F(u)}, l_{F^2(u)}, \ldots$ is stationary. Let $x \in \{w, F(w), \ldots\}$ and $y \in \{u, F(u), \ldots\}$ such that the distance from y to x in the circular order is minimal. If $x = y$, then $l_w = l_x = l_y = l_u$ and we are done. Otherwise we have $x \in (y, F(y))$, and by repeatedly applying Lemma 4.3.12, that

$$F^{l_y-2}(x) \in (F^{l_y-2}(y), F^{l_y-1}(y))$$

and

$$F^{l_y}(x) \in (F^{l_y}(y), F^{l_y+1}(y)).$$

Since $y \in (F^{l_y-1}(y), F^{l_y}(y)]$, we obtain from the minimality that $x \in (y, F^{l_y}(x)]$ and therefore $x \in (F^{l_y-2}(x), F^{l_y}(x)]$. Thus $l_x \leq l_y$, implying $l_w \leq l_u$, proving the statement. □

We remark that the vertex w from Proposition 4.3.16 can help us find a $(k, \lambda_k^T(D))$-colouring of a round digraph D. If we take as cliques the outneighbourhoods of the vertices in W^w, then the remaining vertices are k-colourable by Lemma 4.3.11.

4.3.2 (k, l)-colourings with $l = 0$

The case of $(k, 0)$-colourings (or k-colourings) of round digraphs is more complicated. Several polynomial algorithms for k-colourings have been found [48, 56, 5]. We will give an algebraic approach that ultimately leads to a characterization of k-colourable round digraphs in the form of a set of configurations, from which all maximal k-colourable round digraphs on n vertices can be obtained (Theorem 4.3.32). As intermediate steps toward that goal, several equivalent conditions to k-colourability will be developed in this subsection. We will assume that the maximum outdegree is at most k, as otherwise the digraph contains a $(k+1)$-clique and is clearly not k-colourable.

We will need some basic terminology about permutations. Therefore we give a brief review here. A *permutation* of the set $\{0, 1, \ldots, k-1\}$ is a bijective function from $\{0, 1, \ldots, k-1\}$ to itself. The set of all permutations of $\{0, 1, \ldots, k-1\}$ forms a group under composition (from the right), the *symmetric group* S_k. We will use two notations for permutations. The first is the list notation, writing a permutation σ as a sequence $\sigma(0)\sigma(1)\ldots\sigma(k-1)$ (without commas). For example, the list 2130 stands for the permutation σ with $\sigma(0) = 2$, $\sigma(1) = 1$, $\sigma(2) = 3$ and $\sigma(3) = 0$. The second is the cycle notation, presenting a permutation according to its orbits. For example, $(0, 2, 3)(1)$ is the cycle notation of 2130. A *fixed point* of a permutation σ is an element $i \in \{0, 1, \ldots, k-1\}$ with $\sigma(i) = i$. For the sake of simplicity, we omit any fixed points (cycles of length one) in cycle notation. For $A \subseteq \{0, 1, \ldots, k-1\}$, the *permutation group of* A is the subgroup of S_k containing all permutations with the property that all elements of $\{0, 1, \ldots, k-1\} \setminus A$ are fixed points. An *inversion* of a permutation σ is a pair of elements $i, j \in \{0, 1, \ldots, k-1\}$ with $i < j$ and $\sigma(i) > \sigma(j)$. A *transposition* is a permutation with cycle notation (i, j) for some i, j. The identity permutation is denoted by id and the permutation $(0, 1, \ldots, k-1)$ by z. We also call z the *elementary cycle*.

In the following, the addition performed on the elements of the set $\{0, 1, \ldots, k-1\}$ is understood to be modulo k, that is, the set $\{0, 1, \ldots, k-1\}$ can be seen as the base set of the cyclic group $\mathbb{Z}_k$, generated by z. Recall that addition performed on the vertex indices $0, 1, \ldots, n-1$ is modulo n.

First, we are going to establish a permutation valued function on the vertices such that the permutation associated to every vertex has encoded a colouring of the vertex and its outneighbours. To that purpose, consider a colouring of the round

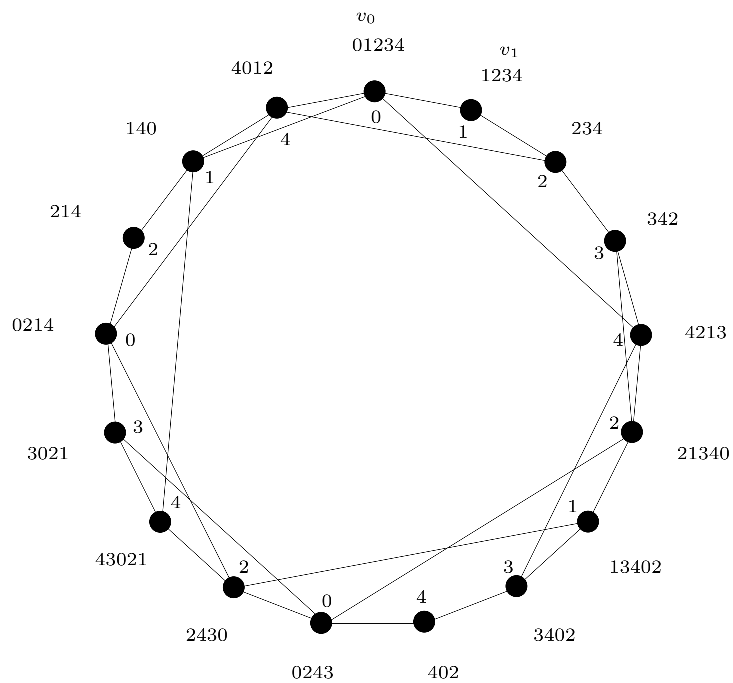

Figure 4.16: A 5-colouring of a round digraph given by its essential arcs.

digraph from earlier, shown on the inside of the circle in Figure 4.16. For every vertex we produce a list of the colours of the vertex and its outneighbours (shown on the outside). We notice that all the elements of the list are distinct, as every vertex with its outneighbours forms a clique in the digraph. In addition, the lists of two consecutive vertices are related, as the list for the second vertex starts with the same elements as the list of the first vertex if we delete the first entry. Lastly every list has length at most 5, since the outdegree is at most 4. Therefore we can complete each such list to a permutation of S_5. We will formalize these notions.

Definition 4.3.17. A *consecutive k-colour function* of D is a permutation-valued function

$$\begin{aligned} g : V(D) &\to S_k \\ v_i &\mapsto g_i. \end{aligned}$$

with the property

$$g_{i+1}(r-1) = g_i(r), \qquad \text{if } 1 \leq r \leq d^+(v_i).$$

Proposition 4.3.18. *D is k-colourable if and only if D admits a consecutive k-colour function.*

Proof. Suppose g is a consecutive k-colour function of D. We will define a colouring on the vertices of D by colouring each vertex v_i with $g_i(0)$. We only need to check that two adjacent vertices receive distinct colours. Assume $v_i v_{i+r}$ is an arc in D, i.e., $r \leq d^+(v_i)$. Then we have, by definition,

$$g_{i+r}(0) = g_{i+r-1}(1) = \cdots = g_i(r) \neq g_i(0).$$

Thus none of the outneighbours of v_i receive the same colour and the colouring is proper.

Now suppose we have given a proper colouring $c : V(D) \to \{0, 1, \ldots, k-1\}$. Then we define a function $g : V(D) \to S_k$ partly by

$$g_i(r) = c(v_{i+r}), \qquad \text{if } r \leq d^+(v_i),$$

every value $g_i(r)$ being distinct for fixed i, since c is a proper colouring. The remaining values of g_i we fill up in an arbitrary way that produces a permutation in S_k. It remains to check whether g satisfies the condition for a consecutive k-colour function. Let $1 \leq r \leq d^+(v_i)$. Then $r \leq d^+(v_{i+1}) + 1$, and we obtain

$$g_{i+1}(r-1) = c(v_{i+1+r-1}) = c(v_{i+r}) = g_i(r). \qquad \square$$

By definition, the two permutations g_i, g_{i+1} of a consecutive k-colour function are closely related for any i. The sequence

$$g_i(1), g_i(2), \ldots, g_i(d^+(v_i))$$

equals the sequence

$$g_{i+1}(0), g_{i+1}(1), \ldots, g_{i+1}(d^+(v_i) - 1).$$

In particular, if $d^+(v_i) = k-1$, then the two permutations are equal but for a cyclic shift of one. In general, we can obtain g_{i+1} from g_i by shifting the permutation by one (i.e., composing it with the elementary cycle z^{-1}) and permuting the later elements (after $g_{i+1}(d^+(v_i)-1)$). To make these changes to the permutations more apparent we are going to eliminate the cyclic shift in the definition of the consecutive k-colour function. To that end, we make the following definition.

Definition 4.3.19. For every vertex v_i in $V(D)$, let

$$F_i = \left\{\sigma \in S_k \,\middle|\, \forall\, 1 \le r \le d^+(v_i) : \sigma(i+r) = i+r \right\}.$$

Observe that each F_i is the fixgroup of a set of elements or equivalently the permutation group of the remaining elements $i + d^+(v_i) + 1, i + d^+(v_i) + 2, \ldots, i$ (recall that addition is modulo k). Therefore every F_i is closed under composition and taking inverses.

Definition 4.3.20. A *k-permutation labelling* of D is a permutation-valued function

$$\begin{aligned} h: V(D) &\to S_k \\ v_i &\mapsto h_i. \end{aligned}$$

with the property

$$(h_{i+1})^{-1} h_i \in F_i, \qquad 0 \le i \le n-2$$

and

$$z^n (h_0)^{-1} h_{n-1} \in F_{n-1}.$$

Figure 4.17 shows an example of a 5-permutation labelling alongside a consecutive 5-colour function for the example from earlier on. As can be seen, a k-permutation labelling of D is basically a reformulation of a consecutive k-colour function in that the respective g_i and h_i will just differ by a cyclic shift of i positions in list notation. The difference in the second condition above is a consequence of that property since n need not be divisible by k. Therefore a k-permutation labelling depends on the choice

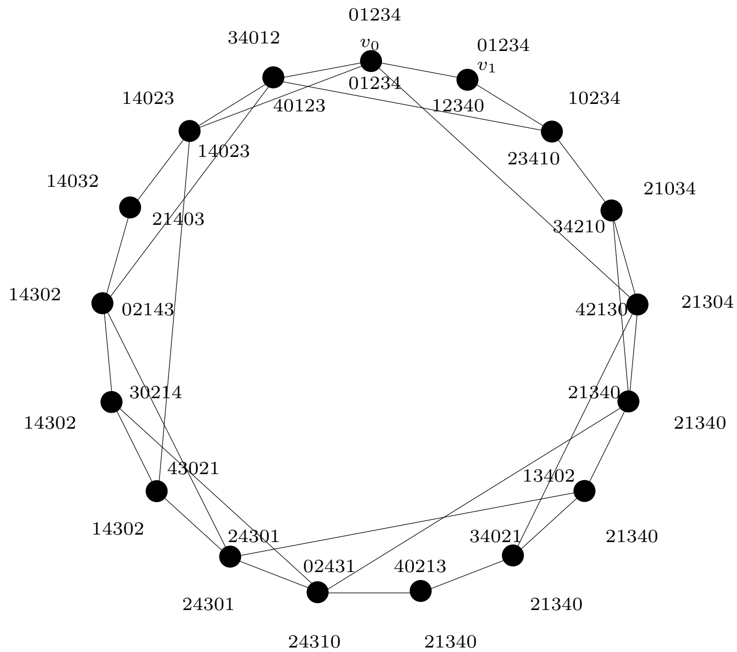

Figure 4.17: A consecutive 5-colouring (inside) and a 5-permutation labelling (outside) of a round digraph given by its essential arcs.

of the starting vertex v_0, whereas the consecutive k-colour function was independent of that choice. The details of the equivalence between the two concepts are shown in the following proposition.

Proposition 4.3.21. *D admits a k-permutation labelling if and only if D admits a consecutive k-colour function.*

Proof. Suppose g is a consecutive k-colour function of D. Then we set

$$h_i = g_i z^{-i}.$$

We are going to prove that those h_i define a k-permutation labelling. First assume that $i \leq n-2$. We have to show that $(h_i)^{-1}h_{i+1} \in F_i$, i.e., that $(h_i)^{-1}h_{i+1}$ has fixed points $i+r$ for $1 \leq r \leq d^+(v_i)$ (recall that all these numbers are taken modulo k).

