CLASSES OF SIGN NONSINGULAR MATRICES

WITH A SPECIFIED NUMBER OF ZERO ENTRIES

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Classes of Sign Nonsingular Matrices With a Specified Number of Zero Entries

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Abstract

An n-by-n matrix B_n is *sign nonsingular* (SNS) if every matrix with the same sign pattern as B_n is nonsingular. A given SNS matrix determines an equivalence class (with respect to transposition and multiplication by permutation and signature matrices) of SNS matrices, all of which have the same number of zero entries. Such a matrix is *maximal* if no zero entry can be set nonzero so that the resulting matrix is SNS, and is *fully indecomposable* if it does not have an $(n-k)$-by-k zero submatrix for some k, where $1 \leq k \leq n-1$. For fixed n, the Hessenberg matrix is known to represent the unique equivalence class with the minimum number of zero entries, namely $\binom{n-1}{2}$. We prove that for $n \geq 5$, there is exactly one equivalence class of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}+1$ zero entries. Similarly, for $n \geq 5$, we prove that there are exactly two such equivalence classes having $\binom{n-1}{2}+2$ zero entries. For these proofs, we identify two new infinite classes of fully indecomposable maximal SNS matrices, which can be obtained by stretching known SNS matrices.

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1 Introduction

We begin with notation and definitions and then collect together results, mainly from the literature, which we use in subsequent sections.

1.1 Definitions

We denote an m-by-n real matrix by $A_{mn} \equiv [a_{ij}]$ and abbreviate this to A_n if $m = n$. We let $|A_{mn}| \equiv [|a_{ij}|]$ and $\nu(A_{mn})$ denote the number of nonzero entries in A_{mn}. A matrix A_{mn} is *contained* in B_{mn} if $a_{ij} \neq 0$ implies $b_{ij} = a_{ij}$; note that $\nu(A_{mn}) \leq \nu(B_{mn})$. For an index set $\alpha \subseteq \{1, \ldots, n\}$, the *principal submatrix* of A_n lying in rows and columns indicated by α is denoted by $A[\alpha]$. We write $A[1, \ldots, k]$ rather than $A[\{1, \ldots, k\}]$.

A *signature matrix* is a diagonal matrix in which every diagonal entry is ± 1. The n-by-n *identity matrix* is denoted by I_n. It is often convenient to specify a *permutation matrix* P_n by a permutation of $\{1, \ldots, n\}$. Thus the matrix $P_n = (i_1, i_2, \ldots, i_n)$ has $p_{j,i_j} = 1$ for $j = 1, \ldots, n$ and all other entries zero [6]. For example, the permutation matrix $P_4 = (4, 1, 3, 2)$ has nonzero entries $p_{1,4} = p_{2,1} = p_{3,3} = p_{4,2} = 1$. In some cases, when specifying a permutation matrix with this notation, a segment of the permutation may be vacuous. For example, if $p = 1$ and $q = 0$, then the permutation matrix $P_{p+q+1} = (p+1, p+2, \ldots, p+q, p, 1, 2, \ldots, p-1, p+q+1) = (1, 2)$ as the segments $p+1, p+2, \ldots, p+q$ and $1, 2, \ldots, p-1$ are both vacuous.

There is an obvious one-to-one correspondence between n-by-n matrices and directed graphs (digraphs) on n vertices. Given A_n, the *associated digraph* $D(A_n)$

has vertices $\{1, \ldots, n\}$ and an arc (i, j) from vertex i to vertex j if and only if $a_{ij} \neq 0$; self loops are allowed but multiple arcs are not. As in [15], the *total degree* of vertex i in $D(A_n)$ is $d(i) = \sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |a_{ji}| - 2|a_{ii}|$. In the *associated signed digraph* $SD(A_n)$, the arc (i, j) is labelled with $\pm$ according as $sgn(a_{ij}) = \pm 1$. Note that the *sgn* of a matrix entry returns an integer from $\{-1, 0, +1\}$, whereas the *sgn* of an arc returns a sign from $\{-, +\}$.

For a digraph D, we let $V(D)$ and $E(D)$ denote the vertices and the arcs, respectively. A digraph D' is a *subdigraph* of D if $V(D') \subseteq V(D)$, $E(D') \subseteq E(D)$, and for every $(u, v) \in E(D')$ the vertices $u, v \in V(D')$; a subdigraph of a signed digraph is defined similarly, the signs of the arcs in D' being the same as in D.

A (*simple directed*) *path* of length $k \geq 1$, say $((v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1}))$, is a sequence of k arcs in which all the vertices v_i are distinct. In the case that $v_1 = v_{k+1}$, the sequence is a (*directed*) *cycle* of length $k \geq 1$. Paths and cycles are represented by their vertex sequences rather than their arc sequences. The (*directed*) *path sign* of a path in a signed digraph is a sign ascribed to the entire path and is defined as $(-1)^m$, where m is the number of arcs in the path that are negatively signed. The (*directed*) *cycle sign* is defined similarly.

A matrix A_n is *fully indecomposable* if it does not have an $(n-k)$-by-k zero submatrix for some k, where $1 \leq k \leq n-1$, otherwise it is *partly decomposable.* A matrix A_n with $n \geq 2$ is *irreducible* if there does not exist a permutation matrix P_n such that

$$P_n A_n P_n^T = \begin{bmatrix} D_r & E_{r,n-r} \\ O_{n-r,r} & F_{n-r} \end{bmatrix}$$

where $1 \le r \le n-1$ and $O_{n-r,r}$ is a zero matrix. It is well known that a matrix A_n is irreducible if and only if $D(A_n)$ is strongly connected. If, in addition, every entry on the main diagonal is nonzero, then A_n is fully indecomposable (see e.g. [3, Theorem 4.2.3]).

Definition 1.1.1 *A matrix A_n is* sign nonsingular (SNS) *if every matrix B_n is nonsingular where $sgn(b_{ij}) = sgn(a_{ij})$ for all i and j.*

Since the magnitude of each entry of a SNS matrix A_n is inconsequential, throughout this paper we take $a_{ij} \in \{-1, 0, +1\}$. A SNS matrix A_n is *normalized* if $a_{ii} = -1$ for all $i = 1, \ldots, n$. A zero entry a_{ij} in a SNS matrix A_n is an *essential zero* if when a_{ij} is replaced by ± 1, the matrix A_n is no longer SNS. If every zero entry in the SNS matrix A_n is essential, then the matrix is *maximal.*

Two matrices A_n and B_n are *equivalent* (or in the *same equivalence class*) if A_n can be transformed into B_n by any combination of transposition, multiplication by permutation matrices, and multiplication by signature matrices (see [2]). Thus, within each equivalence class of SNS matrices, there is a normalized SNS matrix. For small values of n, the number of equivalence classes is known (see e.g. [14] and Table 1).

1.2 Preliminary Results

The following digraph characterization of SNS matrices is due to Bassett, Maybee and Quirk.

Theorem 1.2.1 ([1, Lemma 3]) *A matrix A_n with every diagonal entry equal to -1 is* SNS *if and only if every cycle in $SD(A_n)$ has negative cycle sign.*

Using this characterization, Lady and Maybee [8] proved that each essential zero in the matrix corresponds to a path condition in the associated signed digraph.

