Monomino-Domino Tatami Coverings

by

Alejandro Erickson
B.Sc., Simon Fraser University, 2007
M.Math., University of Waterloo, 2008

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

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University of Victoria

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ABSTRACT

We present several new results on the combinatorial properties of a locally restricted version of monomino-domino coverings of rectilinear regions. These are monomino-domino tatami coverings, and the restriction is that no four tiles may meet at any point. The global structure that the tatami restriction induces has numerous implications, and provides a powerful tool for solving enumeration problems on tatami coverings. Among these we address the enumeration of coverings of rectangles, with various parameters, and we develop algorithms for exhaustive generation of coverings, in constant amortised time per covering. We also consider computational complexity on two fronts; firstly, the structure shows that the space required to store a covering of the rectangle is linear in its longest dimension, and secondly, it is NP-complete to decide whether an arbitrary polyomino can be tatami-covered only with dominoes.
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A diagonal flip in a $9 \times 9$ vertical bond.

(c), If both diagonals are blocked, then $c < r$. The covering is at least this tall and at most this wide.

Each vortex and vee is associated with segments of monomino-free grid squares shown in purple. (a) Segments associated with vortices have length at least three. Those associated with vees have at least two 0s. (b) The two types of updates to sequences $P$ and $Q$. The upper sequences are before the updates and the lower are after updates. The symbol $\times$ represents a deletion from the sequence.

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A sequence of 5 diagonal flips, shown in blue, beginning with a bond, results in this covering. Flipped monominoes are coloured red.

The magenta diagonal contains 5 tiles and it is flipped up. The grey diagonal contains 6 tiles and it is flipped down.

When $w$ is flipped up (magenta), there are $n - 3$ independently flippable diagonals (grey).

Pairs of monominoes in the respective superimposed coverings are associated if they share an edge. Their respective up and down diagonals are also associated.

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DEDICATION

To Marina Marshall Silva.
ΕΠΙΓΡΑΦΗ

Μὴ μου τοὺς κύκλους τάραττε.

– Archimedes\(^1\)

\(^1\)Noted in one of several different accounts of Archimedes’ death.
Chapter 1

Introduction

Tatami mats are a common floor furnishing, originating in aristocratic Japan, during the Heian period (794-1185). These thick mats, once hand-made with a rice straw core and a soft, woven rush straw exterior, are now machine-produced in a variety of materials, and are available in mass-market stores. They are so integral to Japanese culture, that a standard sized mat is the unit of measurement in many architectural applications (see [18]).

Here we depart considerably from traditional layouts, but we retain two essential features. The first of these is aspect ratio; a full mat is a $1 \times 2$ domino, and a half mat is a $1 \times 1$ monomino. The second is the 17th century rule for creating auspicious arrangements: no four mats may meet.

Counting domino coverings is a classic area of enumerative combinatorics and theoretical computer science, but less attention has been paid to problems where the local interactions of the dominoes are restricted in some fashion. The tatami restriction is perhaps the most natural of these, and it imposes a visually appealing structure with nice combinatorial properties (see Figure 1.1). It is the subject of exercise 7.1.4.215 in Volume 4 of “The Art of Computer Programming” (see [19]), where Knuth reprints a diagram from Jinkôki, by the renowned 17th century Japanese mathematician, Mitsuyoshi Yoshida, and recently the tatami restriction has been studied in several research papers (see [1, 10–12, 14, 26]).

The integer grid is the set of unit grid squares arranged on the integer lattice with their corners on lattice points.

**Definition 1.1** (Monomino-domino tatami covering). Let $R$ be a subset of the integer grid. A monomino-domino tatami covering of $R$ is an exact covering of $R$ with non-overlapping $1 \times 1$ monominoes, $1 \times 2$ horizontal dominoes, and $2 \times 1$ vertical dominoes.
in which no four tiles meet. The terms covering and tatami covering refer to monomino-dominino tatami covering in the remainder of this dissertation, except in Chapter 6, where only domino tatami coverings are considered.