We have

$$\begin{aligned}(h_{i+1})^{-1}h_i(i+r) &= z^{i+1}(g_{i+1})^{-1}g_i z^{-i}(i+r)\\ &= z^{i+1}(g_{i+1})^{-1}g_i(r)\\ &= z^{i+1}(g_{i+1})^{-1}g_{i+1}(r-1)\\ &= z^{i+1}(r-1)\\ &= i+r.\end{aligned}$$

Similarly, for $1 \leq r \leq d^+(v_{n-1})$,

$$\begin{aligned}z^n(h_0)^{-1}h_{n-1}(n-1+r) &= z^n(g_0)^{-1}g_{n-1}z^{1-n}(n-1+r)\\ &= z^n(g_0)^{-1}g_{n-1}(r)\\ &= z^n(g_0)^{-1}g_0(r-1)\\ &= z^n(r-1)\\ &= n-1+r.\end{aligned}$$

Now assume that h is a k-permutation labelling. Set

$$g_i = h_i z^i.$$

We are going to prove that those g_i define a consecutive k-colour function. For $i \leq n-2$ and $1 \leq r \leq d^+(v_i)$ we have, for some $\sigma_i \in F_i$,

$$\begin{aligned}g_{i+1}(r-1) &= h_{i+1}z^{i+1}(r-1)\\ &= h_{i+1}(i+r)\\ &= h_i\sigma_i(i+r)\\ &= g_i z^{-i}(i+r)\\ &= g_i(r).\end{aligned}$$

Similarly for $1 \leq r \leq d^+(v_{n-1})$ we have for some $\sigma_{n-1} \in F_{n-1}$,

$$\begin{aligned} g_0(r-1) &= h_0(r-1) \\ &= h_{n-1}\sigma_{n-1}z^n(r-1) \\ &= g_{n-1}z^{1-n}\sigma_{n-1}(n-1+r) \\ &= g_{n-1}z^{1-n}(n-1+r) \\ &= g_{n-1}(r). \end{aligned}$$

□

We remark that we can obtain a k-colouring of a digraph from a k-permutation labelling by colouring every vertex by $h_i(i)$. This follows directly from the proofs of the equivalences of k-permutation labellings, consecutive k-colour functions and k-colourings.

While both the consecutive k-colour function and the k-permutation labelling have local conditions for their existence, we are now ready to state a global condition for their existence (and therefore for the existence of a k-colouring) using the F_i defined earlier. We note that for two sets P, Q of permutations, the product PQ is defined as the set $\{pq \mid p \in P,\, q \in Q\}$.

Proposition 4.3.22. *D admits a k-permutation labelling if and only if*

$$z^n \in F_{n-1}F_{n-2}\cdots F_0.$$

Proof. Suppose h is a k-permutation labelling. Then we have

$$\begin{aligned} z^n &= z^n(h_0)^{-1}h_{n-1}(h_{n-1})^{-1}h_{n-2}\cdots h_1(h_1)^{-1}h_0 \\ &\in F_{n-1}F_{n-2}\cdots F_0. \end{aligned}$$

On the other hand assume that $z^n \in F_{n-1}F_{n-2}\cdots F_0$. Then there exist $\sigma_i \in F_i$ such that

$$z^n = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_0.$$

We set

$$h_i = (\sigma_0^{-1}\cdots\sigma_{i-1}^{-1}).$$

With this definition we obtain for $i \leq n-2$,

$$\begin{aligned}(h_{i+1})^{-1}h_i &= (\sigma_i\sigma_{i-1}\cdots\sigma_0)(\sigma_0^{-1}\sigma_1^{-1}\cdots\sigma_{i-1}^{-1}) \\ &= \sigma_i \\ &\in F_i,\end{aligned}$$

and

$$\begin{aligned}z^n(h_0)^{-1}h_{n-1} &= (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_0)\cdot id\cdot(\sigma_0^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}) \\ &= \sigma_{n-1} \\ &\in F_{n-1}.\end{aligned}$$

□

This last proposition gives rise to an $O(n)$ algorithm (for fixed k) for checking the k-colourability of round digraphs, since the size of every F_i is bounded by $k!$ and every F_i can be computed in time $O(1)$. An algorithm produced like that would still be exponential in k, though.

A summary of the various characterizations is given in the following theorem.

Theorem 4.3.23. *Let D be a round digraph and k a positive integer. Then the following are equivalent:*

(i) D is k-colourable;

(ii) D admits a consecutive k-colour function;

(iii) D admits a k-permutation labelling;

(iv) $z^n \in F_{n-1}F_{n-2}\cdots F_0$. □

The strength of the last characterization lies in the ability to produce a structural characterization of maximally k-colourable round digraphs (maximal with respect to the edge set).

4.3.3 Maximally k-colourable round digraphs

We continue to use the terminology from Subsection 4.3.2. We will employ characterization (iv) from Theorem 4.3.23 Proposition 4.3.22 heavily in obtaining the characterization of maximally k-colourable round digraphs. For clarity, a maximally k-colourable round digraph is a round digraph D that is k-colourable, and every round digraph D' with $V(D') = V(D)$ and $E(D') \supseteq E(D)$ is not k-colourable. For the sake of simplicity, we make the following definitions.

Definition 4.3.24. Let D be a round digraph and k a positive integer. Let $\tilde{\sigma}$: $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ be a sequence of permutations of S_k. We say that $\tilde{\sigma}$ is a *round sequence for* D if $\sigma_i \in F_i$ for all i, and $\sigma_{n-1} \cdots \sigma_1 \sigma_0 = z^n$.

Definition 4.3.25. Let D be a round digraph and k a positive integer. For $0 \leq i \leq n-1$, we define s_i, t_i as

$$s_i = i + d^+(v_i) + 1 \,(\text{mod } k), \qquad t_i = i \,(\text{mod } k).$$

With this notation, F_i is the permutation group of $\{s_i, s_i+1, \ldots, t_i\}$ (viewed as a subgroup of S_k). Recall that an essential vertex is a vertex with $d^+(v_i) \leq d^+(v_{i+1})$. We will be particularly interested in essential vertices of outdegree less than $k-1$, the reason being that a vertex v_i is an essential vertex with outdegree less than $k-1$ if and only if the arc $v_i v_{i+d^+(v_i)+1}$ can be added to obtain a round digraph without creating a k-clique. Also, if v_i is not an essential vertex then $d^+(v_i) = d^+(v_{i+1}) + 1$. Therefore, if $v_{i'}$ is the next essential vertex after v_i, then

$$s_{i'} = \cdots = s_{i+1} = s_i \leq t_i < t_{i+1} < \cdots < t_{i'},$$

where these inequalities are to mean that $t_i, \ldots, t_{i'-1}, t_{i'}$ appear in that order in the round enumeration.

Lemma 4.3.26. *Let D be a k-colourable round digraph. Then D is maximally k-colourable if and only if every round sequence $\tilde{\sigma}$ for D has the property that*

$$\sigma_i(s_i) \neq s_i$$

for all essential vertices v_i with $d^+(v_i) < k-1$.

Proof. Suppose first that there exist a round sequence $\tilde{\sigma}$ for D and an essential vertex v_i with $d^+(v_i) < k-1$ such that

$$\sigma_i(s_i) = s_i.$$

Let D' be the round digraph obtained from D by adding the arc $v_i v_{i+d^+(v_i)+1}$. Then $F'_j = F_j$ for $j \neq i$, while F'_i is the permutation group of $\{s_i+1, s_i+2, \ldots, t_i\}$. Since $\sigma_i(s_i) = s_i$, we obtain

$$\sigma_i \in F'_i$$

and we can conclude that $\tilde{\sigma}$ is also a round sequence for D'. Therefore D' is k-colourable and D is not maximally k-colourable.

On the other hand assume that D is k-colourable but not maximally k-colourable. Then there exists an essential vertex v_i with $d^+(v_i) < k-1$ such that the round digraph D' obtained from D by adding the arc $v_i v_{i+d^+(v_i)1}$ is k-colourable. Therefore we obtain

$$d^+_{D'}(v_j) = \begin{cases} d^+_D(v_j) & j \neq i \\ d^+_D(v_j)+1 & j = i \end{cases}.$$

In particular this means that $F'_j \subseteq F_j$ (with strict inequality for $j = i$) and $s'_i = s_i+1$. Since D' is k-colourable there exists a round sequence $\tilde{\sigma}$ for D'. As $F'_j \subseteq F_j$ for all j, we get that $\tilde{\sigma}$ is also a round sequence for D. Furthermore, since F'_i is the permutation group of $\{s_i+1, s_i+2, \ldots, t_i\}$, we have $\sigma_i(s_i) = s_i$, proving the claim. □

We are now proceeding to establish a concrete classification for all maximally k-colourable round digraphs. To this purpose, we will develop a concept called *path diagrams* which will be helpful in simplifying the proofs.

Definition 4.3.27. Let $\tilde{\sigma}$ be a round sequence for D. The *path diagram* of $\tilde{\sigma}$ is the graph on the vertex set $\{0, 1, \ldots, n\} \times \mathbb{Z}_k$ with edges $(i,t)(i+1, \sigma_i(t))$ for all i and t.

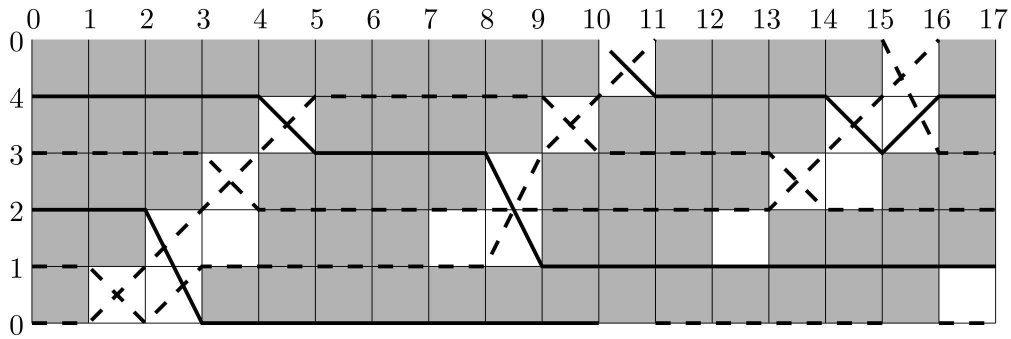

Figure 4.18: A path diagram.

It follows from the definition that the path diagram for $\tilde{\sigma}$ consists of k vertex-disjoint paths. We can imagine this graph drawn on a cylindrical grid of length n and circumference k. As we are going to use the geometry of this cylinder, we specify how we embed the edges. We draw all edges as straight lines. However, on a cylinder, this does not define the embedding. If t is an element fixed by F_i, then we will draw the edge $(i, r)(i + 1, r)$ as the shortest possible straight line. The remaining edges defined by σ_i we will draw as the unique straight lines that do not intersect that first line. Figure 4.18 shows a path diagram for the round digraph from earlier. The top line is identified with the bottom one. The white underlay corresponds to the range between s_i and t_i for each i. The distinction between the paths drawn solid or dashed corresponds to the direction, in which the path rotates around the cylinder (which we will formalize in a moment).

The cylindrical embedding of path diagrams motivates the following definitions.

Definition 4.3.28. Let a path diagram of a round sequence $\tilde{\sigma}$ for D be given. We define p_r to be the path originating at $(0, r)$. Let $(i, a_i)(i + 1, \sigma_i(a_i))$ and $(i, b_i)(i + 1, \sigma_i(b_i))$ be two edges lying on paths p_r and $p_{r'}$ respectively. If $s_i \leq a_i < b_i \leq t_i$ and $s_i \leq \sigma_i(b_i) < \sigma_i(a_i) \leq t_i$, then we say that p_r and $p_{r'}$ *intersect* at σ_i. We call the path diagram *reduced* if any two paths intersect at most once. We call the sequence $\tilde{\sigma}$ a *reduced round sequence* if its path diagram is reduced. We define the *weight* of an edge $(i, s_i + h)(i + 1, s_i + j)$ as $j - h$. The *weight of a path* p_r is the sum over the weights of its edges.

The intersections of a path diagram are precisely the crossings of the paths in the cylindrical embedding of the path diagram. We can also interpret the weight of an edge as a measure, how far the path moves around the cylinder in clockwise direction (or up in the flat representation as in Figure 4.18) along that edge. For example, the edge $(2, 2)(3, 0)$ in the figure has weight -2, while the two edges $(2, 1)(3, 2)$ and $(2, 0)(3, 1)$ have weight 1 each.

We observe that a path p_r consists of the vertices

$$\{(0, r), (1, \sigma_O(r)), (2, \sigma_1\sigma_0(r)), \ldots, (n, \sigma_{n-1} \cdots \sigma_0(r))\}.$$

From this we see another property of the path diagrams. If we write down the paths on which the vertices $(i, 0), (i, 1), \ldots, (i, k-1)$ lie in this order, then their indices yield a permutation in S_k. In fact those permutations form a k-permutation labelling of the round digraph. Thereby, if we colour every vertex v_i of the round digraph by the index of the path containing (i, i), we obtain a proper k-colouring.

Another interesting observation is that the number of intersections occurring at σ_i is the number of inversions of σ_i, if interpreted as a permutation of $\{s_i, s_{i+1}, \ldots, t_i\}$.

Proposition 4.3.29. *Let D be a k-colourable round digraph. Then there exists a reduced round sequence $\tilde{\sigma}$ for D. Furthermore, all round sequences for D with the fewest intersections are reduced.*

Proof. Suppose D is a k-colourable round digraph. Assume that $\tilde{\sigma}$ is a round sequence for D with as few intersections as possible. Assume $\tilde{\sigma}$ is not reduced. We are going to construct a round sequence for D with fewer intersections than $\tilde{\sigma}$, a contradiction to our assumption.

