EXPECTED LENGTH OF THE GAME
(American Mathematical Monthly Problem E3213)

By

BRUCE R. JOHNSON

DM-448-IR SEPTEMBER 1987

EXPECTED LENGTH OF THE GAME
(American Mathematical Monthly Problem E3213)

By

BRUCE R. JOHNSON

DM-448-IR SEPTEMBER 1987

Bruce R. Johnson
Department of Mathematics, University of Victoria, Victoria, B.C., Canada V8W 2Y2.

EXPECTED LENGTH OF THE GAME

(American Mathematical Monthly Problem E3213)

E3213. *Proposed by D.M. Bloom, Brooklyn College, CUNY.*

As in Problem E2276 [1971, 78; 1972, 90] we consider the "Ehrenfest urn" game in which players A and B have between them n cards labelled 1, 2, ..., n. At each move, one of the numbers 1, 2, ..., n is chosen at random and the player who has the card with that number must give it to the other player. The game continues until one player has all the cards. Prove that the expected length of the game is
(a) $2^{n-1} - 1$ if A initially has exactly one card,
(b) $2^{n-1} - (n+1)/2$ if A is equally likely to start with any number of cards in $\{1,2,\ldots,n\}$.

Solution by Bruce R. Johnson, University of Victoria, Canada. In the process of proving parts (a) and (b), we will derive a family of difference equations that later will be solved recursively to obtain the expected length of the game if player A initially has exactly k cards, for $k = 2, 3, \ldots, n-1$.

We begin by defining the notation

e_j = expected length of the game, given that player A initially has j cards.

By conditioning on the outcome of the first move, we obtain

(1) $$e_j = (1+e_{j+1})\frac{n-j}{n} + (1+e_{j-1})\frac{j}{n}, \text{ for } j = 1, 2, \ldots, n-1.$$

To prove part (a) we first subtract $e_j(n-j)/n + e_{j-1}j/n$ from both sides of (1) and then multiply both sides by n/j to obtain

(2) $$e_j - e_{j-1} = \frac{n}{j} + \frac{n-j}{j}(e_{j+1} - e_j), \text{ for } j = 1, 2, \ldots, n-1.$$

Using (2) recursively as j increases from 1 to $n-1$, we find

$$e_1 - e_0 = \frac{n}{1} + \frac{n-1}{1}\left[\frac{n}{2} + \frac{n-2}{2}\left[\frac{n}{3} + \frac{n-3}{3}\left[\cdots\left[\frac{n}{n-2} + \frac{2}{n-2}\left[\frac{n}{n-1} + \frac{1}{n-1}\left[e_n - e_{n-1}\right]\right]\right]\cdots\right]\right]\right]$$

$$= \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} + (e_n - e_{n-1}).$$

Bruce R. Johnson
Department of Mathematics, University of Victoria, Victoria, B.C., Canada V8W 2Y2.

Since $e_0 = 0 = e_n$, $e_{n-1} = e_1$, and $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, the above simplifies to $e_1 = 2^{n-1} - 1$, which proves part (a).

To prove part (b) we multiply both sides of (2) by j and then sum over j to obtain

$$\sum_{j=1}^{n-1} j(e_j - e_{j-1}) = (n-1)n + \sum_{j=1}^{n-1} (n-j)(e_{j+1} - e_j),$$

which simplifies to

$$-\sum_{j=0}^{n-1} e_j + ne_{n-1} = (n-1)n - ne_1 + \sum_{j=1}^{n} e_j.$$

Using $e_n = e_0 = 0$ and $e_{n-1} = e_1 = 2^{n-1} - 1$, collecting like terms, and dividing by $2n$, we obtain

(3) $$(1/n) \sum_{j=1}^{n} e_j = 2^{n-1} - (n+1)/2.$$

Part (b) follows immediately from (3) because $(1/n) \sum_{j=1}^{n} e_j$ is seen to be the desired expectation obtained by conditioning on the initial uniform distribution.

Remark: The initial distribution of cards for part (b) is favorable to player A because it is possible for player A to win immediately but not lose immediately from the initial distribution of cards. Changing the initial distribution by having player A equally likely to start with any number of cards in $\{1,2,...,n-1\}$ would make the game fair and insure the need for at least one move. Then the expected length of the game would be

$$\frac{1}{n-1} \sum_{j=1}^{n-1} e_j = \frac{n}{n-1} \left[2^{n-1} - \frac{n+1}{2}\right],$$

obtained from (3) and the fact that $e_n = 0$.

To derive an expression for e_k when $k = 2, 3, ..., n-1$, we first solve (2) for $e_{j+1} - e_j$, obtaining

(4) $$e_{j+1} - e_j = \frac{j}{n-j} (e_j - e_{j-1}) - \frac{n}{n-1}, \quad \text{for } j = 1, 2, ..., n-1.$$

Bruce R. Johnson
Department of Mathematics, University of Victoria, Victoria, B.C., Canada V8W 2Y2.

From part (a) and (4) with $j = 1$, we find

$$e_2 - e_1 = \frac{1}{n-1}(2^{n-1}-1) - \frac{n}{n-1}.$$

Substituting this expression for $e_2 - e_1$ into (4) with $j = 2$ yields

$$e_3 - e_2 = \frac{(2)(1)}{(n-2)(n-1)}(2^{n-1}-1) - n\left[\frac{2}{(n-2)(n-1)} + \frac{1}{n-2}\right].$$

Continuing recursively in this way, we obtain the family of formulas

(5) $$e_{j+1} - e_j = \begin{bmatrix} n-1 \\ j \end{bmatrix}^{-1}(2^{n-1}-1) - n\sum_{r=0}^{j-1}(j)_r/(n-j+r)_{r+1}, \text{ for } j = 1, 2, \ldots, n-1,$$

where $(i)_m = (i)(i-1) \ldots (i-m+1)$ and $(i)_0 = 1$.

We now sum both sides of (5) from $j = 1$ to $j = k-1$, giving

$$e_k - e_1 = (2^{n-1}-1)\sum_{j=1}^{k-1}\begin{bmatrix} n-1 \\ j \end{bmatrix}^{-1} - n\sum_{j=1}^{k-1}\sum_{r=0}^{j-1}(j)_r/(n-j+r)_{r+1},$$

or

$$e_k = (2^{n-1}-1)\sum_{j=0}^{k-1}\begin{bmatrix} n-1 \\ j \end{bmatrix}^{-1} - n\sum_{j=1}^{k-1}\sum_{r=0}^{j-1}(j)_r/(n-j+r)_{r+1}, \text{ for } k = 2, 3, \ldots, n-1.$$

Of course, these formulas are needed only for $k \le [(n+1)/2]$ because $e_{n-j} = e_j$ for $j = 0, 1, \ldots, n$.

This game tends to take a relatively large number of moves to complete because the player who is currently behind is always the favorite to win the next move — the further behind the stronger the favorite. Also, the expected number of moves required to complete a game does not change very much whether the cards are initially divided equally or one of the players starts with only one card. For example, if $n = 10$ cards are used, then $e_1 = 511 = e_9$, $e_2 = 566.67 = e_8$, $e_3 = 579.33 = e_7$, $e_4 = 583.33 = e_6$, $e_5 = 584.33$.