Theorem 1.2.2 ([8, Corollary of Lemma 3]) *Let A_n be a normalized* SNS *matrix. Then $a_{ij} = 0$ is an essential zero if and only if there exist two paths in $SD(A_n)$ from vertex j to vertex i having opposite path signs.*

In Section 3, we use the following result, which shows that certain essential zeros in a principal submatrix of a normalized SNS matrix B_n cause certain other entries in B_n also to be essential zeros. Note that a principal submatrix of a normalized SNS matrix is necessarily normalized and SNS.

Lemma 1.2.3 *Let $n \geq 4$, $k \geq 2$ and B_n be a normalized* SNS *matrix. If $b_{1c} \neq 0$ and b_{ic} is an essential zero in $B[k, \dots, n]$, where $k \leq i, c \leq n$, then b_{i1} is an essential zero in B_n. If $b_{r1} \neq 0$ and b_{rj} is an essential zero in $B[k, \dots, n]$, where $k \leq r, j \leq n$, then b_{1j} is an essential zero in B_n.*

Proof: Assume b_{ic} is an essential zero in $B[k, \dots, n]$, where $k \leq i, c \leq n$. By Theorem 1.2.2, there exist two paths, $(c, u_1, u_2, \dots, i)$ and $(c, v_1, v_2, \dots, i)$, in $SD(B[k, \dots, n])$ and also in $SD(B_n)$ with opposite path signs. Appending $(1, c)$ onto both paths yields two new paths, $(1, c, u_1, u_2, \dots, i)$ and $(1, c, v_1, v_2, \dots, i)$, with opposite path signs in $SD(B_n)$. Thus, by Theorem 1.2.2, b_{i1} is an essential zero in B_n. A similar argument proves the second statement. ∎

Seymour and Thomassen [13] showed that there is a correspondence between SNS matrices and even digraphs. From their paper, we extract the following definitions and corollary (see [13, Theorem 4.1]).

Let C_3^* denote the digraph with vertices $\{v_1, v_2, v_3\}$ and arcs (v_i, v_j) for $i, j = 1, 2, 3$ and $i \neq j$. An *edge subdivision* of a digraph D is a digraph where, for $u \neq v$, some arc $(u, v) \in E(D)$ is replaced by a path L from u to v so that $V(D) \cap V(L) = \{u, v\}$. A *subdivision of* D is a digraph with some (possibly none) edge subdivisions of D.

Corollary 1.2.4 *Let A_n be a matrix with every diagonal entry equal to -1. If $D(A_n)$ has a subdigraph that is a subdivision of C_3^*, then A_n is not* SNS.

Brualdi and Shader [4], in their work on convertible matrices, showed that there is essentially a unique way to sign a given convertible matrix to obtain a SNS matrix. We use the following slight generalization of Theorem 2.1 of [4].

Corollary 1.2.5 *Let B_n and A_n be fully indecomposable* SNS *matrices such that $|B_n|$ is contained in $|A_n|$. Then there exist signature matrices T_n and U_n such that B_n is contained in $T_n A_n U_n$. If, in addition, B_n is maximal, then $B_n = T_n A_n U_n$.*

An n-by-n SNS matrix can be obtained from an $(n-1)$-by-$(n-1)$ SNS matrix by bordering. One of the common ways is a row/column stretch of a matrix (see [11, 12]). We modify the definition given there so that the bordering row and column are the first rather than the last.

Definition 1.2.6 *Let A_{n-1} be a matrix with entries from $\{-1, 0, +1\}$ and let r be a fixed integer, $1 \leq r \leq n-1$. Then the* row stretch *of A_{n-1}* on row *r is the* *matrix* $P_nB_nP_n^T$, *where* $P_n = (n, 1, 2, \ldots, n-1)$ *and* B_n *is defined as follows:*

$$\begin{aligned} B[1,\ldots,n-1] &= A_{n-1} \\ b_{nj} &= b_{rj} \quad \text{for } j = 1,\ldots,n-1 \\ b_{nn} &= -1 \\ b_{rn} &= +1 \\ b_{in} &= 0 \quad \text{for } i = 1,\ldots,n-1 \text{ and } i \neq r. \end{aligned}$$

The column stretch of A_{n-1} on column r *is similarly defined. Note that* $(P_nB_nP_n^T)[2,\ldots,n] = A_{n-1}$.

Proof of the following can be found in [7]; see [4, 12, 14] for the necessity.

Theorem 1.2.7 *Let* B_n *be the row «column» stretch of* A_{n-1} *on an arbitrary row «column». Then* A_{n-1} *is a fully indecomposable maximal* SNS *matrix if and only if* B_n *is a fully indecomposable maximal* SNS *matrix.*

2 New Classes of SNS Matrices

In this section, two new infinite classes of n-by-n SNS matrices are identified. Both classes are defined in terms of the well known Hessenberg class.

Definition 2.0.1 *The Hessenberg matrix* H_n *is defined by:*

$$h_{ij} = \begin{cases} 0 & \text{for } i > j+1 \\ +1 & \text{for } i = j+1 \\ -1 & \text{for } i \leq j. \end{cases}$$

If $n \geq 1$ and B_n is SNS, then Gibson [5, Corollary 2] showed that B_n has at least $\binom{n-1}{2}$ zero entries, with exactly this number if and only if there exist permutation matrices P_n and Q_n such that $|P_n B_n Q_n| = |H_n|$. Combining this with Theorem 2.1 of [4], we have the following.

Theorem 2.0.2 *For $n \geq 1$, B_n is a fully indecomposable maximal* SNS *matrix with $\binom{n-1}{2}$ zero entries if and only if B_n is equivalent to H_n.*

Related to the minimum number of zero entries in a SNS matrix, Thomassen [15] defined a digraph called an *extended caterpillar.* Lundgren and Maybee [10] showed how to sign the adjacency matrix of this digraph on n vertices so that when each diagonal entry is -1, this matrix is maximal and SNS. Since this matrix has $\binom{n-1}{2}$ zero entries, by Theorem 2.0.2 it is equivalent to H_n.

2.1 The $G_n^{(r)}$ Class

Definition 2.1.1 *Let r be an integer, $1 \leq r \leq n-3$, and H_n denote the Hessenberg matrix. Then $G_n^{(r)} \equiv [g_{ij}^{(r)}]$ is defined by:*

$$
\begin{array}{lll lll}
g_{r+2,r}^{(r)} & = & +1 & g_{r+1,r+2}^{(r)} & = & 0 \\
g_{r+3,r+1}^{(r)} & = & +1 & g_{r+2,j}^{(r)} & = & 0 \quad \text{for } j = r+3, \ldots, n \\
g_{i,r+1}^{(r)} & = & 0 \quad \text{for } i = 1, \ldots, r & g_{ij}^{(r)} & = & h_{ij} \quad \text{otherwise.}
\end{array}
$$

Remark: The matrix $G_n^{(r)}$ is obtained from H_n by replacing the zeros in positions $(r+2, r)$ and $(r+3, r+1)$ by $+1$ and, in addition, replacing certain negative entries of H_n by 0 so that the resulting matrix is SNS. Here, the parameter r represents the column in which to place the leftmost of the two new +1 entries.

Property 2.1.2 *Each matrix* $G_n^{(r)}$ *is fully indecomposable maximal and* SNS *with exactly* $\binom{n-1}{2} + (n-3)$ *zero entries.*

Property 2.1.3 *Each matrix* $G_n^{(r)}$ *can be obtained from* $G_4^{(1)}$ *by a sequence of row and/or column stretches and/or permutation similarities.*

Property 2.1.4 *Let* P_n *be the permutation matrix with* $p_{i,n-i+1} = 1$ *for* $i = 1, 2, \ldots, n$. *Then* $P_n G_n^{(r)T} P_n^T = G_n^{(n-r-2)}$.

