Total Roman Domination Edge-Supercritical and Edge-Removal-Supercritical Graphs

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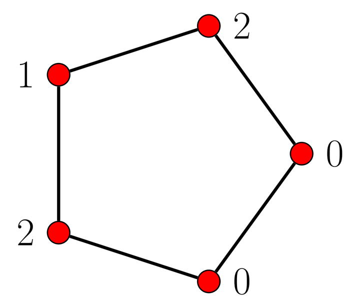
The impact of removing or adding edges on the total domination number of a graph was studied in [?] and [?]. We consider the same processes with respect to total Roman domination, and discover a connection which exists between the two processes.

Total Roman Domination

A *total Roman dominating function* (abbr. *TRD-function*) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that:

- (i) Every vertex v with $f(v) = 0$ is adjacent to some vertex u with $f(u) = 2$;
- (ii) The subgraph of G induced by the set of all vertices w such that $f(w) > 0$ has no isolated vertices.

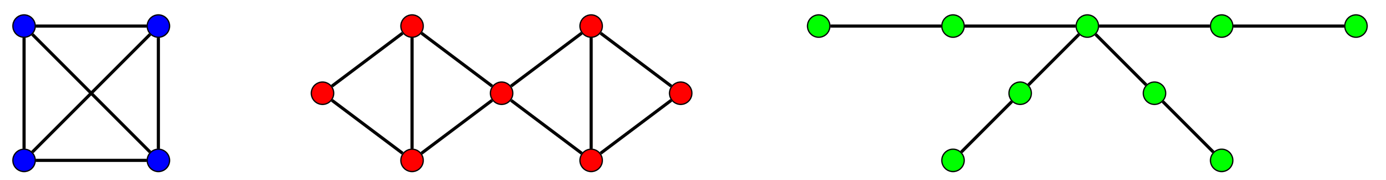


We define the *weight* of f to be $\Sigma_{v \in V(G)} f(v)$.

The *total Roman domination number* (abbr. *TRD-number*) $\gamma_{tR}(G)$ is the minimum weight of a TRD-function on G.

A TRD-function f such that $\omega(f) = \gamma_{tR}(G)$ is a *$\gamma_{tR}(G)$-function.*

Your Turn: What is the TRD-number for each of the following graphs? Can you find a γ_{tR}-function for each?

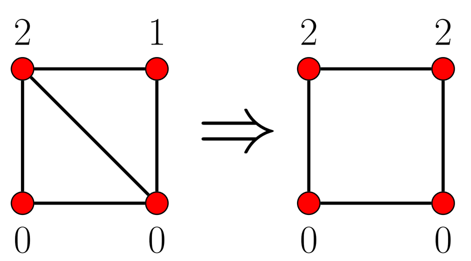


Removing an Edge

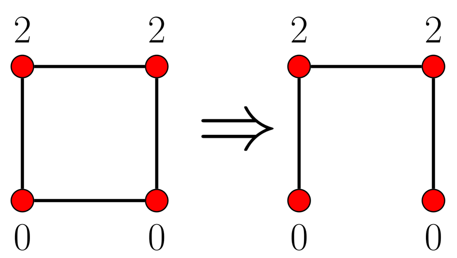
The removal of an edge from a graph G has the potential to increase its total Roman domination number.

For an edge $e \in E(G)$ incident with a degree 1 vertex, define $\gamma_{tR}(G - e) = \infty$.

An edge $e \in E(G)$ is *removal-critical* with respect to total Roman domination if $\gamma_{tR}(G) < \gamma_{tR}(G - e)$.



If every such edge is removal-critical, then G is *γ_{tR}-ER-critical.*



An edge $e \in E(G)$ is *removal-stable* with respect to total Roman domination if $\gamma_{tR}(G) = \gamma_{tR}(G - e)$.