Further insight may be gained from the following graph theoretic interpretation. Let $G$ be a graph, and let $H$ be a subgraph of $G$. A matching in $G$ meets $H$, if $H$ contains at least one edge in the matching. A tatami covering is a matching on an induced subgraph of the infinite grid-graph, which meets all 4-cycles. In this setting, tatami coverings are a special case of $H$-transverse matchings, in which a matching of the graph, $G$, meets every instance of the subgraph, $H$, that occurs in $G$. Ross Churchley and Jing Huang show, in [4], that deciding whether or not $G$ has a $C_4$ transverse matching is NP-complete. Consider the physical properties of a matching in which matched edges are rigid bonds between vertices, while unmatched edges are weaker bonds. Intuitively, a tatami restricted matching has some structural advantage over any non-tatami restricted matching, because there are no 4-cycles consisting only of weak bonds.

Our primary concern, however, is with the enumerative combinatorics of tatami coverings. This is the subject of Chapters 3 and 4, followed by natural extensions to combinatorial algorithms, in Chapter 5. Chapter 6 diverges from this pattern, and relates a computational complexity result similar to the one mentioned above.

1.1 Main results

The combinatorial richness of tatami coverings lies in the surprising global structure that is imposed by the tatami restriction. Tatami coverings of rectangular grids are determined by four local configurations of tiles, called features, which
may be rotated or reflected. The features themselves “force” the placement of other tiles, and must be arranged such that the forced tiles are not in conflicting places.

In Chapter 2 we describe the structure in detail, which was largely unknown before the publication of [12]. Prior to this, the structure of domino-only tatami coverings was resolved by Ruskey and Woodcock, in [26]. However, there was expressed interest — in the literature and by the present author — about the extension to monomino-domino coverings. Specifically, Alhazov et al. followed up on the aforementioned domino-only research with a treatment of odd-area tatami coverings which include a single monomino (see [1]). They closed with this remark:

However, the variety of coverings with arbitrary number of monominoes is quite “wild” in [the] sense that such coverings cannot be easily decomposed, see Figure 11; therefore, most results presented here do not generalise to arbitrary number of monominoes, the techniques used here are not applicable, and it is expected that any characterisation or enumeration of them would be much more complicated.

The structure discovered by the present author (see [12]), however, reveals the opposite; the coverings with an arbitrary number of monominoes are easily decomposed. The decomposition has a satisfying symmetry, it is amenable to inductive arguments, and it shows that the space complexity of a tatami covering is linear in the dimensions of the grid (see Figures 2.1 and 2.8).

The main results of this thesis are listed below. Let \( T(r, c, m) \) be the number of tatami coverings of the \( r \times c \) rectangle, with \( m \) monominoes.

**Theorem 3.2** (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]). If \( T(r, c, m) > 0 \), then \( m \) has the same parity as \( rc \) and

\[
m \leq \max\{r + 1, c + 1\}.
\]

**Theorem 3.11**. If \( n \) and \( m \) have the same parity, and \( m < n \), then \( T(n, n, m) = m2^m + (m + 1)2^{m+1} \).

This result is by Mark Schurch, but we give a new proof using the transfer matrix method (see [28]).

**Theorem 3.17** (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]).
Let \( T(r, c) = \sum_{m \geq 0} T(r, c, m) \). The generating function
\[
T_r(\lambda) = \sum_{c \geq 0} T(r, c) \lambda^c,
\]
is a rational function.

In Section 3.2.1 we show that every \( n \times n \) covering with \( n \) monominoes has monominoes in exactly two corners, and those corners share a boundary. Let \( T_n \) be those coverings with monominoes in their top corners. Let \( V(n, k) \) and \( H(n, k) \) be the number of coverings of \( T_n \) with exactly \( k \) vertical and horizontal dominoes, respectively.

**Theorem 4.2** (Erickson, Ruskey, 2013, [11]). The generating polynomial
\[
VH_n(z) := 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{n-i-2}(z)S_{i-1}(z)z^{n-i-1} + \left( \frac{S_{\lfloor \frac{n-2}{2} \rfloor}(z)}{z} \right)^2,
\]
where \( S_n(z) = \prod_{i=1}^{n} (1 + z^i) \), is equal to
\[
\sum_{k \geq 0} V(n, k)z^k \quad \text{and} \quad \sum_{k \geq 0} H(n, k)z^k,
\]
for even and odd \( n \), respectively.

Don Knuth was the first to produce \( VH_n(z) \) for small \( n \), and we have generalised his observations.