2.2 The $H_n^{(p,q,r,s)}$ Class

Definition 2.2.1 *Let* $p, q, r, s \geq 0$ *be integers such that* $p+q+r+s \leq n-1$ *and* H_n *denote the Hessenberg matrix. Then* $H_n^{(p,q,r,s)} \equiv [h_{ij}^{(p,q,r,s)}]$ *is defined as follows:*

if $p > 0$ *and* $q > 0$, *then for* $i = 1, \ldots, q$

$$h_{p+i+1,p}^{(p,q,r,s)} = +1$$

$$h_{j,p+i}^{(p,q,r,s)} = 0 \qquad \text{for } j = 1, \ldots, p;$$

if $r > 0$ *and* $s > 0$, *then for* $i = 1, \ldots, s$

$$h_{n-r+1,n-r-i}^{(p,q,r,s)} = +1$$

$$h_{n-r-i+1,n-r+j}^{(p,q,r,s)} = 0 \qquad \text{for } j = 1, \ldots, r;$$

otherwise

$$h_{ij}^{(p,q,r,s)} = h_{ij}.$$

Remark: For $p > 0$ and $r > 0$, the matrix $H_n^{(p,q,r,s)}$ is obtained from H_n by replacing q zeros in column p by $+1$ and s zeros in row $n - r + 1$ by $+1$. In addition, certain negative entries of H_n are replaced by 0 so that $H_n^{(p,q,r,s)}$ is a SNS matrix. Note that $H_n^{(p,0,r,0)} = H_n^{(0,q,0,s)} = H_n^{(0,0,0,0)} = H_n$ for any p and r satisfying $p + r \leq n - 1$ and any q and s satisfying $q + s \leq n - 1$.

Property 2.2.2 *Each matrix $H_n^{(p,q,r,s)}$ is fully indecomposable maximal and* SNS *with exactly $\binom{n-1}{2} + q \cdot \max\{ p - 1, 0 \} + s \cdot \max\{ r - 1, 0 \}$ zero entries.*

Property 2.2.3 *Each matrix $H_n^{(p,q,r,s)}$ can be obtained from H_1 by a sequence of row and/or column stretches and/or permutation similarities.*

Property 2.2.4 *Let P_n be the permutation matrix with $p_{i,n-i+1} = 1$ for $i = 1, 2, \ldots, n$. Then $P_n H_n^{(p,q,r,s)^T} P_n^T = H_n^{(r,s,p,q)}$.*

For example, $H_n^{(0,0,2,1)}$ is equivalent to $H_n^{(2,1,0,0)}$.

Property 2.2.5 *Let $p > 0$ and $q, r, s \geq 0$ with $p + q + r + s \leq n - 1$. Let S_n be the signature matrix with $s_{q+1,q+1} = -1$ and $s_{ii} = +1$ otherwise, $P_n = (p + 1, p + 2, \ldots, p + q + 1, 1, 2, \ldots, p) \oplus I_{n-(p+q+1)}$ and $Q_n = (p + 1, p + 2, \ldots, p + q, p, 1, 2, \ldots, p - 1, p + q + 1) \oplus I_{n-(p+q+1)}$. Then $P_n H_n^{(p,q,r,s)} Q_n^T S_n = H_n^{(q+1,p-1,r,s)}$.*

For example, $H_n^{(2,2,0,0)}$ is equivalent to $H_n^{(3,1,0,0)}$. Also, taking $p = 1$ in the above gives the equivalence of $H_n^{(1,q,r,s)}$ and $H_n^{(0,0,r,s)}$.

3 Uniqueness

The results of this section are motivated by the data on fully indecomposable maximal SNS matrices given in Table 1, compiled from the results of a computer search [12]. The evidence from the table suggests that for $n \geq 5$ there is only one equivalence class with exactly $\binom{n-1}{2} + 1$ zero entries, and there are only two equivalence classes with exactly $\binom{n-1}{2} + 2$ zero entries. These results are proved in Theorems 3.1.2 and 3.2.4. One direction is trivial and the other follows the general framework below, which uses a combination of properties of submatrices of SNS matrices, uniqueness of other SNS matrices and simple counting.

- Begin with a normalized fully indecomposable maximal SNS matrix.
- Symmetrically permute the matrix so that the total degree of vertex 1 is minimal.
- By properties of submatrices of SNS matrices, the total degree is bounded and can assume only a few values.
- For each of these values, determine the number of nonzero entries in the rest of the matrix (excluding row and column 1). This number either is used to show that the matrix is equivalent to a known matrix (which is unique by some previous argument) or leads to a contradiction.

In the following subsections, when a path is represented by a vertex sequence and if equality of two adjacent parameters occurs, these are coalesced to yield a path. For example, suppose a path from j to i passing through vertex 1 is represented by the vertex sequence $(j, v', v, 1, u, u', i)$, where $j \leq v' \leq v \leq n$ and $2 \leq u \leq u' \leq i$. Then the vertex sequence $(j, j, j, 1, i, i, i)$ yields the path $(j, 1, i)$.

3.1 Uniqueness of $H_n^{(2,1,0,0)}$

The first theorem of this subsection identifies all the n-by-n normalized fully indecomposable SNS matrices that have H_{n-1} in rows and columns 2 through n. The second theorem gives the desired uniqueness result.

Theorem 3.1.1 *Let $n \geq 2$ and B_n be a normalized fully indecomposable* SNS *matrix with $B[2, \ldots, n] = H_{n-1}$. Then B_n is contained in a matrix A_n that is equivalent to either a row stretch or a column stretch of H_{n-1}. Furthermore, B_n is maximal if and only if $B_n = A_n$.*

Proof: Let $j_s, j_l \geq 2$ be the smallest and largest integers, respectively, such that $b_{1,j_s} \neq 0$ and $b_{1,j_l} \neq 0$ and let $i_s, i_l \geq 2$ be the smallest and largest integers, respectively, such that $b_{i_s,1} \neq 0$ and $b_{i_l,1} \neq 0$.

If $i_s = i_l$, then let $\tilde{A}_n$ be the row stretch of H_{n-1} on row $i_l - 1$. It can be verified that $|B_n|$ is contained in $|\tilde{A}_n|$. By Theorem 1.2.7 and Corollary 1.2.5, there exist signature matrices T_n and U_n such that B_n is contained in $T_n \tilde{A}_n U_n \equiv A_n$. If $j_s = j_l$, then the preceding argument can be repeated with $\tilde{A}_n$ equal to the column stretch of H_{n-1} on column $j_s - 1$. Furthermore, in both cases, B_n is maximal if and only if $B_n = A_n$.