Since $\tilde{\sigma}$ is not reduced, its path diagram has two paths, say p_r and $p_{r'}$ which intersect at least twice. Assume these intersections occur at σ_i and σ_j $(i < j)$ and the edges involved are $(a_i, \sigma_i(a_i))$, $(a_j, \sigma_j(a_j))$ lying on p_r and $(b_i, \sigma_i(b_i))$, $(b_j, \sigma_j(b_j))$ lying on p'_r. We now define the sequence $\tilde{\tau} : \tau_0, \tau_1, \ldots, \tau_{n-1}$ by being equal to $\tilde{\sigma}$ everywhere but for the indices i and j for which we set

$$\tau_i = \sigma_i \cdot (a_i, b_i), \qquad \tau_j = \sigma_j \cdot (a_j, b_j).$$

We need to show that τ is a round sequence for D and that its path diagram has fewer intersections than the path diagram for $\tilde{\sigma}$. First, we know by the definition of the intersection of paths that $(a_i, b_i) \in F_i$, and since F_i is a group, we obtain $\tau_i \in F_i$. Similarly for τ_j. Now consider the changes to the path diagram of $\tilde{\sigma}$ if we replace σ_i and σ_j by τ_i and τ_j. All the edges on paths except p_r and $p_{r'}$ remain unchanged. As for the path originating at $(0, r)$, it contains the vertices of p_r up to (i, a_i), then the vertices of $p_{r'}$ from $(i+1, \sigma_i(b_i))$ to (j, b_j) and then the vertices of p_r from $(j, \sigma_j(a_j))$ up to the endvertex of p_r. Similarly the path originating at $(0, r')$ ends at the same vertex as $p_{r'}$ did. Therefore

$$\tau_{n-1} \cdots \tau_0 = \sigma_{n-1} \cdots \sigma_0 = z^n.$$

Finally, we consider the number of intersections of the path diagrams of $\tilde{\sigma}$ and $\tilde{\tau}$. Since both sequences are identical except at indices i and j, we only need to consider those indices. The number of intersections at σ_i is equal to the number of inversions of σ_i, interpreted as a permutation of $\{s_i, s_{i+1}, \ldots, t_i\}$. By the definition of a_i and b_i, the pair $(\sigma_i(a_i), \sigma_i(b_i))$ is an inversion of σ_i. Therefore if we swap these two elements in σ_i, i.e., consider the permutation $\sigma_i \cdot (a_i, b_i) = \tau_i$, we obtain a permutation with fewer inversions. The same argument can be made for j and we conclude that the number of intersections of the path diagram of $\tilde{\tau}$ is smaller than the number of intersections of the path diagram of $\tilde{\sigma}$, which concludes the proof. □

Figure 4.19 shows a reduced path diagram obtained from the path diagram in Figure 4.18 by performing the uncrossing operation from the previous proof. Note that this reduced path diagram is not unique, as a different order of uncrossings might return a different reduced path diagram. We observe that the paths of negative weight (drawn in solid lines) do not cross each other, and neither do the paths of positive weights (drawn in dashed lines). It turns out that this is a characteristic of all reduced path diagrams.

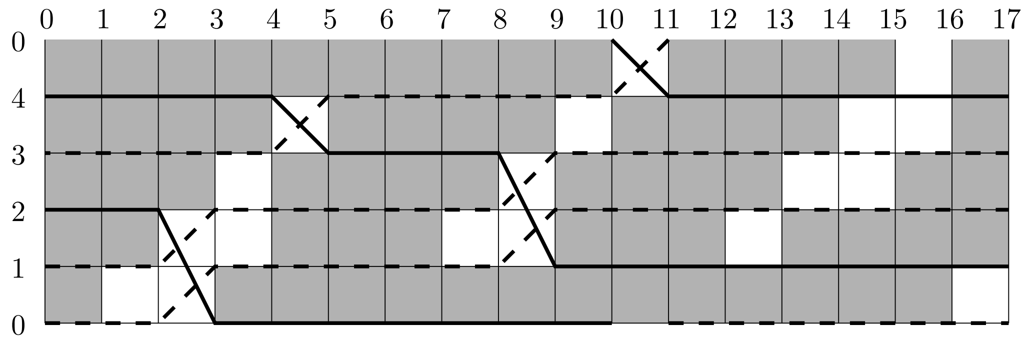

Figure 4.19: A reduced path diagram.

Proposition 4.3.30. *Let $\tilde{\sigma}$ be a reduced round sequence for D. Let $m = n \,(mod\ k)$. Then the path diagram for $\tilde{\sigma}$ has $k - m$ paths of weight m and m paths of weight $-(k-m)$. Furthermore, every path intersects every path of differing weight exactly once and does not intersect any of the paths of equal weight.*

Proof. First we observe that by the definition, the weight of the edge $(i, t)(i+1, \sigma_i(t))$ is congruent to $\sigma_i(t) - t$ modulo k. Therefore the weight of a path p_r is congruent to

$$\begin{aligned}&(\sigma_0(r) - r) + (\sigma_1\sigma_0(r) - \sigma_0(r)) + \cdots + (\sigma_n\sigma_{n-1}\cdots\sigma_0(r) - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_0(r))\\ &\equiv \sigma_n\cdots\sigma_0(r) - r\\ &\equiv r + n - r\\ &\equiv m.\end{aligned}$$

Consider the embedding of the path diagram on the cylinder of length n and circumference k that was mentioned earlier. Then the weight of p_r divided by k is equal to the number of revolutions around the cylinder in k-direction. Therefore, since all the weights of the paths differ by multiples of k, the number of times two paths intersect is at least the difference of the weights of the paths divided by k (more precisely, it is equal to the difference of the weights of the paths divided by k plus a nonnegative even integer). Since $\tilde{\sigma}$ is a reduced round sequence, no two paths intersect more than once. Therefore the weights of any two paths differ by 0, in which case they do not intersect or by k, in which case they intersect once. Now consider the sum over the weights of all paths. This is equal to the sum over all weights of all edges. Consider the edges induced by σ_i. The weight of an edge $(i, s_i + h)(i + 1, s_i + j)$ is defined as $j - h$. If we sum up the weights of all these edges, all values between 0 and $k - 1$ appear once each for both j and h. Therefore the sum over all the weights of all the edges is 0, and we conclude that the sum over the weights of all paths is 0 as well. Since the weights of two paths differ by 0 or k and are all congruent to m, we must have $(k - m)$ paths of weight m and m paths of weight $-(k - m)$. □

Proposition 4.3.31. *Let D be a maximally k-colourable round digraph. Then the only reduced round sequence for D is the sequence $\tilde{\sigma}$ defined by*

$$\sigma_i = \begin{cases} (s_i, s_{i+1}, \ldots, t_i), & v_i \textit{ essential vertex} \\ id, & \textit{otherwise.} \end{cases}$$

Proof. Assume that D is a maximally k-colourable round digraph. By Proposition 4.3.29, there exists a reduced round sequence $\tilde{\sigma}$ for D. Consider an essential vertex v_j. Assume that $\sigma_j(t_j) \neq s_j$ (thus $d^+(v_i) < k-1$). Then there exists an r with $s_j \leq r < t_j$ and $\sigma_j(r) = s_j$. By the definition of s_j and t_j (and the fact that $d^+(v_{j-1}) \leq d^+(v_j)+1$) we obtain that

$$s_{j-1} \leq s_j \leq r \leq t_{j-1} < t_j.$$

Therefore the transposition (r, s_j) is an element of both F_{j-1} and F_j. We now define the sequence $\tilde{\tau}$ by

$$\tau_i = \begin{cases} (r, s_j)\sigma_{j-1} & i = j - 1 \\ \sigma_j(r, s_j) & i = j \\ \sigma_i & \text{otherwise.} \end{cases}$$

For every i, we have $\tau_i \in F_i$ and it is

$$\begin{aligned} \tau_{n-1}\tau_{n-2}\cdots\tau_j\tau_{j-1}\cdots\tau_0 &= \sigma_{n-1}\sigma_{n-2}\cdots\sigma_j(r, s_j)(r, s_j)\sigma_{j-1}\cdots\sigma_0 \\ &= \sigma_{n-1}\sigma_{n-2}\cdots\sigma_0 \\ &= z^n. \end{aligned}$$

Therefore $\tilde{\tau}$ is a round sequence for D. However, we have $\tau_j(s_j) = s_j$, which is a contradiction to Lemma 4.3.26. Thus we can assume that $\sigma_j(t_j) = s_j$. Now consider the paths on which the vertices $\{(j, s_j), (j, s_j + 1), \ldots, (j, t_j - 1)\}$ lie. Every one of these paths intersects the path using the edge $(j, t_j)(j + 1, s_j)$. Therefore, by Proposition 4.3.30, all of these paths have the same weight and cannot intersect each other. Hence

$$\sigma_j = (s_j, s_{j+1}, \ldots, t_j).$$

Now consider a vertex v_j that is not essential, i.e., $d^+(v_j) = d^+(v_{j+1}) + 1$. Then

$$s_j = j + d^+(v_j) + 1 = j + 1 + d^+(v_{j+1}) + 1 = s_{j+1}.$$

If we let m be the first index after j such that v_m is an essential vertex we obtain

$$s_j = s_{j+1} = \cdots = s_m$$

while by definition

$$t_j < t_{j+1} < \cdots < t_m.$$

Thus

$$F_j \subset F_{j+1} \subset \cdots \subset F_{m-1} \subset F_m$$

and we obtain for every $r \in \{s_j, s_{j+1}, \ldots, t_j\}$ that

$$\sigma_{m-1}\sigma_{m-2}\cdots\sigma_j(r) \in \{s_m, s_m + 1, \ldots, t_m - 1\}.$$

Therefore all the paths with edges of nonzero weight at σ_j are contained in the paths using the vertices $\{(m, s_m), (m, s_{m+1}), \ldots, (m, t_m - 1)\}$. Since we showed earlier that no two of these paths can intersect, there can be no intersections at σ_j. Thus

$$\sigma_j = id. \qquad \square$$

Theorem 4.3.32. *Let D be a round digraph with maximum outdegree at most $k - 1$ and let $m = n \,(mod\ k)$. Let $\sigma_i \in S_k$ be defined as*

$$\sigma_i = \begin{cases} (s_i, s_{i+1}, \ldots, t_i) & v_i \textit{ essential} \\ id & \textit{otherwise.} \end{cases}$$

Then D is maximally k-colourable if and only if

$$\sigma_{n-1}\sigma_{n-2}\cdots\sigma_0 = z^n$$

and

$$\sum(k-1-d^+(v_i)) = m(k-m),$$

where the sum is over all essential vertices v_i.

Proof. First, we note that the sum in the hypothesis is precisely the number of intersections of the path diagram of the sequence $\tilde{\sigma} : \sigma_0, \sigma_1, \ldots, \sigma_{n-1}$, as for all essential vertices v_i,

$$k-1-d^+(v_i) = t_i - s_i,$$

and $t_i - s_i$ is the number of inversions of σ_i as a permutation of $\{s_i, s_{i+1}, \ldots, t_i\}$, which equals the number of intersections at σ_i in the path diagram of $\tilde{\sigma}$. Therefore the two conditions in the hypothesis are equivalent to the statement that $\tilde{\sigma}$ is a reduced round sequence for D.

Now let D be maximally k-colourable. By Proposition 4.3.31, the sequence $\tilde{\sigma}$ defined by the σ_i's is a reduced round sequence for D.

On the other hand, assume D is a round digraph for which $\tilde{\sigma} : \sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ is a reduced round sequence for D. By Lemma 4.3.26 it suffices to show that $\tilde{\sigma}$ is the only reduced round sequence for D, as $\tilde{\sigma}$ satisfies the condition $\sigma_i(s_i) \neq s_i$ for all essential vertices v_i.

We assume in the following, without loss of generality, that v_{n-1} is an essential vertex. We are going to compare the weights of paths of different reduced round sequence, so for clarity, we let $p_r(\tilde{\tau})$ denote the path in the path diagram of a round reduced sequence $\tilde{\tau}$ starting at $(0, r)$.

We first show a property of $\tilde{\sigma}$ that will be very helpful later. Consider a path $p_r(\tilde{\sigma})$ with negative weight. By Proposition 4.3.30, the path has no edges of positive weight. Suppose (i, a) is a vertex on $p_r(\tilde{\sigma})$ with $a \in \{s_i, s_i + 1, \ldots, t_i\}$. If v_i is an essential vertex, then we must have $a = t_i$ and $\sigma_i(a) = s_i$. If v_i is not an essential vertex, then $\sigma_i(a) = a$. In this case, consider the next index i' after i such that $v_{i'}$ is an essential vertex. By the definition of $\tilde{\sigma}$, the vertex (i', a) must also lie on $p_r(\tilde{\sigma})$. Thus

$$s_{i'} = \cdots = s_i \leq a \leq t_i < \cdots < t_{i'},$$

and so $\sigma_{i'}(a) = a + 1$, a contradiction to the fact that $p_r(\tilde{\sigma})$ contains no edges of positive weight. Thus if (i, a) is a vertex on a path $p_r(\tilde{\sigma})$ with negative weight, then $a \in \{s_i, s_{i+1}, \ldots, t_i\}$ implies $a = t_i$ and $\sigma_i(a) = s_i$. We can conclude from this that if we have a reduced round sequence $\tilde{\tau}$ for D, then we cannot have either of the cases

$$s_i \leq \tau_i(b) \leq \sigma_i(a) \leq a < b \leq t_i$$

or

$$s_i \leq \tau_i(b) < \sigma_i(a) \leq a \leq b \leq t_i.$$

In particular, this proves that if $p_r(\tilde{\sigma})$ and $p_r(\tilde{\tau})$ both have negative weight (equal to $-(k - m)$ by Proposition 4.3.30) then they are equal, since the first and last vertex of both paths are the same.