**Theorem 4.6** (Erickson, Ruskey, 2013, [11]). The generating polynomial \( VH_n(z) \) has the factorisation
\[
VH_n(z) = P_n(z)D_n(z)
\]
where \( P_n(z) \) is a polynomial and
\[
D_n(z) = \prod_{j \geq 1} S_{\lfloor \frac{n-2}{2j} \rfloor}(z).
\]

Three classes of tatami coverings, which are related to those we have enumerated, can be exhaustively generated in constant amortised time per covering; the number of operations required to generate all the coverings in each class is a con-
stant factor of the size of the class. Each theorem follows from the construction of a data structure and an algorithm.

**Theorem 5.2.** There exists a Gray code for listing the elements of $T_n$ such that successive coverings differ by exactly one diagonal flip. The list can be created in constant amortised time per covering.

**Theorem 5.4.** The coverings in $T_n$ with exactly $k$ vertical dominoes can be exhaustively generated in constant amortised time.

A rectangular grid of height $r$ and infinite width is called a *strip*. Tatami coverings of the height $r$ strip — called *strip coverings* — encapsulate some of the combinatorial properties of coverings of the rectangle. Here, isomorphism refers to the topological arrangement of certain structural features, by ignoring their precise horizontal location.

**Theorem 5.5.** If $S(r, n)$ is the number of non-isomorphic strip coverings with exactly $n$ features, then it satisfies the system of homogeneous linear recurrence relations,

\[
\begin{align*}
V_r(n) &= 4(r - 1)V_r(n - 1) + 2H_r(n - 1), \text{ where } V_r(0) = 1, V_r(1) = 4r - 2; \\
H_r(n) &= 2V_r(n - 1), \text{ where } H_r(0) = 1; \\
S(r, n) &= V_r(n) + H_r(n).
\end{align*}
\]

Our final result is on the computational complexity of Domino Tatami Covering, defined below.

**INSTANCE:** A rectilinear region $R$, on the integer lattice, represented, say, as $n$ line segments joining the corners of the polygon, which need not be simply connected.

**QUESTION:** Can $R$ be covered by dominoes such that no four of them meet at any one point?

The solution makes use of SAT-solvers to find the gadgets used in a polynomial reduction from an NP-complete problem called planar 3SAT.

**Theorem 6.2.** Domino tatami covering is NP-complete
1.2 Outline

Chapter 2 describes the structure of tatami coverings of rectangles, and a useful abstraction, called the T-diagram. We prove that such a covering can be recovered from the tiles on its boundary, and thereby show that the data required to store a covering is at most linear in the length of its perimeter.

Chapter 3 gives all of the new and known formulas for $T(r, c, m)$, defined in Section 1.1. In particular, we give closed form formulae for all instances where $r = c$, in Section 3.2. When $r < c$, general formulae for $T(r, c, m)$ are unknown for fixed $m$, besides certain cases where $T(r, c, m) = 0$, by Theorem 3.2, and $m = 0$, which is the result of [26], and $m = 1$, from [1]. We do, however, give conjectured formulae for $T(r, c, m)$ when $m$ is maximum, via Jennifer Woodcock (private communication), in Conjecture 3.14. The values of $T(r, c)$, defined above, are given as a rational generating function. The chapter concludes with the aforementioned alternate proof of Theorem 3.17.

Chapter 4 expands on Theorem 3.6 — that is, $T(n, n, n) = n2^{n-1}$ — to derive the generating polynomial in Theorem 4.2 which counts coverings with $k$ vertical dominoes. From this, we describe a natural partition of the $n2^{n-1}$ coverings into $n$ classes of size $2^{n-1}$, and we pursue the factorisation of $\mathcal{VH}_n(z)$, observed for small $n$ by Knuth. The polynomial, $P_n(z)$ mentioned above in Theorem 4.6, has several compelling properties, which are demonstrated both with proof and empirical evidence in its coefficients.

Chapter 5 develops some of the enumeration results of the previous two chapters into combinatorial algorithms. The first three of these are based on the results of Chapter 4, generating all coverings counted by $T(n, n, n)$, or the subset of these with exactly $k$ vertical dominoes. The fourth and final combinatorial algorithm generates strip coverings.