Now assume $i_s \neq i_l$ and $j_s \neq j_l$. If $i_l > j_s + 1$, then in $D(B_n)$ the paths

$$(i_l, 1, j_s), (j_s, i_l), (j_s, j_s + 1), (j_s + 1, j_s), (j_s + 1, i_l), (i_l, i_l - 1, \ldots, j_s + 1)$$

form a subdivision of C_3^*; if $i_l = j_s + 1$, then in $D(B_n)$ the paths

$$(1, j_l, j_l - 1, \ldots, i_l), (i_l, 1), (1, j_s), (j_s, j_s - 1, \ldots, i_s, 1), (i_l, j_s), (j_s, i_l)$$

form a subdivision of C_3^*. By Corollary 1.2.4, B_n is not SNS, giving a contradiction; thus, $i_l \leq j_s$. Let $P_n = (j_s + 1, 2, 3, \ldots, j_s, j_s + 2, j_s + 3, \ldots, n, 1)$, $Q_n = (j_s, 2, 3, \ldots, j_s - 1, j_s + 1, j_s + 2, \ldots, n, 1)$ and $\tilde{A}_n$ be the column stretch of H_{n-1} on column $n-1$. Then it can be verified that $|B_n|$ is contained in $|P_n \tilde{A}_n Q_n^T|$. By Corollary 1.2.5, there exist signature matrices T_n and U_n such that B_n is contained in $T_n P_n \tilde{A}_n Q_n^T U_n \equiv A_n$. Furthermore, B_n is maximal if and only if $B_n = A_n$. Equality is possible only when $i_l = j_s$ as $b_{k1} = 0$ for $k = i_l + 1, \ldots, n$ and $a_{k1} \neq 0$ for $k = 2, \ldots, j_s$. ∎

For $n \leq 3$, there are no fully indecomposable maximal SNS matrices with $\binom{n-1}{2} + 1$ zero entries. However, for $n = 4$, there are two such equivalence classes, represented by $H_4^{(2,1,0,0)}$ and $G_4^{(1)}$ (see Table 1). The case $n \geq 5$ is now considered.

Theorem 3.1.2 *For $n \geq 5$, B_n is a fully indecomposable maximal* SNS *matrix with $\binom{n-1}{2} + 1$ zero entries if and only if B_n is equivalent to $H_n^{(2,1,0,0)}$.*

Proof: The sufficiency follows from Property 2.2.2, which shows that $H_n^{(2,1,0,0)}$ is a fully indecomposable maximal SNS matrix having $\binom{n-1}{2} + 1$ zero entries.

For the necessity, assume B_n is a fully indecomposable maximal SNS matrix having $\binom{n-1}{2} + 1$ zero entries. Also, without loss of generality, assume B_n is normalized and $d(1) = \min_{1 \leq i \leq n} d(i)$. If $d(1) < n - 1$, then $B[2, \ldots, n]$ is not SNS as it contains at least $(n-1)^2 - \binom{n-2}{2} + 1$ nonzero entries. This contradicts the sign nonsingularity of B_n and thus $d(1) \geq n-1$. Also, $d(1) \leq n$ by Proposition 2.3 of [15], so $d(1)$ equals n or $n-1$.

By assuming $d(1) = n$, Theorem 2.7 of [15] implies (since $n \geq 5$) that $D(B_n)$ is a subdigraph of an extended caterpillar on n vertices. Thus, by the remark following Theorem 2.0.2, without loss of generality $D(B_n)$ is a subdigraph of $D(H_n)$. Since $\nu(B_n) = \nu(H_n) - 1$, B_n cannot be maximal; thus, the assumption $d(1) = n$ is false.

With $d(1) = n-1$ and $b_{11} = -1$, $B[2,\ldots,n]$ must be SNS and have exactly $(n-1)^2 - \binom{n-2}{2}$ nonzero entries. But this submatrix must be equivalent to H_{n-1} by Theorem 2.0.2, and thus, without loss of generality $B[2,\ldots,n] = H_{n-1}$. Now Theorem 3.1.1 can be applied with the proviso that B_n is maximal. Since $\nu(B_n) = \binom{n-1}{2}+1$, B_n is equivalent to a matrix A_n that is a stretch of H_{n-1} on either row 3 or column $n-3$. If A_n is the row stretch of H_{n-1} on row 3 and $P_n = (2,3,1,4,5,\ldots,n)$, then $B_n = P_n A_n P_n^T = H_n^{(2,1,0,0)}$. If A_n is the column stretch of H_{n-1} on column $n-3$ and $P_n = (2,3,\ldots,n-2,1,n-1,n)$, then $B_n = P_n A_n P_n^T = H_n^{(0,0,2,1)}$, which is equivalent to $H_n^{(2,1,0,0)}$ by Property 2.2.4. ■

3.2 Uniqueness of $H_n^{(3,1,0,0)}$ and $H_n^{(2,1,2,1)}$

The first and second theorems in this subsection identify all n-by-n normalized fully indecomposable SNS matrices that have rows and columns 2 through n identical to either $H_{n-1}^{(2,1,0,0)}$ or a matrix equal to H_{n-1} with one nonzero changed to zero. The final theorem (Theorem 3.2.4) gives the desired uniqueness result.

Theorem 3.2.1 *Let $n \geq 5$ and B_n be a normalized fully indecomposable* SNS *matrix with $B[2,\ldots,n] = H_{n-1}^{(2,1,0,0)}$. Then B_n is contained in a matrix A_n that is* *equivalent to either a row stretch or a column stretch of* $H_{n-1}^{(2,1,0,0)}$. *Furthermore,* B_n *is maximal if and only if* $B_n = A_n$.

Proof: Let j_s, j_l, i_s and i_l be defined as in the proof of Theorem 3.1.1. As in that proof, when $i_s = i_l$ «$j_s = j_l$», B_n is contained in $T_n \tilde{A}_n U_n \equiv A_n$, where now $\tilde{A}_n$ is the row «column» stretch of $H_{n-1}^{(2,1,0,0)}$ on row $i_l - 1$ «column $j_s - 1$». Furthermore, B_n is maximal if and only if $B_n = A_n$.

We now assume that $i_s \neq i_l$ (and thus $i_l \geq 3$) and $j_s \neq j_l$, and consider four cases.

Case 1. Assume that $j_s = 2$. As $b_{1,2} \neq 0$ and b_{k2} is an essential zero in $B[2,\ldots,n]$ for $k \geq 4$, Lemma 1.2.3 implies that b_{k1} is an essential zero. Therefore, $i_l = 3$. However, in $D(B_n)$ the paths

$$(1, j_l, j_l - 1, \ldots, 3), (3,1), (1,2), (2,1), (2,3), (3,2)$$

form a subdivision of C_3^* contradicting the sign nonsingularity of B_n by Corollary 1.2.4.

Case 2. Assume that $j_s = 3$. Since $b_{1,3} \neq 0$ and b_{k3} is an essential zero in $B[2,\ldots,n]$ for $k \geq 6$, Lemma 1.2.3 implies b_{k1} is an essential zero. Thus, $3 \leq i_l \leq 5$.

Subcase 2a. Assume that $i_l = 3$. Since $b_{3,1} \neq 0$ and $b_{3,4}$ is an essential zero in $B[2,\ldots,n]$, Lemma 1.2.3 implies $b_{1,4}$ is an essential zero. Let $P_n = (2,1,3,4,\ldots,n)$ and $\tilde{A}_n$ be the row stretch of $H_{n-1}^{(2,1,0,0)}$ on row 2.

Subcase 2b. Assume that $i_l = 4$ or 5. If $i_s = 2$ or 3, then Lemma 1.2.3 implies $b_{1,4}$ is an essential zero, and in $D(B_n)$ the paths

$$(1, j_l, j_l - 1, \ldots, 5), (5, i_l, 1), (5, 3), (3, 5), (1, 3), (3, i_s, 1)$$

form a subdivision of C_3^*. Thus, $i_l = 5$ and $i_s = 4$. If $b_{1,4} \neq 0$, then the paths

$$(1, 3, 5), (5, 1), (1, 4), (4, 1), (4, 5), (5, 4)$$

form a subdivision of C_3^*. Thus, $b_{1,4} = 0$ and $j_l \geq 5$. Let $P_n = (4, 2, 3, 1, 5, 6, \ldots, n)$ and $\tilde{A}_n$ be the row stretch of $H_{n-1}^{(2,1,0,0)}$ on row 4.

