NEW CHARACTERIZATIONS OF CERTAIN STARLIKE AND
CONVEX GENERALIZED HYPERGEOMETRIC FUNCTIONS

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DM-378-IR

JUNE 1985
Applying various properties of a certain class of linear integral operators, the authors prove a number of theorems which provide interesting characterizations of starlike and convex generalized hypergeometric functions. Several useful corollaries are also deduced.

1980 Mathematics Subject Classification. Primary 30C45, 33A30.

1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1}
\]

\((a_1 = 1)\), \hspace{1cm} (1)

which are analytic in the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \). We denote by \( \mathcal{S}^* \) and \( \mathcal{K} \) the subclasses of \( \mathcal{A} \) consisting of all starlike functions in \( \mathbb{D} \) and of all convex functions in \( \mathbb{D} \), respectively. Note that \( f(z) \in \mathcal{K} \) if and only if \( zf'(z) \in \mathcal{S}^* \).

Let \( \alpha_j \) (\( j = 1, \ldots, p \)) and \( \beta_j \) (\( j = 1, \ldots, q \)) be complex numbers with

\[
\beta_j \neq 0, -1, -2, \ldots; \quad j = 1, \ldots, q.
\]

Then the generalized hypergeometric function \( \sum_{\binom{p}{q}}^{\binom{p}{q}} \) is defined by (see, e.g., [6], p. 19)

\[
\sum_{\binom{p}{q}}^{\binom{p}{q}} (a_1)_n \ldots (a_p)_n z^n \sum_{\binom{\beta_1}{\beta_q}}^{\binom{\beta_1}{\beta_q}} \sum_{\binom{\beta_1}{\beta_q}}^{\binom{\beta_1}{\beta_q}} (\beta_1)_n \ldots (\beta_q)_n n!
\]

\((p \leq q + 1)\), \hspace{1cm} (2)
where \((\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)\) is the Pochhammer symbol.

In order to prove our characterization theorems for these generalized hypergeometric functions, we need the following lemmas due to Owa and Srivastava [4].

**LEMMA 1.** Let the generalized hypergeometric function \( \frac{\pFqF{p}{q}}{z} \) defined by (2) satisfy the condition:

\[
\left| \frac{z \frac{\pFqF{p}{q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)}{\pFqF{p}{q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)}} < 1 \right|
\]

for \( z \in \mathbb{U} \). Then

\( z \frac{\pFqF{p}{q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)}{\mathbb{S}^*} \in \mathbb{S}^* \).

**LEMMA 2.** Let the generalized hypergeometric function \( \frac{\pFqF{p}{q}}{z} \) defined by (2) satisfy the condition (3) for \( z \in \mathbb{U} \). Then

\( z \frac{\pFqF{p+1}{q+1}(\alpha_1, \ldots, \alpha_p, 1; \beta_1, \ldots, \beta_q, 2; z)}{\mathbb{S}^*} \in \mathbb{S}^* \).

**LEMMA 3.** Let the generalized hypergeometric function \( \frac{\pFqF{p}{q}}{z} \) defined by (2) satisfy the condition:

\[
\left| \frac{z \frac{\pFqF{p}{q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)}{\pFqF{p}{q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)}} < 1 \right|
\]

for \( z \in \mathbb{U} \) and \( \prod_{j=1}^{p} a_j \neq 0 \).

Then
2. PROPERTIES OF A CERTAIN LINEAR INTEGRAL OPERATOR

Let $\mathcal{J}_\gamma(f)$ be a linear integral operator defined by

$$
\mathcal{J}_\gamma(f) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) \, dt \quad (\gamma > -1)
$$

for $f(z) \in \mathcal{A}$. The operator $\mathcal{J}_\gamma(f)$, when $\gamma \in \mathcal{N} = \{1, 2, 3, \ldots\}$ was studied by Bernardi [1]. In particular, the operator $\mathcal{J}_1(f)$ was considered earlier by Libera [2] and Livingston [3].