Now let $\tilde{\tau}$ be any reduced round sequence for D. We first show that if $p_r(\tilde{\sigma})$ has positive weight, then $p_r(\tilde{\tau})$ has positive weight as well. Suppose to the contrary that $p_r(\tilde{\tau})$ has negative weight. Let i be the smallest index such that $r \in \{s_i, s_{i+1}, \ldots, t_i\}$ and $s_i \neq t_i$. Then the vertex (i, r) lies on $p_r(\tilde{\tau})$. Let i' be the next index after i such that $v_{i'}$ is an essential vertex and let $(i' + 1, a)$ be the vertex lying on $p_r(\tilde{\tau})$. By the definition of i and i' we have

$$s_{i'} = s_i < t_i < t_{i'},$$

and we obtain $s_{i'} \leq a \leq t_{i'}$. Now consider the path $p_{r'}(\tilde{\sigma})$ containing the edge $(i, t_{i'})(i + 1, s_{i'})$. Since this edge has negative weight, the path $p_{r'}(\tilde{\sigma})$ must have negative weight by Proposition 4.3.30. Since, by the same proposition, $p_{r'}(\tilde{\sigma})$ contains no edge of positive weight, the total weight of the first $i+1$ edges of $p_{r'}(\tilde{\sigma})$ is less than the total weight of the first $i + 1$ edges of $p_r(\tilde{\tau})$ (equality would imply $r = r'$, which is a contradiction since $p_r(\tilde{\sigma})$ has positive weight). Since the weights of the paths are equal, there must be an index j with (j, b) and (j, b') being the vertices lying on $p_r(\tilde{\tau})$ and $p_{r'}(\tilde{\sigma})$ respectively such that

$$s_j \leq \tau_j(b) < \sigma_j(b') \leq b' \leq b \leq t_j,$$

a contradiction to the property we established earlier for paths of $\tilde{\sigma}$ with negative weight. Thus $p_r(\tilde{\tau})$ has positive weight.

Since $\tilde{\sigma}$ and $\tilde{\tau}$ have the same number of paths by Proposition 4.3.30, we obtain that $p_r(\tilde{\sigma})$ has the same weight as $p_r(\tilde{\tau})$ for all r. From what we have shown earlier, the paths of negative weight of $\tilde{\sigma}$ and $\tilde{\tau}$ are then identical. As the paths of positive weight do not intersect each other, they are completely defined by the paths of negative weight and we conclude $\tilde{\sigma} = \tilde{\tau}$. □

As an example we will use Theorem 4.3.32 to find all maximally 3-colourable round digraphs. The easiest case arises when $n \equiv 0 \pmod 3$. The last condition of Theorem 4.3.32 then reads

$$\sum (2 - d^+(v_i)) = 0,$$

implying $d^+(v_i) = 2$ for all essential vertices and therefore all vertices v_i. The only such graph is C_n^2 (recall that D^m is the graph on the same vertex set as D with two vertices adjacent if and only if their distance is at most m in D) with the natural orientation of the edges (which we will also assume for other graphs in the following).

Now consider the case $n \equiv 1 \pmod 3$. Then we have the condition

$$\sum (2 - d^+(v_i)) = 0.$$

This can be obtained by having one essential vertex of outdegree 0 or two essential vertices of outdegree 1 (all other essential vertices having outdegree 2). Suppose we have an essential vertex of outdegree 0, say v_{n-1} without loss of generality. Then

$$\begin{aligned} \sigma_{n-1} &= (n, n+1, n+2) = (1, 2, 0) = (0, 1, 2), \\ \sigma_j &= id \qquad\qquad (j \neq n-1), \end{aligned}$$

and we obtain

$$\sigma_{n-1} \cdots \sigma_0 = \sigma_{n-1} = (0, 1, 2) = (0, 1, 2)^n,$$

so we indeed have a maximally 3-colourable graph. The graph defined by this is P_n^2. Suppose we have two essential vertices of outdegree 1, say v_i and v_{n-1} without loss of generality. Then

$$\begin{aligned} \sigma_i &= (i-1, i), \\ \sigma_{n-1} &= (n+1, n+2) = (2, 0), \\ \sigma_j &= id \qquad\qquad (j \notin \{i, n-1\}). \end{aligned}$$

We obtain

$$\sigma_{n-1} \cdots \sigma_0 = \sigma_{n-1}\sigma_i = (2, 0)(i-1, i).$$

Since $z^n = (0, 1, 2)^n = (0, 1, 2)$, we need $i \equiv 1 \pmod 3$. The graphs satisfying this condition can be described as follows. Take the two graphs $P_{m_1}^2, P_{m_2}^2$, where $m_1 \equiv m_2 \equiv 0 \pmod 3$ and identify the last vertex of $P_{m_1}^2$ with the first vertex of $P_{m_2}^2$ and the last vertex of $P_{m_2}^2$ with the first vertex of $P_{m_1}^2$. For short, we denote this graph by $C(P_{m_1}^2, P_{m_2}^2)$.

Finally, consider the case of $n \equiv 2 \pmod 3$. Again, we have the condition

$$\sum (2 - d^+(v_i)) = 0.$$

If we have an essential vertex of outdegree 2, say v_{n-1} without loss of generality, then

$$\begin{aligned} \sigma_{n-1} &= (n, n+1, n+2) = (2, 0, 1) = (0, 1, 2), \\ \sigma_j &= id \qquad\qquad (j \neq n-1), \end{aligned}$$

but

$$\sigma_{n-1} \cdots \sigma_0 = \sigma_{n-1} = (0, 1, 2) \neq (0, 2, 1) = (0, 1, 2)^n.$$

Thus we do not obtain a maximally 3-colourable round digraph. So suppose, we have two essential vertices of outdegree 1, say v_i, v_{n-1} without loss of generality. Then

$$\begin{aligned} \sigma_i &= (i-1, i), \\ \sigma_{n-1} &= (n+1, n+2) = (0, 1), \\ \sigma_j &= id \qquad\qquad (j \notin \{i, n-1\}), \end{aligned}$$

and with the second condition

$$(0,2,1) = (0,1,2)^n = \sigma_{n-1} \cdots \sigma_0 = \sigma_{n-1}\sigma_i = (0,1)(i-1,i).$$

Thus $i \equiv 0 \pmod 3$. The graphs satisfying this condition can be described (similarly to the previous case) as $C(P^2_{m_1}, P^2_{m_2})$, where $m_1 \equiv m_2 \equiv 2 \pmod 3$.

In summary, the maximally 3-colourable round digraphs are

$$\begin{aligned} C_n^2 &\quad (n \equiv 0 \pmod 3), \\ P_n^2 &\quad (n \equiv 1 \pmod 3), \\ C(P^2_{m_1}, P^2_{m_2}) &\quad (m_1 \equiv m_2 \equiv 0 \pmod 3), \\ C(P^2_{m_1}, P^2_{m_2}) &\quad (m_1 \equiv m_2 \equiv 2 \pmod 3). \end{aligned}$$

4.3.4 Forbidden subgraphs

We finish this section by giving a few partial results regarding forbidden subgraph characterizations of (k, l)-colourable round digraphs.

The easiest case is the one of $(2, 0)$-colourings, that is, 2-colourings. A graph is $(2, 0)$-colourable if and only if it is bipartite. Therefore a round digraph is $(2, 0)$-colourable if and only if it does not contain any C_n with n odd as an induced subgraph.

The case of $(3, 0)$-colourable round digraphs is slightly more complicated. To find the forbidden subgraph characterization, we will use the list of maximally 3-colourable round digraphs established at the end of the previous subsection. Clearly, any round digraph containing K_4 as an induced subgraph is not 3-colourable. So suppose we have a round digraph D with maximum outdegree at most 2. Consider the number n of vertices of D. If $n \equiv 0 \pmod 3$, then D is a subgraph of C_n^2 and therefore 3-colourable.

If $n \equiv 1 \pmod 3$ and D has a vertex with outdegree 0, then D is a subgraph of P_n^2 and is therefore 3-colourable. So suppose that all vertices of D have outdegree 1 or 2. Consider the vertices of outdegree 1. If the distance of any two of them in the circular order is congruent to 2 modulo 3, then D is a subgraph of $C(P_{m_1}^2, P_{m_2}^2)$ for suitable $m_1 \equiv m_2 \equiv 0 \pmod 3$ and is therefore 3-colourable. Otherwise D must be of the form $C(P_{m_1}^2, P_{m_2}^2, \ldots, P_{m_r}^2)$, where

$$m_i \equiv \begin{cases} 1 \pmod 3 & i \leq r-1 \\ 2 \pmod 3 & i = r. \end{cases}$$

Finally, consider $n \equiv 2 \pmod 3$. If D has a vertex of outdegree 0, then D is a subgraph of $C(P_n^2, P_2^2)$ and is therefore 3-colourable. So suppose that all vertices of D have outdegree 1 or 2 and consider the vertices of outdegree 1. If the distance of any two of them in the circular order is congruent to 1 modulo 3, then D is a subgraph of $C(P_{m_1}^2, P_{m_2}^2)$ for suitable $m_1 \equiv m_2 \pmod 3$ and is therefore 3-colourable. Otherwise D must be of the form $C(P_{m_1}^2, P_{m_2}^2, \ldots, P_{m_r}^2)$, where

$$m_i \equiv \begin{cases} 1 \pmod 3 & i \leq r-1 \\ 0 \pmod 3 & i = r. \end{cases}$$

Note that if we remove the penultimate vertex of $P_{m_r}^2$, then we obtain the graph $C(P_{m_1}^2, \ldots, P_{m_{r-1}}^2, P_{m_r-2}^2, P_2^2)$, which already is not 3-colourable. Therefore we obtain the following statement regarding forbidden subgraphs.

Proposition 4.3.33. *A round digraph is $(3,0)$-colourable if and only if it does not contain K_4 or $C(P_{m_1}^2, \ldots, P_{m_r}^2)$ with*

$$m_i \equiv \begin{cases} 1 \pmod 3 & i \leq r-1 \\ 2 \pmod 3 & i = r \end{cases}$$

as an induced subgraph. □

Any characterization of $(k,0)$-colourable graphs for $k \geq 4$ can be expected to be more complicated. However, we can use Lemma 4.3.7 to obtain a forbidden subgraph characterization of $(0,l)$-colourable round digraphs.

Proposition 4.3.34. *Let $l \geq 2$. A round digraph is $(0,l)$-colourable if and only if it does not contain C_{lr+1}^{r-1} as an induced subgraph for any $r \geq 1$.*

Proof. The clique number of C_{lr+1}^{r-1} is r, while the number of vertices is $lr + 1 > lr$. Therefore C_{lr+1}^{r-1} cannot be $(0,l)$-coloured. Now suppose that D is a forbidden subgraph for $(0,l)$-colouring, that is, D cannot be $(0,l)$-coloured but any induced subgraph can. Consider an arbitrary vertex $v_i \in V(D)$ and a $(0,l)$-colouring of $D - v_i$ with transitive cliques (existing by Lemma 4.3.7). Consider the clique containing v_{i+1}. Assume without loss of generality that this clique is maximal. If v_{i+1} is not the source vertex (the vertex with indegree 0) of the clique, we can extend this clique in D by adding v_i, contradicting that D is not $(0,l)$-colourable. Thus v_{i+1} must be the source vertex. If $d^+(v_i) > d^+(v_{i+1})$ then again we can extend this clique by adding v_i. Therefore $d^+(v_i) \leq d^+(v_{i+1})$. As we chose v_i arbitrarily, we obtain that the outdegree of every vertex of D is the same. The result follows. □

As a final remark, we note that in fact the graphs C_{lr+1}^{r-k-1} with $r \geq 2k + 1$ are forbidden subgraphs for the (k,l)-colourability of round digraphs if $l \geq 2$. However, these turn out not to be the only forbidden subgraphs if $k \geq 1$.

Chapter 5

Fractional versions and the lexicographic product

This chapter considers fractional versions of (k, l)-colourings, the cochromatic number and the bichromatic number and investigates how the lexicographic product influences the various graph parameters considered here.

Section 5.1 considers fractional analogues of the various types of colourings. In Subsection 5.1.1, a fractional analogue of (k, l)-colouring is presented, while Subsections 5.1.2 and 5.1.3 give fractional analogues of the cochromatic number and the bichromatic number based on fractional (k, l)-colourings. Proposition 5.1.4 gives a bound for the fractional cochromatic number in terms of the number of vertices, the independence number and the clique number, which turns out to be a precise formula for vertex-transitive graphs. In Corollary 5.1.5, the fractional cochromatic number is calculated for square graphs. In Proposition 5.1.8, it is shown that the fractional bichromatic number is completely defined by the fractional chromatic number and the fractional clique covering number. A comparison of the various graph parameters and their fractional versions is provided in Proposition 5.1.10 and Figures 5.2 and 5.3.

In Section 5.2, the lexicographic product of graphs is considered, in particular with respect to the bichromatic number, the cochromatic number and the fractional cochromatic number. A lower bound for the cochromatic number of a lexicographic product of graphs is given in Proposition 5.2.9, while an upper found for the bichromatic number can be found in Proposition 5.2.11. In Subsection 5.2.1, it is shown that both box cographs and square graphs are closed under lexicographic products. Finally, Subsection 5.2.2 calculates various graph parameters for lexicographic products of a few select graphs, in particular lexicographic powers of P_4 and C_5.

5.1 Fractional versions

In this section, we will define and investigate fractional versions of (k, l)-colouring, the cochromatic number and the bichromatic number. While it turns out that the fractional bichromatic number can be completely described by the fractional chromatic number and the fractional clique-covering number, the fractional cochromatic number is an interesting concept that will appear again in relation to the lexicographic product in Section 5.2. At the end of this section, we provide a complete comparison of the various graph parameters defined here for general graphs, perfect graphs and vertex-transitive graphs.