Then, in both subcases, it can be verified that $|B_n|$ is contained in $|P_n \tilde{A}_n P_n^T|$. By Corollary 1.2.5, there exist signature matrices T_n and U_n such that B_n is contained in $T_n P_n \tilde{A}_n P_n^T U_n \equiv A_n$. Furthermore, B_n is maximal if and only if $B_n = A_n$.

Case 3. Assume that $j_s = 4$. As $b_{1,4} \neq 0$ and b_{k4} is an essential zero in $B[2, \ldots, n]$ for $k = 2$, 3, or $k \geq 6$, Lemma 1.2.3 implies that b_{k1} is an essential zero. Thus $i_s = 4$ and $i_l = 5$. However, the paths

$$(1, j_l, j_l - 1, \ldots, 5), (5, 1), (1, 4), (4, 1), (4, 5), (5, 4)$$

form a subdivision of C_3^*.

Case 4. Assume that $j_s \geq 5$. If $i_l > j_s + 1$ or $i_l = j_s + 1$, then as in the proof of Theorem 3.1.1, a subdivision of C_3^* exists contradicting the sign nonsingularity of B_n. Thus, $i_l \leq j_s$ and from the same proof, B_n is contained in $T_n P_n \tilde{A}_n Q_n^T U_n \equiv A_n$, where now $\tilde{A}_n$ is the column stretch of $H_{n-1}^{(2,1,0,0)}$ on column $n-1$. The matrix B_n is maximal if and only if $B_n = A_n$. ■

Theorem 3.2.2 *Let $n \geq 5$ and B_n be a fully indecomposable maximal* SNS *matrix with $b_{1,1} = -1$, $B[2,\ldots,n]$ contained in H_{n-1} and $\nu(B[2,\ldots,n]) = \nu(H_{n-1}) - 1$. Then B_n is equivalent to either H_n or $G_n^{(r)}$ for some r, $1 \leq r \leq n-3$.*

Proof: Let $B[2,\ldots,n]$ be H_{n-1} with the b_{ij} entry replaced by zero. Clearly b_{ij} is not an essential zero in $B[2,\ldots,n]$ and $i \leq j+1$. The proof consists of four cases based on the location of this zero: $i = j+1$, $i = j$, $i = j-1$ and $i \leq j-2$.

Case 1. Assume that $i = j+1$. Then every zero entry b_{kl} in $B[2,\ldots,n]$, except b_{ij}, is essential due to paths (l,k) and $(l,l+1,k)$ of opposite path sign in $SD(B[2,\ldots,n])$.

Since B_n is fully indecomposable but $B[2,\ldots,n]$ is not, there exists some path in $D(B_n)$ from v to u for all v,u, where $i \leq v \leq n$ and $2 \leq u \leq j = i-1$, which implies the existence of a path $(v,1,u)$ for any one pair v,u. If $v \geq i+1$, then as each b_{vk} is an essential zero in $B[2,\ldots,n]$ for $k = 2,\ldots,v-2$, Lemma 1.2.3 implies $b_{1k} = 0$. This contradicts the necessity that $b_{1u} \neq 0$ and therefore, $v = i$. Since $b_{i1} \neq 0$ and b_{ik} is an essential zero in $B[2,\ldots,n]$ for $k = 2,\ldots,j-1$, Lemma 1.2.3 implies b_{1k} is an essential zero. Thus $u = j$. For $2 \leq j \leq n-2$, since $b_{1j} \neq 0$ and b_{kj} is an essential zero in $B[2,\ldots,n]$ for $k = j+2,\ldots,n$, Lemma 1.2.3 implies b_{k1} is an essential zero. With $P_n = (j,1,2,\ldots,j-1,j+1,j+2,\ldots,n)$, it can be verified that $|B_n|$

is contained in $|P_n H_n P_n^T|$. Since B_n is maximal, B_n is equivalent to H_n by Corollary 1.2.5.

Case 2. Assume that $i = j$. When $i = n$, the matrix B_n is equivalent to $\tilde{B}_n = R_n B_n^T R_n^T$, where $R_n = (1, n, n-1, \ldots, 2)$ and $\tilde{B}[2, \ldots, n] = H_{n-1}$ except that $\tilde{b}_{2,2} = 0$. Thus, without loss of generality, $2 \leq i \leq n-1$. We can replace B_n by $B_n Q_n^T S_n$, where $Q_n = (1, n, 2, 3, \ldots, n-1)$ and $S_n = I_2 \oplus -I_{n-2}$, so that the zero that is not essential in $B[2, \ldots, n]$ is in position $(i', i'+1)$, where $2 \leq i' \leq n-1$. Since $b_{i',i'+1}$ is not an essential zero in $B[2, \ldots, n]$, all paths from $i'+1$ to i' in $SD(B[2, \ldots, n])$ have the same path sign, which is negative. However, since $b_{i',i'+1}$ is an essential zero in B_n, without loss of generality, there exists a path represented by one of the following vertex sequences and having a positive path sign:

$$(i'+1, v, 1, u, i') \quad \text{for} \quad i'+1 \leq v \leq n \quad \text{and} \quad 3 \leq u \leq i' \tag{3.1}$$

$$(i'+1, 2, v, 1, u, i') \quad \text{for} \quad 2 \leq v \leq i'-1 \quad \text{and} \quad 3 \leq u \leq i' \tag{3.2}$$

$$(i'+1, v, 1, u, 2, i') \quad \text{for} \quad i'+1 \leq v \leq n \quad \text{and} \quad \begin{array}{c} i'+2 \leq u \leq n \\ \text{or } u = 2. \end{array} \tag{3.3}$$

Subcase 2a. Assume that $i' = 2$. Then every zero entry in $B[2, \ldots, n]$ is essential except for $b_{2,3}$. Since $i' = 2$, vertex sequences (3.1) or (3.2) do not exist and (3.3) becomes $(3, v, 1, u, 2)$ for $3 \leq v \leq n$ and $4 \leq u \leq n$ or $u = 2$. Since $b_{1,3} \neq 0$ (otherwise B_n is partly decomposable) and b_{k3} is an essential zero for $k = 4, 5, \ldots, n$, Lemma 1.2.3 implies that b_{k1} is an essential zero. Thus, $v = 3$ and (3.3) reduces further to just

$(3,1,u,2)$ for $4 \leq u \leq n$ or $u = 2$ with $sgn\,(3,1) = -\,sgn\,(1,3)$. Since B_n is fully indecomposable, $b_{2,1} \neq 0$ with $sgn\,(2,1) = -\,sgn\,(1,3,2)$. With $P_n = (2,3,1,4,5,\ldots,n)$ and $Q_n = (2,n,1,3,4,\ldots,n-1)$, it can be verified that $|B_n|$ is contained in $|P_n H_n Q_n^T|$. Since B_n is maximal, B_n is equivalent to H_n by Corollary 1.2.5.

Subcase 2b. Assume that $3 \leq i' \leq n-1$. Then every zero entry in $B[2,\ldots,n]$ is essential except for $b_{i',i'+1}$ and $b_{i'+1,i'}$. Vertex sequences (3.2) and (3.3) induce the positively signed cycles $(v,1,u,i',2,v)$ and $(v,1,u,2,i'+1,v)$, respectively, contradicting the sign nonsingularity of B_n by Theorem 1.2.1. With (3.1), if $u \neq i'$ or $v \neq i'+1$, then either $(v,1,u,2,v)$ or $(v,1,u,v)$ has a positive cycle sign. This leaves (3.1) as the path $(i'+1,1,i')$ with a positive path sign.