If every such edge is removal stable, then G is *γ_{tR}-ER-stable.*

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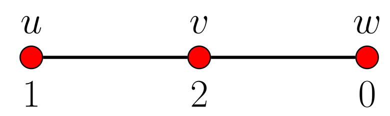
Characterizations

Observation 1. *If $uv \in E(G)$ is removal-critical, then, for any $\gamma_{tR}(G)$-function f, $\{f(u), f(v)\} \in \{\{0, 2\}, \{1, 2\}, \{2, 2\}, \{1, 1\}\}$.*

Proposition 2. *For a γ_{tR}-ER-critical graph G and any $\gamma_{tR}(G)$-function f, if $f(w) = 0$, then $\deg(w) = 1$. Moreover, $\delta(G) = 1$.*



Let $\mathcal{F}_n$ be the family of graphs constructed from the star graph S_n by appending $k_1, k_2, ..., k_n \geq 0$ pendant vertices to each pendant vertex of S_n.

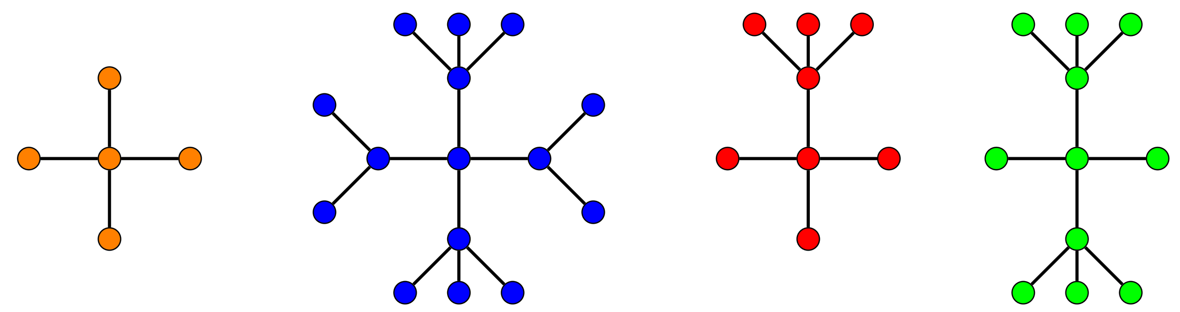


Figure 1: Examples of graphs in $\mathcal{F}_4$

Theorem 3. *A connected graph G is γ_{tR}-ER-critical if and only if G is a member of $\mathcal{F}_n$, for some $n \geq 1$, with $k_1, k_2, ..., k_n \neq 1$.*

An edge $e \in E(G)$ is *removal-supercritical* with respect to total Roman domination if $\gamma_{tR}(G) + 2 \leq \gamma_{tR}(G - e)$.

If every such edge is removal-supercritical, then G is *γ_{tR}-ER-supercritical.*

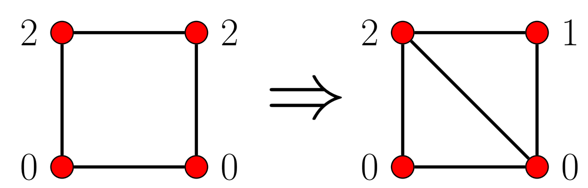
Your Turn: Which of the graphs in Figure 1 are γ_{tR}-ER-supercritical?

Theorem 4. *A connected graph G is γ_{tR}-ER-supercritical if and only if G is either a non-trivial star, or a double star where each non-pendant vertex has degree at least 3.*

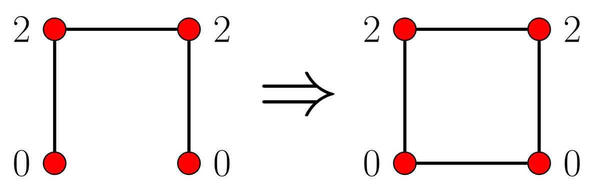
Adding an Edge

The addition of an edge to a graph G can decrease its TRD-number.

An edge $e \in E(\overline{G})$ is *critical* with respect to total Roman domination if $\gamma_{tR}(G+e) < \gamma_{tR}(G)$.