For more background on fractional graph parameters, we refer to [53]. We will mainly follow the terminology from [36].

Definition 5.1.1. For a graph G, let $\mathcal{S}(G)$ be the family of independent sets of G and $\mathcal{C}(G)$ the family of cliques of G. For any vertex v of G, let $\mathcal{S}(G, v)$ be the family of independent sets containing v and $\mathcal{C}(G, v)$ the family of cliques containing v.

We give a brief review of fractional colourings and clique coverings. A *fractional k-colouring* of a graph G is a nonnegative real-valued weight function w on the set $\mathcal{S}(G)$ with the property that for all $v \in V(G)$

$$\sum_{S \in \mathcal{S}(G,v)} w(S) \geq 1$$

and

$$\sum_{S \in \mathcal{S}(G)} w(S) = k.$$

The *fractional chromatic number* $\chi_f(G)$ is the minimum k over all fractional k-colourings of G. We remark that if we only allow weights 0 and 1 then the definition given can be interpreted as a covering of G with k independent sets (the sets with weight 1) and the minimum possible weight over all those k is precisely the cochromatic number of G.

We remark that the fractional chromatic number $\chi_f(G)$ can be formulated as a standard linear program (see for example [36]). Using the methods of linear programming, it is easy to show that $\chi_f(G)$ is bounded between $\omega(G)$ and $\chi(G)$, which in particular implies that $\chi_f(G) = \chi(G)$ for all perfect graphs G.

Similarly to fractional colourings, we can obtain *fractional clique coverings* and the *fractional clique covering number* $\theta_f(G)$ by replacing all independent sets in the definition of fractional colourings by cliques.

5.1.1 Fractional (k, l)-colouring

We define fractional (k, l)-colourings analogously to fractional colourings and fractional clique coverings.

Definition 5.1.2. A *fractional* (k, l)*-colouring* is a nonnegative real-valued weight function on the set $\mathcal{S}(G) \cup \mathcal{C}(G)$ with the property that for all $v \in V(G)$

$$\sum_{S \in \mathcal{S}(G,v)} w(S) + \sum_{C \in \mathcal{C}(G,v)} w(C) \geq 1$$

and

$$\sum_{S \in \mathcal{S}(G)} w(S) = k, \qquad \sum_{C \in \mathcal{C}(G)} w(C) = l.$$

We remark that a fractional $(k, 0)$-colouring can be interpreted as a fractional k-colouring, while a fractional $(0, l)$-colouring corresponds to a fractional l-clique-covering.

An example of a fractional $(1, \frac{2}{3})$-colouring of $K_3 + 2K_1$ is given in Figure 5.1. The four coordinates of each vector correspond to three independent sets and a clique, where a minus indicates that the vertex is not contained in the set, while the numbers are the weights of the sets.

5.1.2 Fractional cochromatic number

Definition 5.1.3. The *fractional cochromatic number* of a graph G is defined by

$$\chi_f^c(G) = \min \{ r \mid \exists k, l \geq 0, k + l = r : G \text{ admits a fractional } (k, l)\text{-colouring} \}.$$

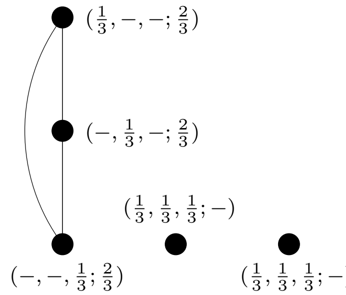

Figure 5.1: A fractional $(1, \frac{2}{3})$-colouring of $K_3 + 2K_1$.

For example, Figure 5.1 shows that the fractional cochromatic number of K_3+2K_1 is at most $\frac{5}{3}$, thus strictly less than the cochromatic number, which is 2. We remark that we see a difference to the fractional chromatic number. The fractional chromatic number is equal to the chromatic number for all perfect graphs. However, as K_3+2K_1 is a perfect graph, we see that the same does not hold for the fractional cochromatic number.

To obtain bounds on the fractional cochromatic number, we generalize two results for the fractional chromatic number. The fractional chromatic number of a graph G can be bounded from below by $\frac{|V(G)|}{\alpha(G)}$ with equality for vertex-transitive graphs [53].

Proposition 5.1.4. *Let G be a graph. Then*

$$\chi_f^c(G) \geq \frac{|V(G)|}{\max\{\alpha(G), \omega(G)\}},$$

with equality if G is vertex-transitive.

Proof. Suppose we have a fractional (k, l)-colouring of G with $k + l = \chi_f^c(G)$. Denote $\mathcal{T}(G) = \mathcal{S}(G) \cup \mathcal{C}(G)$ and $\mathcal{T}(G, v) = \mathcal{S}(G, v) \cup \mathcal{C}(G, v)$ for all $v \in V(G)$. We calculate

$$\sum_{v \in V(G)} \sum_{T \in \mathcal{T}(G,v)} w(T) \geq \sum_{v \in V(G)} 1 = |V(G)|.$$

On the other hand,

$$\begin{aligned}
\sum_{v\in V(G)} \sum_{T\in\mathcal{T}(G,v)} w(T) &= \sum_{T\in\mathcal{T}(G)} \sum_{v\in T} w(T) \\
&= \sum_{T\in\mathcal{T}(G)} w(T)\,|T| \\
&\leq \max\{\alpha(G),\omega(G)\} \sum_{T\in\mathcal{T}(G)} w(T) \\
&= \max\{\alpha(G),\omega(G)\}\,\chi_f^c(G).
\end{aligned}$$

Combining these two inequalities yields

$$\chi_f^c(G) \geq \frac{|V(G)|}{\max\{\alpha(G),\omega(G)\}}.$$

Now suppose G is vertex-transitive. We need to find a fractional (k,l)-colouring with $k+l = \frac{|V(G)|}{\max\{\alpha(G),\omega(G)\}}$. To that purpose we choose the family of sets $\mathcal{T}$ consisting of all the independent sets and cliques of order $\max\{\alpha(G),\omega(G)\}$, each with weight $\frac{|V(G)|}{|\mathcal{T}|\cdot\max\{\alpha(G),\omega(G)\}}$. Given a vertex v, we have, by counting in two ways,

$$|\{T\in\mathcal{T} \mid v\in T\}|\cdot|V(G)| = |\mathcal{T}|\cdot\max\{\alpha(G),\omega(G)\}.$$

Therefore the total weight of all the sets covering a vertex is equal to 1 and we have defined a fractional cocolouring of G. The total weight over all the sets in $\mathcal{T}$ is

$$|\mathcal{T}|\cdot\frac{|V(G)|}{|\mathcal{T}|\cdot\max\{\alpha(G),\omega(G)\}} = \frac{|V(G)|}{\max\{\alpha(G),\omega(G)\}}.$$ □

Considering Figure 5.1 yet again, this proves that $\chi_f^c(K_3+2K_1)=\frac{5}{3}$.

Corollary 5.1.5. *For any square graph G on r^2 vertices,*

$$\chi_f^c(G)=\chi^b(G)=r.$$

Proof. The second equality is the definition of square graphs (see Section 3.2). By Lemma 3.3.1, $\chi^b(G)=\chi^c(G)$ for all square graphs G and therefore $\chi(G)=\theta(G)=r$, implying $\alpha(G)=\omega(G)$. Using Proposition 5.1.4, we then obtain

$$r=\chi^c(G)\geq\chi_f^c(G)\geq\frac{r^2}{r}=r.$$

The result follows. □

Corollary 5.1.6. *Let G be a vertex-transitive graph. Then*

$$\chi_f^c(G) = \min\{\chi_f(G), \theta_f(G)\}.$$

Proof. For vertex-transitive graphs we have $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$ and $\theta_f(G) = \frac{|V(G)|}{\omega(G)}$ [53]. Therefore

$$\chi_f^c(G) = \frac{|V(G)|}{\max\{\alpha(G), \omega(G)\}} = \min\left\{\frac{|V(G)|}{\alpha(G)}, \frac{|V(G)|}{\omega(G)}\right\} = \min\{\chi_f(G), \theta_f(G)\}. \quad \square$$

In general, equality does not hold in the previous corollary. For example, consider the graph $G = K_m + (m-1)K_1$ for $m \geq 2$. For a fractional $(1, 1 - \frac{1}{m})$-colouring of G we choose the m independent sets containing $(m-1)K_1$ and one vertex of K_m, each with weight $\frac{1}{m}$. If we add the clique consisting of K_m with weight $\frac{m-1}{m}$, each vertex is covered exactly once. The total weight of the selected sets is $\frac{2m-1}{m}$, which equals the lower bound from Proposition 5.1.4. Thus,

$$\chi_f^c(G) = \frac{2m-1}{m} < 2.$$

However, since $\chi(G) = \alpha(G) = m$ and $\theta(G) = \omega(G) = m$, we have

$$\chi_f(G) = \theta_f(G) = m.$$

The gap between the fractional cochromatic number and the minimum of the fractional chromatic number and the fractional clique covering number can therefore be arbitrarily large.

5.1.3 Fractional bichromatic number

Definition 5.1.7. The *fractional bichromatic number* of a graph G is defined by

$$\chi_f^b(G) = \min\{r \mid \forall k, l \geq 0, k + l = r : G \text{ admits a fractional } (k,l)\text{-colouring}\}.$$

It turns out that the fractional bichromatic number only depends on the fractional chromatic and the fractional clique covering number.

Proposition 5.1.8. *The fractional bichromatic number of a graph G is*

$$\chi_f^b(G) = \max\{\chi_f(G), \theta_f(G)\}.$$

Proof. By definition, we have $\chi_f^b(G) \geq \max\{\chi_f(G), \theta_f(G)\}$. For the other inequality, let

$$r = \max\{\chi_f(G), \theta_f(G)\}$$

and let w_1 be a fractional $(r, 0)$-colouring and w_2 a fractional $(0, r)$-colouring of G. Given $k + l = r$, we will construct a fractional (k, l)-colouring w of G from w_1 and w_2. For every independent set $S \in \mathcal{S}(G)$, we set

$$w(S) = \frac{k}{r} w_1(S)$$

and for every clique $C \in \mathcal{C}(G)$, we set

$$w(C) = \frac{l}{r} w_2(C).$$

Then we have for each vertex $v \in V(G)$,

$$\begin{aligned}
\sum_{S \in \mathcal{S}(G,v)} w(S) + \sum_{C \in \mathcal{C}(G,v)} w(C) &= \sum_{S \in \mathcal{S}(G,v)} \frac{k}{r} w_1(S) + \sum_{C \in \mathcal{C}(G,v)} \frac{l}{r} w_2(C) \\
&\geq \frac{k}{r} + \frac{l}{r} \\
&= 1
\end{aligned}$$

and

$$\sum_{S \in \mathcal{S}(G)} w(S) = \sum_{S \in \mathcal{S}(G)} \frac{k}{r} w_1(S) = \frac{k}{r} r = k$$

$$\sum_{C \in \mathcal{C}(G)} w(C) = \sum_{C \in \mathcal{C}(G)} \frac{l}{r} w_2(C) = \frac{l}{r} r = l.$$

Therefore w is a fractional (k, l)-colouring of G. □

Corollary 5.1.9. *Let G be a graph. Then*

$$\chi^b(G) \geq \chi_f^b(G).$$

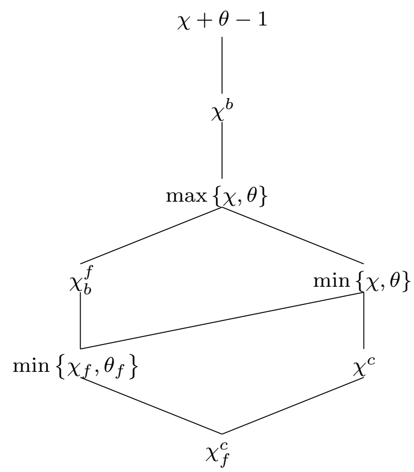

Figure 5.2: Relationships among the colouring parameters.

Proof. Since $\chi^b(G) \geq \max\{\chi(G), \theta(G)\}$ and both the chromatic number and the clique covering number are bounded from below by their fractional versions, the result follows. □

Combining the results for the fractional bichromatic number with the results for the fractional cochromatic number, we obtain the following proposition.

Proposition 5.1.10. *Let G be a graph. Then*

$$\chi^b(G) \geq \max\{\chi(G), \theta(G)\} \geq \max\{\chi_f(G), \theta_f(G)\} =$$
$$= \chi_f^b(G) \geq \min\{\chi_f(G), \theta_f(G)\} \geq \chi_f^c(G).$$ □

Figure 5.2 summarizes the observations about the comparisons among the chromatic number, clique covering number, cochromatic number, bichromatic number and their respective fractional versions, whereas Figure 5.3 does the same for the special cases of vertex-transitive respectively perfect graphs.

No other comparabilities exist in general for these classes of graphs, as can be seen from the examples in Table 5.1. To read Table 5.1, consider the first row. The cycle C_6 is a graph for which $\chi + \theta - 1 > \chi^b$, while for C_4, we have $\chi + \theta - 1 = \chi^b$. There exists no graph with $\chi + \theta - 1 < \chi^b$.