Since $b_{i'+1,i'}$ is an essential zero in B_n but not in $B[2,\ldots,n]$, some path exists in $SD(B_n)$ from vertex i' to vertex $i'+1$ passing through vertex 1 and having path sign equal to $-\,sgn\,(i',2,i'+1)$, which is always positive. Without loss of generality, this path is represented by one of the following vertex sequences:

$$(i',w,1,x,i'+1) \quad \text{for} \quad \begin{array}{c} i'+1 < w \leq n \\ \text{or } w = 1 \end{array} \quad \text{and} \quad \begin{array}{c} 3 \leq x < i' \\ \text{or } x = 1 \end{array} \tag{3.4}$$

$$(i',2,w,1,x,i'+1) \quad \text{for} \quad \begin{array}{c} 3 \leq w < i' \\ \text{or } w = 1 \end{array} \quad \text{and} \quad \begin{array}{c} 3 \leq x < i' \\ \text{or } x = 1 \end{array} \tag{3.5}$$

$$(i', w, 1, x, 2, i'+1) \quad \text{for} \quad \begin{matrix} i'+1 < w \leq n \\ \text{or } w = 1 \end{matrix} \quad \text{and} \quad \begin{matrix} i'+1 < x \leq n \\ \text{or } x = 1. \end{matrix} \tag{3.6}$$

Vertex sequences (3.5) and (3.6) induce the positively signed cycles $(i'+1, 2, w, 1, x, i'+1)$ and $(i', w, 1, x, 2, i')$, respectively. With (3.4), the following two cycles exist and have positive cycle sign: $(i'+1, w, 1, x, i'+1)$ for $w \neq 1$ and $(i', 1, x, i')$ for $w = 1$ and $x \neq 1$. Therefore, both w and x are restricted to being 1 and (3.4) becomes the path $(i', 1, i'+1)$ with a positive path sign. Since $(i', 1)$ and $(1, i')$ must be oppositely signed, $sgn\,(i', 1) = sgn\,(1, i'+1) = -\,sgn\,(i'+1, 1) = -\,sgn\,(1, i')$. Let $P_n = (i', 2, 1, 3, 4, \ldots, i'-1, i'+1, i'+2, \ldots, n)$. When $i' = 3$, let $Q_n = (3, n, 1, 2, 4, 5, \ldots, n-1)$ and when $i' \geq 4$, let $Q_n = (i'-1, n, 1, 2, \ldots, i'-2, i', i'+1, \ldots, n-1)$. Then it can be verified that $|P_n G_n^{(i'-2)} Q_n^T|$ is contained in $|B_n|$. Since $P_n G_n^{(i'-2)} Q_n^T$ is maximal, B_n is equivalent to $G_n^{(i'-2)}$ by Corollary 1.2.5.

In Subcase 2a, B_n is equivalent to H_n and in Subcase 2b, B_n is equivalent to $G_n^{(r)}$ for some r, $1 \leq r \leq n-3$.

Case 3. Assume that $i = j - 1$. When $i = n - 1$, the matrix B_n is equivalent to $\tilde{B}_n = R_n B_n^T R_n^T$, where $R_n = (1, n, n-1, \ldots, 2)$ and $\tilde{B}[2, \ldots, n] = H_{n-1}$ except that $\tilde{b}_{2,3} = 0$. Thus, without loss of generality, $2 \leq i \leq n-2$.

Since b_{ij} is not an essential zero in $B[2, \ldots, n]$, all paths from j to i in $SD(B[2, \ldots, n])$ have the same path sign, which is positive. However, since b_{ij} is an essential zero in B_n, there exists a negatively signed path from vertex j to i passing through vertex 1. Without loss of generality, the path is represented by the following vertex sequence:

$$(j, v', v, 1, u, u', i) \text{ for } 2 \leq u \leq u' \leq i \text{ and } j \leq v' \leq v \leq n. \qquad (3.7)$$

Subcase 3a. Assume that $i = 2$. Then every zero entry in $B[2, \ldots, n]$ is essential except for entries $b_{2,3}$ and $b_{4,2}$. Consideration of (3.7) shows that the path must be $(3, v, 1, 2)$ for $3 < v \leq n$ as all other possible paths induce positive cycles in $SD(B_n)$. As $b_{1,2} \neq 0$ and b_{k2} is an essential zero in $B[2, \ldots, n]$ for $5 \leq k \leq n$, Lemma 1.2.3 implies $b_{k1} = 0$. Thus $v = 4$.

Since $b_{4,2}$ is an essential zero in B_n but not in $B[2, \ldots, n]$, some path exists in $SD(B_n)$ from vertex 2 to vertex 4 passing through vertex 1 and having path sign equal to $-\,sgn\,(2, 4)$, which is positive. Without loss of generality, the path is represented by one of the following three vertex sequences:

$$(2, 1, x, x-1, \ldots, 4) \text{ for } 4 \leq x \leq n \qquad (3.8)$$

$$(2, w', w, 1, x, x-1, \ldots, 4) \text{ for } 5 \leq w' \leq w \leq n \text{ and } 4 \leq x < w' \qquad (3.9)$$

$$(2, 1, 3, 4). \qquad (3.10)$$

With (3.8) and (3.9), the positively signed cycles $(2, 1, x, x-1, \ldots, 2)$ and $(w, 1, x, w', w)$ exist, respectively, and thus, (3.10) must be the path.

Therefore, the two paths through vertex 1 that cause $b_{2,3}$ and $b_{4,2}$ to be essential zeros in B_n are $(3,4,1,2)$ and $(2,1,3,4)$, respectively. With $P_n = (3,1,2,4,5,\ldots,n)$, it can be verified that $|P_n G_n^{(1)} P_n^T|$ is contained in $|B_n|$. Since $P_n G_n^{(1)} P_n^T$ is maximal, B_n is equivalent to $G_n^{(1)}$ by Corollary 1.2.5.

Subcase 3b. Assume that $3 \leq i \leq n-2$. Then every zero entry in $B[2,\ldots,n]$ is essential except b_{ij}. Consideration of (3.7) shows that the negatively signed path from j to i must be either

$$(j, v, 1, i) \text{ for } j+1 \leq v \leq n \tag{3.11}$$

or

$$(j, 1, u, i) \text{ for } 2 \leq u \leq i-1. \tag{3.12}$$

For path (3.11), since $b_{1i} \neq 0$ and b_{k2} is an essential zero in $B[2,\ldots,n]$ for $k = i+2,\ldots,n$, Lemma 1.2.3 implies $b_{k1} = 0$. But as $i = j-1$, no path (3.11) exists. A similar contradiction can be obtained for path (3.12) since b_{jk} is an essential zero for $k = 2,\ldots,j-2$.

In Subcase 3a, B_n is equivalent to $G_n^{(1)}$ and in Subcase 3b, no such matrix B_n exists.