Chapter 6 describes a polynomial reduction from planar 3-SAT to tatami domino covering (see Definition 6.1). This is a computer aided proof, where SAT-solvers were used to find gadgets in the reduction, however, the gadgets that were found can be verified by hand.

Chapter 7 summarises and discusses open problems encountered, some of which are mentioned in previous chapters.
Chapter 2

Structure of tatami coverings

The classical brick laying pattern is fundamental to tatami coverings, and it is defined precisely below. The \textit{checkered integer grid} is the integer grid with its grid squares coloured black or white, with no adjacent squares of the same colour. Up to a permutation of the colours, the checkered integer is unique. A \textit{checkered grid}, $C$, is a subset of the checkered integer grid. Consider a tatami covering of $C$ in which every black square together with the white square to its right is covered by a horizontal domino, and the remaining squares are covered by monominoes. This is one of the four possible \textit{running bonds}, or simply \textit{bonds}, on $C$. The other three are defined as above, but with the white square above, below, or to the left of the black square. \textit{Vertical bond} is bond with no horizontal dominoes, and \textit{horizontal bond} is bond with no vertical dominoes.

All tatami coverings have an underlying structure which partitions the grid into regions filled with bond, and isolated monominoes. For example, in Figure 2.1 there are 10 regions, plus two isolated monominoes. The partition has some special properties which are described later in the present chapter.

![Figure 2.1](image)

Figure 2.1: A covering showing all four types of sources. Coloured in magenta, from left to right they are, a clockwise vortex, a vertical bidimer, a loner, a vee, and two more loners.
Wherever a horizontal and vertical domino share an edge (□□), either the placement of another domino is forced to preserve the tatami condition, or the tiles make a T with the boundary of the grid (□□). In the former case, the placement of the new domino again causes the sharing of an edge (□□), and so on (□□), until the boundary is reached.

This successive placement of dominoes constructs a skinny herringbone formation, called a ray. Observe that once a ray is started, it propagates to the boundary. But how do they start? In a rectangular grid, we will show that a ray starts at one of four possible types of sources. In our discussion we use inline diagrams to depict the tiles that can cover the grid squares at the start of a ray. We need not consider the case where the innermost square (denoted by the circle in □□) is covered by a vertical domino (□□) because this would simply move the start of the ray.

If it is covered by a horizontal domino (□□), the source, which consists of the two dominoes that share a long edge, is called a bidimer. Otherwise the circle is covered by a monomino (□□) in which case we consider the grid square beside it (□□). If the circle in its new location is covered by a monomino, then the source is called a vee (□□); if the circle is covered by a vertical domino, then the source is called a vortex (□□); if the circle is covered by a horizontal domino, then the source is called a loner (□□). Each of these four types of sources forces at least one ray in the covering and all rays begin at either a bidimer, vee, vortex or loner. The union of a source and the rays propagating from it is called a feature. The different types of features are depicted in Figures 2.2-2.4, where the sources are coloured and the rays are shown in white. A bidimer or vortex may appear anywhere in a covering, as long as the coloured tiles are within its boundaries. The vees and loners, on the other hand, must appear along a boundary, as shown in Figure 2.2.

![Figure 2.2: (a), A loner feature and, (b), a vee feature, each overlaid with its feature diagram. These two types of sources must have their coloured tiles on a boundary, as shown, up to rotational symmetry.](image)

The bold staircase-shaped arrows overlaying each ray in Figures 2.2-2.4 are called ray diagrams. The precise start and finish of the arrows is shown for one
ray diagram in each depicted feature, from which the general definition can be inferred. Ray diagrams of the four possible orientations are represented by the symbols $↗, ↘, ↖, \searrow$, and $\nearrow$ regardless of the length of the ray diagram.

The union of ray diagrams in each of the four types of source-ray drawings in Figures 2.2-2.4 is called a feature diagram. Note that feature diagrams do not intersect. A collection of feature diagrams partitions the rectangle, and those parts with unit area greater than 2 are called regions. A collection of feature diagrams is called a $T$-diagram (see Figure 2.5) if there is a bijective correspondence between the features of a tatami covering and the diagrams of the collection. If a collection of feature diagrams is a $T$-diagram, then its regions can be covered with vertical and horizontal bond to obtain a tatami covering.