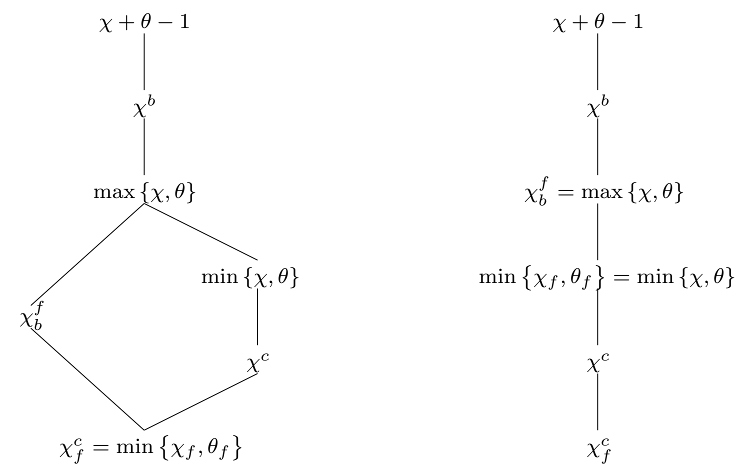

Figure 5.3: Relationships among the colouring parameters for vertex-transitive and perfect graphs.

The only difficult case arising in the table is finding a vertex-transitive graph G with

$$\min\{\chi(G), \theta(G)\} > \chi^c(G).$$

To that end, consider the Kneser graph $K(7,3)$ on 35 vertices (which is defined as the complement of the intersection graph of all 3-subsets of $\{1, 2, \ldots, 7\}$). Its chromatic number is 3 [47], whereas the clique number is 2, showing that the clique covering number must be at least 18. We can $(1, 10)$-colour $K(7,3)$ by choosing a maximum independent set (which corresponds to a family of 3-subsets, all containing a fixed element of $\{1, \ldots, 7\}$). The remaining graph is $K(6,3)$, which is the complement of $10K_2$. Now consider the vertex-transitive graph $K_6[K(7,3)]$ (the lexicographic product of K_6 with $K(7,3)$ - see Section 5.2). We obtain

$$\chi(K_6[K(7,3)]) = 6\chi(K(7,3)) = 18$$

and

$$\theta(K_6[K(7,3)]) = \theta(K(7,3)) \geq 18.$$

However, we can $(6, 10)$-colour $K_6[K(7,3)]$ by blowing up the $(1, 10)$-colouring of

$K(7,3)$ (for details, see Proposition 5.2.15), therefore this graph satisfies the intended inequality.

	$>$	$=$	$<$	
$\chi + \theta - 1$	C_6	C_4	$\times$	χ^b
χ^b	C_4	C_6	$\times$	$\max\{\chi, \theta\}$
$\max\{\chi, \theta\}$	C_5	C_4	$\times$	χ_f^b
$\max\{\chi, \theta\}$	C_6	C_4	$\times$	$\min\{\chi, \theta\}$
χ_f^b	C_6	C_4	$\times$	$\min\{\chi_f, \theta_f\}$
χ_f^b	C_6	C_4	C_5	$\min\{\chi, \theta\}$
χ_f^b	C_6	C_4	C_5	χ^c
$\min\{\chi, \theta\}$	C_5	C_4	$\times$	$\min\{\chi_f, \theta_f\}$
$\min\{\chi, \theta\}$	$K_3 + 2K_1, K_6[K(7,3)]$	C_4	$\times$	χ^c
$\min\{\chi_f, \theta_f\}$	$K_3 + 2K_1$	C_4	C_5	χ^c
$\min\{\chi_f, \theta_f\}$	P_3	C_4	$\times$	χ_f^c
χ^c	C_5, P_3	C_4	$\times$	χ_f^c

Table 5.1: Examples for comparabilities among the colouring parameters.

5.2 Lexicographic product

In this section, we will investigate the behaviour of the bichromatic and cochromatic number with respect to the lexicographic product. Of the four standard graph products (the others being the Cartesian, direct and strong product), the lexicographic product is the most natural to consider in relation to (k, l)-colourings. The main reason is that the lexicographic product is self-complementary, that is, the complement of a lexicographic product of two graphs is a lexicographic product in itself. For a comprehensive introduction to graph products, we refer to [42].

Definition 5.2.1. Let G, H be graphs. The *lexicographic product* $G[H]$ is the graph with vertex set

$$V(G[H]) = V(G) \times V(H)$$

and edge set $E(G[H])$ defined by

$$(u_1, v_1)(u_2, v_2) \in E(G[H]) \Leftrightarrow (u_1u_2 \in V(G)) \text{ or } (u_1 = u_2 \text{ and } v_1v_2 \in E(H)).$$

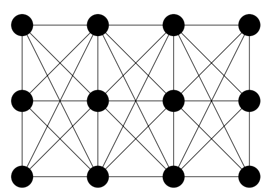

Figure 5.4: The graph $P_4[P_3]$.

The lexicographic product $P_4[P_3]$ is shown in Figure 5.4. The lexicographic product is a special case of the *composition* of graphs. The composition of a graph G with graphs $\{H_v \mid v \in V(G)\}$ is the graph obtained from G by replacing each vertex $v \in V(G)$ by the graph H_v and making each vertex of H_v adjacent to all vertices in all H_w with $vw \in E(G)$. Thus we obtain the lexicographic product $G[H]$ if we let $H_v = H$ for all $v \in V(G)$. An interesting property of composition is that it can be used to provide yet another characterization of cographs. Both disjoint union and join of graphs can be represented via composition. So is the disjoint union of n graphs the same as the composition of $\overline{K_n}$ with those graphs, while the join equals the composition of K_n with the graphs. Therefore we can characterize the class of cographs as the class of graphs that can be obtained by recursively applying composition to independent sets and cliques. Most of the results presented in this section can easily be generalized for the composition of graphs. For the sake of clarity, we only present the results for the lexicographic product, however.

We first give a brief overview of some basic properties of lexicographic products, in particular with respect to colouring parameters, before turning to new results about (k, l)-colourings.

As mentioned, the lexicographic product is self-complementary. To be precise, for any graphs G, H,

$$\overline{G[H]} = \overline{G}[\overline{H}],$$

which can easily be deduced from the definition. As well, the lexicographic product is associative, but not commutative in general.

The following two concepts will be useful in the discussion of lexicographic products.

Definition 5.2.2. Let G, H be graphs. For $v \in V(G)$, we denote by H_v the subgraph of $G[H]$ induced by $\{v\} \times V(H)$. For any set $T \subseteq V(G[H])$, we define the *restriction to H_v* by

$$r_v(T) = T \cap V(H_v)$$

and the *projection onto G* by

$$p_G(T) = \{v \in V(G) \mid r_v(T) \neq \emptyset\}.$$

We consider H_v to be an induced subgraph of G. Even though $H_v \cong H$ for all $v \in V(G)$, it will be useful to distinguish between the different copies of H.

Let us consider independent sets and cliques with respect to the lexicographic product. The definition directly implies that independent sets and cliques are multiplicative, in the sense that if S is an independent set in G and T an independent set in H, then $S \times T$ is an independent set in $G[H]$ (and the same for cliques). While the same can be said for the strong product, cliques are not multiplicative in the Cartesian and direct product, which is another reason that they are less interesting with respect to (k, l)-colourings. Now consider an independent set S in $G[H]$. Clearly, the restriction $r_v(S)$ is an independent set in H_v for all $v \in V(G)$. Similarly the projection $p_G(S)$ is an independent set in G, as two vertices in different copies H_v, H_w are nonadjacent if and only if v and w are nonadjacent. Combining this with the multiplicativity of independent sets, we obtain that the independence number is multiplicative, that is,

$$\alpha(G[H]) = \alpha(G)\alpha(H).$$

Using the same argument for cliques instead of independent sets, we also obtain

$$\omega(G[H]) = \omega(G)\omega(H).$$

Turning to the chromatic number and clique covering number, we observe that the multiplicativity of independent sets and cliques provides us with a construction for the colouring and clique covering of lexicographic products. Suppose $S_1, S_2, \ldots, S_r$ is a colouring of G and $T_1, T_2, \ldots, T_s$ is a colouring of H, then the sets $S_i \times T_j$ are independent sets in $G[H]$ and partition $V(G[H])$, thus form a colouring of $G[H]$. We obtain that the chromatic number is submultiplicative, that is,

$$\chi(G[H]) \leq \chi(G)\chi(H),$$

and similarly

$$\theta(G[H]) \leq \theta(G)\theta(H).$$

We note that these inequalities can be strict. For example,

$$\chi(C_5[K_2]) = 5 < 6 = \chi(C_5)\chi(K_2).$$

Lower bounds for $\chi(G[H])$ are harder to come by. For a few examples, see [42]. We just mention one bound involving the fractional chromatic number:

$$\chi(G[H]) \geq \chi_f(G)\chi(H).$$

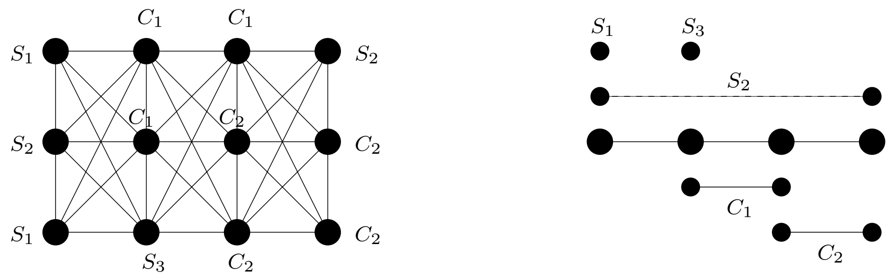

Figure 5.5: A $(3, 2)$-colouring of $P_4[P_3]$ and the projection of its sets onto P_4.

We will find an equivalent of this bound for the cochromatic number. We remark that for perfect graphs G, H, we always have $\chi(G[H]) = \chi(G)\chi(H)$, as the chromatic number equals the clique number, which itself is multiplicative. As an interesting aside, the fractional chromatic number itself is multiplicative [53], that is

$$\chi_f(G[H]) = \chi_f(G)\chi_f(H).$$

The analogous result does not hold for the fractional cochromatic number, as we will see.

The situation becomes more complicated when considering (k, l)-colourings. The construction for a colouring of $G[H]$ combining independent sets of G and H cannot be repeated for (k, l)-colourings, as we are dealing with independent sets and cliques at the same time. Thus a construction must be more subtle. To start, we make observations about the effect of the restriction and projection on (k, l)-colourings.

Lemma 5.2.3. *Let $S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$ be a (k, l)-colouring of a graph $G[H]$ and $v \in V(G)$. Then the nonempty sets among $r_v(S_1), \ldots, r_v(S_k), r_v(C_1), \ldots, r_v(C_l)$ form a (k_v, l_v)-colouring of H_v, where*

$$k_v = |\{i \mid r_v(S_i) \neq \emptyset\}|, \qquad l_v = |\{j \mid r_v(C_j) \neq \emptyset\}|.$$

Proof. Every vertex of H_v appears in exactly one of $S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$, thus in exactly one of $r_v(S_1), \ldots, r_v(S_k), r_v(C_1), \ldots, r_v(C_l)$. Thus the nonempty sets among these form a (k_v, l_v)-colouring of H_v. □

For illustration, consider the left side of Figure 5.5, in which a $(3, 2)$-colouring has been depicted. Each copy of P_3 shows a different (k, l)-colouring. Now consider the projection onto P_4 of the sets shown. The five sets cover all vertices of P_4, in fact, they cover all vertices more than once. If we look at a particular vertex, the sets containing that vertex originate from sets that form a (k, l)-colouring of the corresponding copy of P_3 in $P_4[P_3]$. We formulate this observation in the following definition and lemma.

Definition 5.2.4. Let G, H be graphs. An *H-projective (k, l)-colouring* of a graph G is a collection of independent sets $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k$ and cliques $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_l$ in G such that H is (k_v, l_v)-colourable for all $v \in V(G)$, where

$$k_v = \left| \left\{ i \;\middle|\; v \in \tilde{S}_i \right\} \right|, \qquad l_v = \left| \left\{ j \;\middle|\; v \in \tilde{C}_j \right\} \right|.$$

We remark that we use the tilde for the independent sets and cliques to distinguish the independent sets and clique in G from those in $G[H]$.

Lemma 5.2.5. *For any graphs G, H, the product $G[H]$ is (k, l)-colourable if and only if G has an H-projective (k, l)-colouring.*

Proof. Suppose $S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$ is a (k, l)-colouring of $G[H]$. Set

$$\tilde{S}_i = p_G(S_i), \qquad \tilde{C}_j = p_G(C_j)$$

for all i, j. Then for all $v \in V(G)$,

$$k_v = \left\{ i \;\middle|\; v \in \tilde{S}_i \right\} = \{ i \mid r_v(S_i) \neq \emptyset \}, \quad l_v = \left\{ j \;\middle|\; v \in \tilde{C}_j \right\} = \{ j \mid r_v(C_j) \neq \emptyset \}.$$

By Lemma 5.2.3, $H \cong H_v$ is (k_v, l_v)-colourable.

On the other hand, suppose that $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k, \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_l$ is an H-projective (k, l)-colouring of G. For every $v \in V(G)$, let

$$I^v = \left\{ i \;\middle|\; v \in \tilde{S}_i \right\}, \qquad J^v = \left\{ j \;\middle|\; v \in \tilde{C}_j \right\}.$$

For each v, let $\{ S_i^v \mid i \in I^v \}$ be independent sets and $\{ C_j^v \mid j \in J^v \}$ be cliques forming a (k_v, l_v)-colouring of H_v. For all i, j, set

$$S_i = \bigcup_{v \in V(G)} S_i^v, \qquad C_j = \bigcup_{v \in V(G)} C_j^v.$$

to obtain independent sets and cliques forming a (k, l)-colouring of $G[H]$. □

Lemma 5.2.5 gives us an easy proof of an analogue of a result for the chromatic number [42], stating that if graphs H, H' have the same chromatic number, then $\chi(G[H]) = \chi(G[H'])$ for all graphs G. Recall that $\boldsymbol{\kappa}(G) = (\kappa_0(G), \kappa_1(G), \dots)$, where $\kappa_l(G)$ is the minimum k such that G is (k,l)-colourable.