Case 4. Assume that $i \leq j-2$. Since b_{ij} is not an essential zero in $B[2,\ldots,n]$, all paths from j to i in $SD(B[2,\ldots,n])$ have the same path sign, which is positive. However, since b_{ij} is an essential zero in B_n, there exists a negatively signed path from vertex j to i passing through vertex 1. Without loss of generality, the path is represented by one of the following four vertex sequences:

$$(j, v', v, 1, u, u', i) \text{ for } 2 \leq u \leq u' \leq i \text{ and } j \leq v' \leq v \leq n \tag{3.13}$$

$$(j, v', v, 1, u, u-1, \ldots, i) \text{ for } i+1 \leq u < j \leq v' \leq v \leq n \tag{3.14}$$

$$(j, j-1, \ldots, v, 1, u, u', i) \text{ for } 2 \leq u \leq u' \leq i < v \leq j-1 \tag{3.15}$$

$$(j, j-1, \ldots, v, 1, u, u-1, \ldots, i) \text{ for } i+1 \leq u < v \leq j-1. \tag{3.16}$$

Assume the path is represented by (3.13) and suppose $sgn\,(v,1,u) = +$. Then the positively signed cycle $(v, 1, u, i+1, v)$ is formed. On the other hand, suppose $sgn\,(v,1,u) = -$. Then, for $u \neq i$ or $v \neq j$, the cycle $(v,1,u,v)$ has positive cycle sign. For $u = i \neq 2$, the cycle $(v,1,u,u-1,v)$ has positive cycle sign. For $v = j \neq n$, the cycle $(v,1,u,v+1,v)$ has positive cycle sign. Finally, for $u = i = 2$ and $v = j = n$ and since $n \geq 5$, the cycle $(v,1,u,u+1,v-1,v)$ exists and has positive cycle sign. On the other hand, by assuming the path is represented by (3.14), the cycle $(v,1,u,j,v',v)$ exists and has positive cycle sign. Similarly, a path represented by (3.15) induces the positively signed cycle $(v,1,u,u',i,v)$. Finally, with a path represented by (3.16), the cycle $(v,1,u,v)$ exists and has positive cycle sign. Therefore, none of these paths exist, contradicting the fact that b_{ij} is an essential zero in B_n. Thus, no such matrix B_n exists. ∎

In the next theorem, we use the following lemma.

Lemma 3.2.3 *Let* $n \geq 2$ *and* B_n *be a normalized* SNS *matrix. If* $B[2,\ldots,n]$ *is maximal and partly decomposable, then* B_n *is partly decomposable.*

Proof: If $n = 2, 3$ or 4, then the result is vacuously true. For $n \geq 5$, there exist permutation matrices P_n and Q_n such that $P_n B_n Q_n^T \equiv A_n$ is of the form

$$\left[\begin{array}{c|cc} -1 & \multicolumn{2}{c}{W_{1,n-1}} \\ \hline V_{n-1,1} & \begin{array}{c|c} X_k & Y_{k,n-k-1} \\ \hline O_{n-k-1,k} & Z_{n-k-1} \end{array} & \end{array}\right]$$

for $0 < k < n-1$. If $a_{i1} \neq 0$ for any i, where $k+2 \leq i \leq n$, then by Lemma 1.2.3, $a_{1j} = 0$ for $j = 2, \ldots, k+1$ and A_n (and thus B_n) is partly decomposable. However, if $a_{i1} = 0$ for all $i = k+2, \ldots, n$, then A_n and B_n are partly decomposable. ∎

We also define an anomalous SNS matrix

$$\mathcal{M}_6 = \begin{bmatrix} -1 & -1 & -1 & 0 & -1 & -1 \\ +1 & -1 & -1 & 0 & -1 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

which appears to be a degenerate form of the $H_n^{(2,1,2,1)}$ class.

For $n \leq 4$, there are no fully indecomposable maximal SNS matrices with $\binom{n-1}{2}+2$ zero entries. However, for $n=5$, there are two such equivalence classes, represented by $H_5^{(3,1,0,0)}$ and $G_5^{(1)}$ (see Table 1). The case $n \geq 6$ is now considered.

Theorem 3.2.4 *For $n \geq 6$, B_n is a fully indecomposable maximal* SNS *matrix with $\binom{n-1}{2}+2$ zero entries if and only if*

1. *for $n=6$, B_n is equivalent to $H_6^{(3,1,0,0)}$ or $\mathcal{M}_6$;*

2. *for $n \geq 7$, B_n is equivalent to $H_n^{(3,1,0,0)}$ or $H_n^{(2,1,2,1)}$.*

Proof: The sufficiency follows from Property 2.2.2, which shows that both $H_n^{(3,1,0,0)}$ for $n \geq 6$ and $H_n^{(2,1,2,1)}$ for $n \geq 7$ are fully indecomposable maximal SNS matrices having $\binom{n-1}{2}+2$ zero entries. By inspection, $\mathcal{M}_6$ is also a fully indecomposable maximal SNS matrix having $\binom{n-1}{2}+2$ zero entries.

For the necessity, assume B_n is a fully indecomposable maximal SNS matrix having $\binom{n-1}{2}+2$ zero entries. Also, without loss of generality, assume B_n is normalized and $d(1) = \min_{1\leq i\leq n} d(i)$. If $d(1) < n-2$, then $B[2,\ldots,n]$ is not SNS as it contains at least $(n-1)^2 - \binom{n-2}{2}+1$ nonzero entries. This contradicts the sign nonsingularity of B_n and thus $d(1) \geq n-2$. Also, $d(1) \leq n$ by Proposition 2.3 of [15], so $d(1)$ equals $n-2$, $n-1$, or n. By an argument similar to that in the proof of Theorem 3.1.2, $d(1) \neq n$.

With $d(1) = n-2$ and $b_{11} = -1$, $B[2,\ldots,n]$ must be SNS and have exactly $(n-1)^2 - \binom{n-2}{2}$ nonzero entries. But this submatrix must be equivalent to H_{n-1} by Theorem 2.0.2, and thus, without loss of generality, $B[2,\ldots,n] = H_{n-1}$. Now Theorem 3.1.1 can be applied with the proviso that B_n is maximal. Since $\nu(B_n) =$

$\binom{n-1}{2}+2$, B_n is equivalent to a matrix A_n that is a stretch of H_{n-1} on either row 4 or column $n-4$. If A_n is the row stretch of H_{n-1} on row 4 and $P_n = (2,3,4,1,5,6,\ldots,n)$, then $B_n = P_n A_n P_n^T = H_n^{(3,1,0,0)}$. If A_n is the column stretch of H_{n-1} on column $n-4$ and $P_n = (2,3,\ldots,n-3,1,n-2,n-1,n)$, then $B_n = P_n A_n P_n^T = H_n^{(0,0,3,1)}$, which is equivalent to $H_n^{(3,1,0,0)}$ by Property 2.2.4.

Finally, with $d(1) = n-1$, $B[2,\ldots,n]$ has exactly $(n-1)^2 - \binom{n-2}{2} - 1$ nonzero entries. Since B_n is fully indecomposable, Lemma 3.2.3 states that $B[2,\ldots,n]$ is either not maximal or fully indecomposable.

First, assume $B[2,\ldots,n]$ is not maximal. Then, setting some zero entry nonzero gives a SNS matrix with $(n-1)^2 - \binom{n-2}{2}$ nonzero entries and by Theorem 2.0.2, this must be equivalent to H_{n-1}. Therefore, the only way for $B[2,\ldots,n]$ to be not maximal is for $B[2,\ldots,n]$ to be contained in H_{n-1} and $\nu(B[2,\ldots,n]) = \nu(H_{n-1}) - 1$. Since B_n is maximal, Theorem 3.2.2 states that B_n is equivalent to either H_n or $G_n^{(r)}$ for some r, $1 \le r \le n-3$. However, $\nu(B_n) = \nu(H_n) - 2$ and for $n \ge 6$, $\nu(B_n) > \nu(G_n^{(r)})$. Thus, B_n cannot be equivalent to either H_n or $G_n^{(r)}$, and $B[2,\ldots,n]$ must be maximal.