We require the following properties of $\mathcal{J}_\gamma(f)$ obtained by Pascu [5].

**Lemma 4.** If $f(z) \in \mathcal{S}^*$, then $\mathcal{J}_\gamma(f) \in \mathcal{S}^*$ for $0 \leq \gamma \leq 1$.

**Lemma 5.** If $f(z) \in \mathcal{H}$, then $\mathcal{J}_\gamma(f) \in \mathcal{H}$ for $0 \leq \gamma \leq 1$.

3. APPLICATIONS TO GENERALIZED HYPERGEOMETRIC FUNCTIONS

Our characterization theorem for a class of starlike generalized hypergeometric functions is contained in

**Theorem 1.** Let the generalized hypergeometric function $\mathcal{F}_{p, q}^{(z)}$ defined by (2) satisfy the condition (3) for $z \in \mathcal{H}$. Then

$$
\mathcal{J}_\gamma(z \mathcal{F}_{p, q}^{(z)}(a_1, \ldots, a_p, \gamma + 1; \beta_1, \ldots, \beta_q, \gamma + 2; z)) \in \mathcal{S}^*
$$

for $0 \leq \gamma \leq 1$. 


PROOF. First of all, we note that

$$z \begin{pmatrix} F^p q \alpha_1, \ldots, \alpha_p ; \beta_1, \ldots, \beta_q ; z \end{pmatrix} \in \mathcal{S}^*$$

by Lemma 1. Next, applying Lemma 4 to

$$z \begin{pmatrix} F^p q \alpha_1, \ldots, \alpha_p ; \beta_1, \ldots, \beta_q ; z \end{pmatrix},$$

we have

$$\mathcal{J}_\gamma(z \begin{pmatrix} F^p q \end{pmatrix}) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^\gamma \begin{pmatrix} F^p q \alpha_1, \ldots, \alpha_p ; \beta_1, \ldots, \beta_q ; t \end{pmatrix} dt$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{(\gamma+1)_n}{(\gamma+2)_n} \frac{z^{n+1}}{n!}$$

$$= z \begin{pmatrix} F^p q+1 \alpha_1, \ldots, \alpha_p, \gamma+1 ; \beta_1, \ldots, \beta_q, \gamma+2 ; z \end{pmatrix} \in \mathcal{S}^*.$$
By setting $\gamma = 1$ in Theorem 2, we obtain

COROLLARY 2. Let the generalized hypergeometric function $\frac{F}{p^q}(z)$ defined by (2) satisfy the condition (3) for $z \in \mathbb{H}$. Then

$$z^{p+1}F_{q+1}(a_1, \ldots, a_p, 1; \beta_1, \ldots, \beta_q, 3; z) \in \mathbb{H}.$$  

Finally, by applying Lemma 3 and Lemma 4, we have

THEOREM 3. Let the generalized hypergeometric function $\frac{F}{p^q}(z)$ defined by (2) satisfy the condition (4) for

$$z \in \mathbb{H} \text{ and } \prod_{j=1}^{p} \alpha_j \neq 0.$$  

Then

$$z^{p+2}F_{q+2}(a_1+1, \ldots, a_p+1, 1, \gamma+1; \beta_1+1, \ldots, \beta_q+1, 2, \gamma+2; z) \in \mathbb{S}^*$$

for $0 \leq \gamma \leq 1$. 

The special case of Theorem 3 when $\gamma = 1$ is given by

COROLLARY 3. Let the generalized hypergeometric function $\frac{F}{p^q}(z)$ defined by (2) satisfy the condition (4) for

$$z \in \mathbb{H} \text{ and } \prod_{j=1}^{p} \alpha_j \neq 0.$$  

Then

$$z^{p+1}F_{q+1}(a_1+1, \ldots, a_p+1, 1; \beta_1+1, \ldots, \beta_q+1, 3; z) \in \mathbb{S}^*.$$
ACKNOWLEDGEMENTS

The present investigation was carried out at the University of Victoria while the second author was on study leave from Kinki University, Osaka, Japan. This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant A-7353.

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