Proposition 5.2.6. *For any graphs G, H, H' with $\boldsymbol{\kappa}(H) = \boldsymbol{\kappa}(H')$,*

$$\boldsymbol{\kappa}(G[H]) = \boldsymbol{\kappa}(G[H']).$$

Proof. It suffices to show that $G[H]$ being (k,l)-colourable implies $G[H']$ being (k,l)-colourable for all k, l. Suppose that $G[H]$ is (k,l)-colourable. Then by Lemma 5.2.5, G has a projective H-colouring. Since $\boldsymbol{\kappa}(H) = \boldsymbol{\kappa}(H')$, H is (k_v, l_v)-colourable if and only if H' is (k_v, l_v)-colourable. Therefore a projective H-colouring is also a projective H'-colouring, implying that $G[H']$ is (k,l)-colourable. □

The similar result with G and H reversed does not hold. So is $\boldsymbol{\kappa}(P_3) = \boldsymbol{\kappa}(P_4) = (2,1)$, but $\boldsymbol{\kappa}(P_3[P_3]) = (4,3,1,1)$, while $\boldsymbol{\kappa}(P_4[P_3]) = (4,3,2,1)$.

We now consider the cochromatic number. In contrast to the chromatic number and the clique covering number, the cochromatic number is not submultiplicative, that is, there exist graphs G, H with $\chi^c(G[H]) > \chi^c(G)\chi^c(H)$. Consider for example the graph $\overline{K_n}[K_n]$ (the disjoint union of n copies of K_n). The cochromatic number of both $\overline{K_n}$ and K_n is one, while the cochromatic number of the product is n. However, we can give an upper bound involving the chromatic number and clique covering number by adapting the construction for colourings given earlier.

Proposition 5.2.7. *For any graphs G, H,*

$$\chi^c(G[H]) \leq \max\{\chi(G), \theta(G)\} \cdot \chi^c(H).$$

Proof. Let $S_1, S_2, \dots, S_{\chi(G)}$ be a colouring of G and $C_1, C_2, \dots, C_{\theta(G)}$ be a clique covering of G. Furthermore, let $T_1, T_2, \dots, T_k, D_1, D_2, \dots, D_l$ be a (k,l)-colouring of H with $k + l = \chi^c(H)$. Then the vertex sets $S_i \times T_j$ are independent sets in $G[H]$ for all i, j, while the sets $C_i \times D_j$ are cliques in $G[H]$ for all i, j. Combined, they partition the vertex set of $G[H]$, thus form a $(k\chi(G), l\theta(G))$-colouring of $G[H]$. Therefore

$$\begin{aligned} \chi^c(G[H]) &\leq k\chi(G) + l\theta(G) \\ &\leq \max\{\chi(G), \theta(G)\} \cdot (k+l) \\ &= \max\{\chi(G), \theta(G)\} \cdot \chi^c(H). \end{aligned}$$ □

We can swap the roles of G and H in this result, resulting in the following proposition.

Proposition 5.2.8. *For any graphs G, H,*

$$\chi^c(G[H]) \leq \chi^c(G) \cdot \max\{\chi(H), \theta(H)\}.$$

Proof. Repeat the proof of Proposition 5.2.7 with a minimum colouring and minimum clique covering of H and a (k, l)-colouring of G (where $k + l = \chi^c(G)$) instead. □

We remark that the bounds in Propositions 5.2.7 and 5.2.8 are sharp for some graphs. One product for which the bounds are sharp is $P_4[P_4]$. As seen in Section 3.2, the cochromatic number of this graph is 4. Later in this section, we will see a few graph classes that satisfy equality for all the bounds given in this section.

That it is not possible to bound $\chi^c(G[H])$ from below by $\chi^c(G)\chi^c(H)$ will be seen later in Proposition 5.2.21, which proves that $\chi^c(C_5[C_5]) = 8$. Interestingly, we can almost bound $\chi^c(G[H])$ from below by $\chi^c(G)\chi^c(H)$ if we replace the cochromatic number of G by the fractional cochromatic number of G.

Proposition 5.2.9. *For any graphs G, H,*

$$\chi^c(G[H]) \geq \chi^c_f(G)\chi^c(H).$$

Proof. Consider k, l such that $G[H]$ is (k, l)-colourable and $k + l = \chi^c(G[H])$. By Lemma 5.2.5, G has an H-projective (k, l)-colouring $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k, \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_l$. Then we have for all $v \in V(G)$,

$$\left|\left\{i \,\middle|\, v \in \tilde{S}_i\right\}\right| + \left|\left\{j \,\middle|\, v \in \tilde{C}_j\right\}\right| = k_v + l_v \geq \chi^c(H).$$

Now weight each independent set $\tilde{S}_i$ and clique $\tilde{C}_j$ with $\frac{1}{\chi^c(H)}$. For every vertex $v \in V(G)$, we obtain

$$\sum_{S_i \ni v} w(S_i) + \sum_{C_j \ni v} w(C_j) = \frac{k_v}{\chi^c(H)} + \frac{l_v}{\chi^c(H)} \geq 1.$$

Therefore we have defined a fractional $(\frac{k}{\chi^c(H)}, \frac{l}{\chi^c(H)})$-colouring of G. Thus

$$\chi_f^c \leq \frac{k}{\chi^c(H)} + \frac{l}{\chi^c(H)} = \frac{\chi^c(G[H]}{\chi^c(H)}. \qquad \square$$

We will later see that this bound is sharp, for example if $G = P_4$ and H is a graph with $\chi^b(H) = \chi^c(H)$ (see Proposition 5.2.17). The same proof technique as for Proposition 5.2.9 can be used to get a similar result for the fractional cochromatic number. We omit the proof, which is nearly identical to the previous one without providing new insights.

Proposition 5.2.10. *For any graphs G, H,*

$$\chi_f^c(G[H]) \geq \chi_f^c(G)\chi_f^c(H). \qquad \square$$

Finally, we consider the bichromatic number. As was the case for the chromatic number and clique covering number and contrasting to the cochromatic number, the bichromatic number turns out to be submultiplicative, as the next proposition shows.

Proposition 5.2.11. *For any graphs G, H,*

$$\chi^b(G[H]) \leq \chi^b(G)\chi^b(H).$$

Proof. We need to show that $G[H]$ is (k,l)-colourable for all k, l with $k + l = \chi^b(G)\chi^b(H)$, or equivalently by Lemma 5.2.5, that G has an H-projective (k,l)-colouring for all such k, l.

Let k, l be given with $k + l = \chi^b(G)\chi^b(H)$. Then we can write k as

$$k = r\chi^b(G) + s,$$

with $0 \leq r < \chi^b(H)$ and $0 \leq s \leq \chi^b(G)$. By the definition of the bichromatic number, G has a $(\chi^b(G), 0)$-colouring

$$S_1, S_2, \ldots, S_{\chi^b(G)},$$

a $(0, \chi^b(G))$-colouring

$$C_1, C_2, \ldots, C_{\chi^b(G)}$$

and an $(s, \chi^b(G) - s)$-colouring

$$T_1, T_2, \ldots, T_s, D_1, D_2, \ldots, D_{\chi^b(G)-s}.$$

Consider the collection of independent sets and cliques arising by taking r copies of each S_i, $\chi^b(H) - r - 1$ copies of each C_i and one copy of each T_i and D_i. Each vertex $v \in V(G)$ is contained in exactly $\chi^b(H)$ of those sets, implying that H is (k_v, l_v)-colourable for all $v \in V(G)$, where k_v is the number of independent sets and l_v is the number of cliques containing v. Furthermore, we have exactly k independent sets and l cliques, proving that the collection of independent sets and cliques is an H-projective (k, l)-colouring of G. □

We remark that the bichromatic number $\chi^b(G[H])$ is not in general of the order of $\chi^b(G)\chi^b(H)$, as we can see by considering $\overline{K_n}[K_n]$ yet again. The bichromatic number of both $\overline{K_n}$ and K_n is n, while the bichromatic number of the product (being a box cograph) is $2n - 1$. In fact, no cograph attains equality in Proposition 5.2.11, as we will show below.

5.2.1 Classes closed under the lexicographic product

We mentioned in the beginning of the section that cographs can be defined using composition of graphs. Thus the class of cographs is closed under composition, and in particular under taking the lexicographic product. We can show that the same holds for the class of box cographs (see Subsection 3.1.2).

Proposition 5.2.12. *Let $G \in \mathscr{B}(r, s)$ and $H \in \mathscr{B}(r', s')$. Then $G[H] \in \mathscr{B}(rr', ss')$.*

Proof. Since both G and H are cographs, so is $G[H]$. Both chromatic number and clique covering number are multiplicative, therefore

$$\chi(G[H]) = \chi(G)\chi(H) = rr', \qquad \theta(G[H]) = \theta(G)\theta(H) = ss'.$$

Finally,

$$|V(G[H])| = |V(G)| \cdot |V(H)| = rsr's' = (rr')(ss').$$ □

We can now show that cographs do not attain equality in Proposition 5.2.11, unless one cograph is K_1 or either both cographs are complete graphs or both cographs are edgeless graphs. We only need to consider box cographs, as every cograph contains an induced box cograph of the same bichromatic number by Corollary 4.1.4. Consider the difference between $\chi^b(G)\chi^b(H)$ and $\chi^b(G[H])$ for two box cographs $G \in \mathscr{B}(r, s)$ and $H \in \mathscr{B}(r', s')$. A brief calculation yields

$$\begin{aligned}\chi^b(G)\chi^b(H) - \chi^b(G[H]) &= (r + s - 1)(r' + s' - 1) - (rr' + ss' - 1)\\ &= (r - 1)(s' - 1) + (r' - 1)(s - 1).\end{aligned}$$

Since r, r', s, s' are positive, the difference is always nonnegative. Furthermore, the difference is zero if and only one of the graphs is K_1, both graphs are complete or both graphs are edgeless.

Combining the results for the bichromatic number and the cochromatic number allows us to expand on the class of graphs for which the bichromatic number equals the cochromatic number (see Section 3.3).

Proposition 5.2.13. *For any graphs G, H with $\chi^b(G) = \chi^c_f(G)$ and $\chi^b(H) = \chi^c(H)$,*

$$\chi^b(G[H]) = \chi^c(G[H]) = \chi^b(G)\chi^b(H).$$

Proof. The statement follows directly from Propositions 5.2.9 and 5.2.11. □

In particular, we can apply Proposition 5.2.13 to the class of square graphs. Recall that a square graph is a graph on r^2 vertices with $\chi^b(G) = r$ (see Section 3.2).

Proposition 5.2.14. *Let G be a square graph on n^2 vertices and H be a square graph on m^2 vertices. Then $G[H]$ is a square graph on $(mn)^2$ vertices.*

Proof. That $G[H]$ has $(mn)^2$ vertices follows from the definition of the lexicographic product. By 5.1.5, the fractional cochromatic number equals the bichromatic number for all square graphs. Therefore Proposition 5.2.13 implies $\chi^b(G[H]) = mn$, proving that $G[H]$ is a square graph. □

5.2.2 Lexicographic product of particular graphs

We finish this section with a few results regarding the lexicographic product of particular graphs. The easiest case is if the outer graph is a clique or edgeless graph.

Proposition 5.2.15. *For any positive integer n and any graph H,*

$$\kappa_l(K_n[H]) = n\kappa_l(H).$$

Proof. Consider a H-projective (k, l)-colouring of K_n. Without loss of generality, we may assume that all the cliques are maximal, and thus contain every vertex of K_n. To assure that H is (k_v, l_v)-colourable for every $v \in V(K_n)$, we therefore need at least $\kappa_l(H)$ independent sets containing v. As each independent set can only consist of a single vertex, we obtain $k \geq n\kappa_l(H)$. It is clear that $n\kappa_l(H)$ independent sets are also sufficient. □

Before continuing to P_4 and C_5, we define the lexicographic power of a graph.

Definition 5.2.16. For any graph G and positive integer n, the *n-th lexicographic power* of G is defined as

$$G^{[n]} = \underbrace{G[G[...[G]...]]}_{n}$$

Proposition 5.2.17. *For any graph H with $\chi^b(H) = \chi^c(H)$,*

$$\chi^b(P_4[H]) = \chi^c(P_4[H]) = 2\chi^b(H).$$

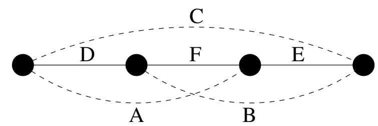

Figure 5.6: Maximal independent sets and cliques in P_4.