Second, assume $B[2,\ldots,n]$ is fully indecomposable and maximal. As this submatrix has $\binom{n-2}{2}+1$ zeros, by Theorem 3.1.2 $B[2,\ldots,n]$ is equivalent to $H_{n-1}^{(2,1,0,0)}$. Without loss of generality, $B[2,\ldots,n] = H_{n-1}^{(2,1,0,0)}$. Now Theorem 3.2.1 can be applied with the proviso that B_n is maximal. Since $\nu(B_n) = \binom{n-1}{2}+2$, B_n is equivalent to a matrix A_n that is a stretch of $H_{n-1}^{(2,1,0,0)}$ on either rows 1, 2, 3, or 4 for $n \ge 6$; or column $n-3$ for $n \ge 7$; or column 2 for $n = 6$. If A_n is the row stretch of $H_{n-1}^{(2,1,0,0)}$ on row 1, then $B_n = A_n = H_n^{(3,1,0,0)}$. If A_n is the row stretch of $H_{n-1}^{(2,1,0,0)}$ on row 2, $Q_n = (3,1,2,4,5,\ldots,n)$ and $S_n = -I_2 \oplus I_{n-2}$, then $B_n = A_n Q_n^T S_n = H_n^{(3,1,0,0)}$. If A_n is the row stretch of $H_{n-1}^{(2,1,0,0)}$ on row 3 and $P_n = (2,3,1,4,5,\ldots,n)$, then $B_n = P_n A_n P_n^T = H_n^{(2,2,0,0)}$, which is equivalent to $H_n^{(3,1,0,0)}$ by Property 2.2.5. If A_n is the row stretch of $H_{n-1}^{(2,1,0,0)}$ on row 4, $P_n = (5,1,4,2,3,6,7,\ldots,n)$, $Q_n = (1,4,3,2,5,6,\ldots,n)$ and $S_n = -I_3 \oplus I_{n-3}$, then $B_n = P_n A_n Q_n^T S_n = H_n^{(3,1,0,0)}$. If $n \geq 7$, A_n is the column stretch of $H_{n-1}^{(2,1,0,0)}$ on column $n-3$ and $P_n = (2,3,\ldots,n-2,1,n-1,n)$, then $B_n = P_n A_n P_n^T = H_n^{(2,1,2,1)}$. Finally, if $n = 6$, A_n is the column stretch of $H_{n-1}^{(2,1,0,0)}$ on column 2 and $P_n = (2,3,1,4,5,6)$, then $B_n = P_n A_n P_n^T = \mathcal{M}_6$. ■

4 Concluding Remarks

This work was initiated in response to the generation of all the fully indecomposable maximal SNS matrices of dimensions 1 through 9 by Lundy, Maybee and Van Buskirk [12]. The results of an analysis of this data with regard to the number of equivalence classes containing a specified number of zero entries are given in Table 1. For fixed n, the known lower bound of $\binom{n-1}{2}$ for the number of zero entries is tight by Gibson [5] but a tight upper bound is not known. The unique equivalence class of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}$ zero entries is represented by H_n (Theorem 2.0.2). The data in Table 1 show that there is exactly one equivalence class of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}+1$ zero entries for $5 \leq n \leq 9$. In Theorem 3.1.2, this is proven for all $n \geq 5$, and moreover this class is represented by $H_n^{(2,1,0,0)}$. Similarly, the data in

Table 1 show that there are exactly two equivalence classes of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}+2$ zeros for $5 \leq n \leq 9$. In Theorem 3.2.4, this is proven for all $n \geq 6$, and moreover $H_n^{(3,1,0,0)}$ and $H_n^{(2,1,2,1)}$ are the representatives for these classes for all $n \geq 7$. The two classes of matrices $H_n^{(p,q,r,s)}$ and $G_n^{(r)}$ classify only a small subset of the SNS matrices known from the data of Lundy et al [12]. Two other classes have recently been identified in [9].

The current status regarding the number of nonequivalent fully indecomposable maximal n-by-n SNS matrices with $\binom{n-1}{2}+k$ zero entries for $0 \leq k \leq 4$ is as given in Table 2. Thus, for example, we conjecture that for each $n \geq 7$, there are 5 such matrices having $\binom{n-1}{2}+4$ zero entries. The method of proof used in Theorems 3.1.2 and 3.2.4 does not easily extend to values of $k \geq 3$.

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Table 1: The categorization of fully indecomposable maximal SNS matrices of dimensions 1 through 9 by dimension and number of zero entries.

Matrix Dimension	Number of Zero Entries	Number of Equivalence Classes	Equivalence Class Membership
1	0	1	H_1
2	0	1	H_2
3	1	1	H_3
4	3	1	H_4
	4	2	$H_4^{(2,1,0,0)}$, $G_4^{(1)}$
5	6	1	H_5
	7	1	$H_5^{(2,1,0,0)}$
	8	2	$H_5^{(3,1,0,0)}$, $G_5^{(1)}$
	9	2	
6	10	1	H_6
	11	1	$H_6^{(2,1,0,0)}$
	12	2	$H_6^{(3,1,0,0)}$, $\mathcal{M}_6$
	13	3	$H_6^{(4,1,0,0)}$, $G_6^{(1)}$, $G_6^{(2)}$
	14	6	$H_6^{(3,2,0,0)}$
	$\vdots$	$\vdots$	$\vdots$
7	15	1	H_7
	16	1	$H_7^{(2,1,0,0)}$
	17	2	$H_7^{(3,1,0,0)}$, $H_7^{(2,1,2,1)}$
	18	2	$H_7^{(4,1,0,0)}$
	19	5	$H_7^{(5,1,0,0)}$, $H_7^{(3,2,0,0)}$, $G_7^{(1)}$, $G_7^{(2)}$
	$\vdots$	$\vdots$	$\vdots$
8	21	1	H_8
	22	1	$H_8^{(2,1,0,0)}$
	23	2	$H_8^{(3,1,0,0)}$, $H_8^{(2,1,2,1)}$
	24	2	$H_8^{(4,1,0,0)}$, $H_8^{(3,1,2,1)}$
	25	5	$H_8^{(5,1,0,0)}$, $H_8^{(3,2,0,0)}$
	$\vdots$	$\vdots$	$\vdots$
9	28	1	H_9
	29	1	$H_9^{(2,1,0,0)}$
	30	2	$H_9^{(3,1,0,0)}$, $H_9^{(2,1,2,1)}$
	31	2	$H_9^{(4,1,0,0)}$, $H_9^{(3,1,2,1)}$
	32	5	$H_9^{(5,1,0,0)}$, $H_9^{(3,2,0,0)}$, $H_9^{(4,1,2,1)}$, $H_9^{(3,1,3,1)}$
	$\vdots$	$\vdots$	$\vdots$

Table 2: The known or conjectured number of equivalence classes of fully indecomposable maximal SNS matrices with $\binom{n-1}{2} + k$ zero entries, where $0 \leq k \leq 4$.

k	Number of Equivalence Classes	Status
0	1	Known for $n \geq 1$ (Theorem 2.0.2)
1	1	Known for $n \geq 5$ (Theorem 3.1.2)
2	2	Known for $n \geq 6$ (Theorem 3.2.4)
3	2	Conjectured for $n \geq 7$
4	5	Conjectured for $n \geq 7$

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