If every edge $e \in E(\overline{G}) \neq \emptyset$ is critical, then G is *γ_{tR}-edge-critical.*



An edge $e \in E(\overline{G})$ is *stable* with respect to total Roman domination if $\gamma_{tR}(G + e) = \gamma_{tR}(G)$.

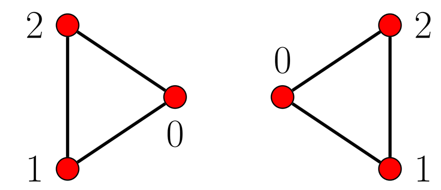
If every edge $e \in E(\overline{G})$ is stable, or if $E(\overline{G}) = \emptyset$, then G is *γ_{tR}-edge-stable.*

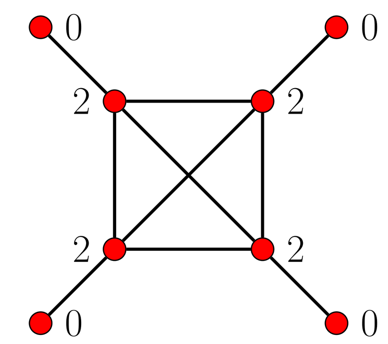
An edge $e \in E(\overline{G})$ is *supercritical* with respect to total Roman domination if $\gamma_{tR}(G + e) \leq \gamma_{tR}(G) - 2$.

If every edge $e \in E(\overline{G}) \neq \emptyset$ is supercritical, then G is *γ_{tR}-edge-supercritical.*

Examples

Proposition 5. [?] *If G is the union of $k \geq 2$ complete graphs, each of order at least 3, then G is γ_{tR}-edge-supercritical.*



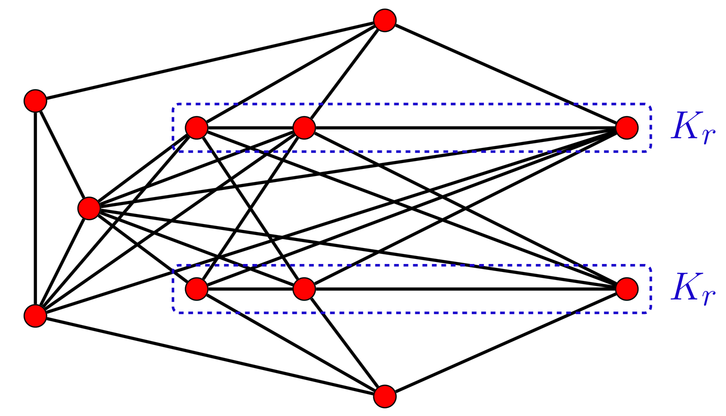


The *corona* of a graph G, denoted $\mathrm{cor}(G)$, is the graph obtained by adding a new pendant vertex to each vertex of G.

Proposition 6. *If $G = cor(K_n)$, $n \geq 4$, then G is γ_{tR}-edge-supercritical.*

Theorem 7. *There are no γ_{tR}-edge-supercritical trees.*

More complex γ_{tR}-edge-supercritical graphs also exist, such as the graph G_r, for $r \geq 2$, shown here.



"Critical" Results

What happens to γ_{tR}-ER-supercritical graphs when an edge is added, or γ_{tR}-edge-supercritical graphs when an edge is removed?

Theorem 8. *If G is a γ_{tR}-ER-supercritical graph, then G is γ_{tR}-edge-stable.*

Theorem 9. *If G is a γ_{tR}-edge-supercritical graph, then every non-pendant edge $e \in E(G)$ is removal-stable. If, in addition, $\delta(G) \geq 2$, then G is γ_{tR}-ER-stable.*

γ_{tR}-ER-supercritical $\Rightarrow$ **γ_{tR}-edge-stable**

γ_{tR}-edge-supercritical $\Rightarrow$ **γ_{tR}-ER-stable**

References

[1] W.J. Desormeaux, T.W. Haynes, M.A. Henning, Total domination critical and stable graphs upon edge removal, *Discrete Applied Mathematics.* **158** (2010), 1587–1592.

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