Proof. Since $\chi^b(P_4) = \chi_f^c(P_4) = 2$, the result follows from Proposition 5.2.13. □

Corollary 5.2.18. *For any positive integer n,*

$$\chi^b(P_4^{[n]}) = \chi^c(P_4^{[n]}) = 2^n.$$

Proof. Repeatedly applying Proposition 5.2.17 proves the statement. □

Proposition 5.2.19. *For any graph H,*

$$\kappa_l(P_4[H]) = \min_{i \leq l}(\kappa_i(H) + \kappa_{l-i}(H)).$$

Proof. By the symmetry of the statement, it is sufficient to consider $i \leq \frac{l}{2}$. We first show that P_4 is (k,l)-colourable for all

$$k = \kappa_i(H) + \kappa_{l-i}(H), \qquad i \leq \frac{l}{2},$$

or equivalently, by Lemma 5.2.5, that P_4 has an H-projective (k,l)-colouring. Consider the maximal independent sets of P_4 as shown in Figure 5.6. Take $\kappa_{l-i}(H)$ of types A and B each, $\kappa_i(H) - \kappa_{l-i}(H)$ independent sets of type C (recall that κ is a nonincreasing sequence), i cliques of types D and E each, and $l - 2i$ cliques of type F. In total we have k independent sets and l cliques. To show that this defines an H-projective (k,l)-colouring of P_4, we observe that we have

$$k_v = \kappa_i(H), \qquad l_v = i$$

for the two endvertices of P_4 and

$$k_v = \kappa_{l-i}(H), \qquad l_v = l - i$$

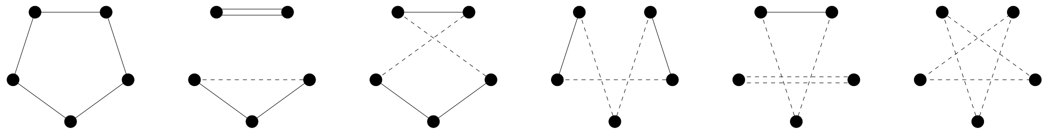

Figure 5.7: Covering C_5 twice with k independent sets and l cliques with $k+l=5$.

for the two central vertices of P_4. In both cases, H is (k_v, l_v)-colourable.

Now on the other hand suppose that we have given an H-projective (k, l)-colouring of P_4. We will show that

$$k \geq \kappa_i(H) + \kappa_{l-i}(H), \qquad i \leq l,$$

implying the statement. Without loss of generality we may assume that the H-projective (k, l)-colouring of P_4 only contains maximal independent sets and cliques. Suppose we have i cliques of type D and j cliques of type E (without loss of generality $i \leq j$). Then we must have $l-i-j$ cliques of type F. Furthermore, suppose we have a independent sets of type A, b independent sets of type B and $k-a-b$ independent sets of type C. As H must be (k_v, l_v)-colourable for each vertex, we obtain from the left of the two central vertices

$$b \geq \kappa_{l-j}(H) \geq \kappa_{l-i}(H),$$

where the last inequality is due to the fact that $\boldsymbol{\kappa}$ is nonincreasing and $i \leq j$. Furthermore, considering the leftmost vertex, we obtain

$$k - b = a + (k-a-b) \geq \kappa_i(H).$$

Combining the inequalities gives

$$k = b + (k-b) \geq \kappa_i(H) + \kappa_{l-i}(H). \qquad \square$$

Next we consider C_5. By Proposition 5.2.11, we know that $\chi^b(C_5[H]) \leq 3\chi^b(H)$ for all graphs H. In fact, we can improve on that result.

Proposition 5.2.20. *For any graph H,*

$$\chi^b(C_5[H]) \leq \left\lceil \frac{5}{2}\chi^b(H) \right\rceil.$$

Proof. By Lemma 5.2.5, we need to find an H-projective (k, l)-colouring for all k, l with $k + l = \left\lceil \frac{5}{2}\chi^b(H) \right\rceil$. It suffices to find a set of independent sets $S_1, S_2, \ldots, S_k$ and cliques $C_1, C_2, \ldots, C_l$ in C_5 that cover each vertex $\chi^b(H)$ times. We will do this five sets at a time (plus three sets at the end if $\chi^b(H)$ is odd). Figure 5.7 shows how to cover each vertex of C_5 exactly twice with five sets in any combination of independent sets and cliques. Similarly we can cover each vertex of C_5 exactly once with three independent sets and/or cliques. Combining these, we obtain an H-projective (k, l)-colouring of C_5. □

This result allows us to find yet more graphs with bichromatic number equal cochromatic number.

Proposition 5.2.21. *For any graph H with $\chi^b(H) = \chi^c(H)$,*

$$\chi^b(C_5[H]) = \chi^c(C_5[H]) = \left\lceil \frac{5}{2}\chi^b(H) \right\rceil.$$

Proof. Since $\chi^c_f(C_5) = \frac{5}{2}$, the result follows from Propositions 5.2.9 and 5.2.20. □

In particular, this allows us to find the bichromatic number and cochromatic number of any lexicographic power of C_5.

Corollary 5.2.22. *For any positive integer n,*

$$\chi^b(C_5^{[n]}) = \chi^c(C_5^{[n]}) = \left\lceil \frac{5}{2}\chi^b(C_5^{[n-1]}) \right\rceil.$$

Proof. The result follows directly from Proposition 5.2.21. □

Bibliography

[1] N. Alon, M. Krivelevich, and B. Sudakov, *Subgraphs with a large cochromatic number*, J. Graph Theory **25** (1997), no. 4, 295–297.

[2] N. Alon and U. Stav, *The maximum edit distance from hereditary graph properties*, J. Combin. Theory Ser. B **98** (2008), no. 4, 672–697.

[3] M. Axenovich, A. Kézdy, and R. Martin, *On the editing distance of graphs*, J. Graph Theory **58** (2008), no. 2, 123–138.

[4] C. Berge, *Les problèmes de coloration en théorie des graphes*, Publ. Inst. Statist. Univ. Paris **9** (1960), 123–160.

[5] B. Bhattacharya, P. Hell, and J. Huang, *A linear algorithm for maximum weight cliques in proper circular arc graphs*, SIAM J. Discrete Math. **9** (1996), no. 2, 274–289.

[6] B. Bollobás and A. Thomason, *Hereditary and monotone properties of graphs*, Algorithms Combin. **14** (1997), 70–78.

[7] A. Brandstädt, *Partitions of graphs into one or two independent sets and cliques*, Discrete Math. **152** (1996), no. 1-3, 47–54.

[8] A. Brandstädt, *Corrigendum: Partitions of graphs into one or two independent sets and cliques*, Discrete Math. **186** (1998), no. 1-3, 295.

[9] A. Brandstädt and D. Kratsch, *On partitions of permutations into increasing and decreasing subsequences*, Elektron. Informationsverarb. Kybernet. **22** (1986), no. 5-6, 263–273.

[10] A. Brandstädt, V.B. Le, and T. Szymczak, *The complexity of some problems related to graph 3-colorability*, Discrete Appl. Math. **89** (1998), no. 1-3, 59–73.

[11] R.S.F. Bravo, S. Klein, L.T. Nogueira, and F. Protti, *Characterization and recognition of P_4-sparse graphs partitionable into k independent sets and l cliques*, Discrete Appl. Math. **159** (2011), no. 4, 165–173.

[12] I. Broere and M. Burger, *Uniquely cocolourable graphs*, Graphs Combin. **2** (1986), no. 3, 201–208.

[13] R.L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. **37** (1941), 194–197.

[14] N.J. Calkin, P. Erdős, and C.A. Tovey, *New Ramsey bounds from cyclic graphs of prime order*, SIAM J. Discrete Math. **10** (1997), 381–387.

[15] D.G. Corneil, H. Lerchs, and L.K. Stewart Burlingham, *Complement reducible graphs*, Discrete Appl. Math **3** (1981), 163–174.

[16] R. de Souza Francisco, S. Klein, and L.T. Nogueira, *Characterizing (k,l)-partitionable cographs*, Electronic Notes in Discrete Math. **22** (2005), 277–280.

[17] M. Demange, T. Ekim, and D. de Werra, *Partitioning cographs into cliques and stable sets*, Discrete Optim. **2** (2005), 145–153.

[18] M. Demange, T. Ekim, and D. de Werra, *(p, k)-coloring problems in line graphs*, Theoret. Comput. Sci. **349** (2005), no. 3, 462–474.

[19] G.A. Dirac, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg **25** (1961), 71–76.

[20] T. Ekim and J. Gimbel, *Partitioning graphs into complete and empty graphs*, Discrete Math. **309** (2009), no. 19, 5849–5856.

[21] D.D.A. Epple and J. Huang, *A note on the bichromatic number of graphs*, J. Graph Theory **65** (2010), no. 2, 263–269.

[22] P. Erdős, P. Frankl, and V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent*, Graphs Combin. **2** (1986), no. 2, 113–121.

[23] P. Erdős and J. Gimbel, *Some problems and results in cochromatic theory*, Ann. Discrete Math. **55** (1993), 261–264.

[24] P. Erdős, J. Gimbel, and H.J. Straight, *Chromatic number versus cochromatic number in graphs with bounded clique number*, Europ. J. Combinatorics **11** (1990), 235–240.

[25] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar **1** (1966), 51–57.

[26] P. Erdős and A.H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091.

[27] T. Feder and P. Hell, *Matrix partitions of perfect graphs*, Discrete Math. **306** (2006), no. 19-20, 2450–2460.

[28] T. Feder, P. Hell, and W. Hochstättler, *Generalized colourings (matrix partitions) of cographs*, Graph Theory in Paris, Trends Math., Birkhäuser, Basel, 2007, pp. 149–167.

[29] T. Feder, P. Hell, S. Klein, and R. Motwani, *List partitions*, SIAM J. Discrete Math. **16** (2003), no. 3, 449–478.

[30] S. Földes and P.L. Hammer, *Split graphs having Dilworth number two*, Canad. J. Math. **29** (1977), no. 3, 666–672.

[31] D.R. Fulkerson and O.A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math. **15** (1965), 835–855.

[32] G. Gao and X. Zhu, *Star-extremal graphs and the lexicographic product*, Discrete Math. **152** (1996), no. 1-3, 147–156.

[33] M.R. Garey and D.S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-completeness. A Series of Books in the Mathematical Sciences*, W. H. Freeman and Co. San Francisco, Calif., 1979.

[34] D. Geller and S. Stahl, *The chromatic number and other functions of the lexicographic product*, J. Combin. Theory Ser. B **19** (1975), no. 1, 87–95.

[35] J. Gimbel, D. Kratsch, and L. Stewart, *On cocolourings and cochromatic numbers of graphs*, Discrete Appl. Math. **48** (1994), 111–127.

[36] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

[37] M. Grötschel, L. Lovász, and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), no. 2, 169–197.

[38] A. Hajnal and J. Surányi, *Über die Auflösung von Graphen in vollständige Teilgraphen*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **1** (1958), 113–121.

[39] F. Harary, *On the group of the composition of two graphs*, Duke Math. J. **26** (1959), 29–34.

[40] F. Harary and E.M. Palmer, *On acyclic simplicial complexes*, Mathematika **15** (1968), 115–122.

[41] P. Hell, S. Klein, L.T. Nogueira, and F. Protti, *Partitioning chordal graphs into independent sets and cliques*, Discrete Appl. Math. **141** (2004), no. 1-3, 185–194.

[42] W. Imrich and S. Klavžar, *Product Graphs, Structure and Recognition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley- Interscience, New York, 2000.

[43] A.E. Kézdy, H.S. Snevily, and C. Wang, *Partitioning permutations into increasing and decreasing subsequences*, J. Combin. Theory Ser. A **73** (1996), no. 2, 353–359.

[44] S. Klavžar and H. Yeh, *On the fractional chromatic number, the chromatic number, and graph products*, Discrete Math. **247** (2002), no. 1-3.

[45] L. Lesniak-Foster and H.J. Straight, *The cochromatic number of a graph*, Ars Combin. **3** (1977), 39–46.

[46] L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Discrete Math. **2** (1972), no. 3, 253–267.

[47] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A **25** (1978), no. 3, 319–324.

[48] J.B. Orlin, M.A. Bonucelli, and D.P. Bovet, *An $O(n^2)$ algorithm for coloring proper circular arc graphs*, SIAM J. Algebraic Disrete Methods **2** (1981), no. 2, 88–93.

[49] L. Ou, *Critically and minimally cochromatic graphs*, Ars Combin. **87** (2008), 23–31.

[50] H.J. Prömel and A. Steger, *Excluding induced subgraphs III: A general asymptotic*, Random Structures Algorithms **3** (1992), no. 1, 19–31.

[51] H.J. Prömel and A. Steger, *Excluding induced subgraphs II: Extremal graphs*, Discrete Appl. Math. **44** (1993), 283–294.

[52] S. P. Radziszowski, *Small Ramsey numbers*, Electronic J. Combinatorics (2009).

[53] E.R. Scheinerman and D.H. Ullman, *Fractional Graph Theory, A Rational Approach to the Theory of Graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1997.

[54] D. Seinsche, *On a property of the class of n-colorable graphs*, J. Combin. Theory Ser. B **16** (1974), 191–193.

[55] H.J. Straight, *Cochromatic number and the genus of a graph*, J. Graph Theory **3** (1979), no. 1, 43–51.

[56] A. Teng and A. Tucker, *An $O(qn)$ algorithm to q-color a proper family of circular arcs*, Discrete Math. **55** (1985), no. 2, 233–243.

[57] A. Tucker, *Characterizing circular-arc graphs*, Bull. Amer. Math. Soc **76** (1970), 1257–1260.

[58] A. Tucker, *Structure theorems for some circular-arc graphs*, Discrete Math. **7** (1974), 167–195.

[59] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, 2nd ed., Cambridge University Press, Cambridge, 2001.

[60] K. Wagner, *Monotonic coverings of finite sets*, Elektron. Informationsverarb. Kybernet. **20** (1984), 633–639.

[61] D.B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Inc., Upper Saddle River, NJ, 2001.