A ray diagram may meet more than two regions of a collection of feature diagrams, so we say that a ray diagram bounds (at most) the two regions of bond containing the tiles of the ray it represents. Specifically, one region contains the vertical tiles and the other region contains the horizontal tiles, provided the regions exist.

If a ray diagram bounds a region, then its ray determines the position of at least one domino in that region. The bond in this region is therefore determined
Figure 2.5: (a) The T-diagram of Figure 2.1. (b) A collection of feature diagrams that is not a T-diagram.

uniquely, and must agree with that of the other ray diagrams bounding the region. Rays bounding the same region are adjacent. Lemma 2.1 gives conditions for a collection of feature diagrams to be a T-diagram; conditions that can each be checked in a constant number of operations.

Consider a region, \( R \), with a ray diagram, \( D \), that bounds it. The orientation of any bond covering \( R \) is completely determined by \( D \) (see Figure 2.6). If all of the ray diagrams that bound \( D \) determine the same orientation of bond inside \( R \), then we say the region \( R \) is consistent.

A path is a sequence of grid squares, with consecutive squares connected at edges, and no repeated squares. The length of a path is the number of grid squares it contains. Let \( p \) be a path that is entirely contained inside a region. If an edge of the first square of \( p \) borders a ray diagram, and an edge of the last square borders another ray diagram, then \( p \) connects these ray diagrams. Given two ray diagrams, the length of every path connecting them has the same parity, which can be checked in constant time by looking at the colour of the grid squares on the checkered grid using the first and last squares.

Lemma 2.1. A collection of feature diagrams is a T-diagram if and only if the following conditions are satisfied.

(a) Every region is consistent (see Figure 2.6).

(b) The lengths of paths connecting adjacent ray diagrams must satisfy the parity requirements tabulated below for each orientation of ray diagram (note that “\( \times \)” entries in the table are impossible by \( [a] \) and see Figure 2.7).

<table>
<thead>
<tr>
<th>Vertical bond</th>
<th>Horizontal bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \uparrow )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( \leftarrow )</td>
<td>( \leftarrow )</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
</tr>
<tr>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>( \times )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>even</td>
<td>even</td>
</tr>
<tr>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
</tr>
<tr>
<td>even</td>
<td>even</td>
</tr>
</tbody>
</table>
Proof. Let $R$ be a rectangular grid and let $\mathcal{F}$ be a collection of feature diagrams in $R$.

Suppose (a) and (b) are satisfied for each region of $\mathcal{F}$, and consider a checkered region, $R$, that must be covered with, say, vertical bond. Let $\alpha$ and $\beta$ be ray diagrams that bound $R$, which are connected by $p$. The colour of the squares bordering $\alpha$ uniquely determines the colour of the squares bordering $\beta$ (and vice versa) by the parity of the length of $p$. Together with the fact that $p$ satisfies (b), we ensure that the same type of vertical bond meets both $\alpha$ and $\beta$ in a way that does not conflict with the tiles of the rays they represent.

Thus every region of $\mathcal{F}$ can be covered with bond that is not in conflict with the tiles belonging to the features represented by $\mathcal{F}$. The tatami condition is satisfied in the bond(s) and in the features (where regions of bond meet), therefore $\mathcal{F}$ is a T-diagram, and this proves sufficiency.

A tatami covering of a T-diagram must cover its regions with bond because there can be no interior monominoes — these would create vortices not accounted for by the T-diagram — or vertical dominoes touching horizontal dominoes — these would create rays not accounted for by the T-diagram. Therefore (a) and (b) must be satisfied.

\[ \square \]

2.1 Storage complexity

The characterisation of Lemma 2.1 has some implications for the space complexity of a covering.
Lemma 2.2. Let $G$ be an $r \times c$ grid, with $r < c$.

(i) A tatami covering of $G$ is uniquely determined by the tiles on its boundary.

(ii) The storage requirement for a tatami covering of $G$ is $O(c)$; that is, a tatami covering can be recovered from $O(c)$ bits.

(iii) Whether a collection of feature diagrams in $G$ is a $T$-diagram can be determined in time $O(c)$.

Proof. To prove (i), we need to show that we can recover the $T$-diagram from the tiles that touch the boundary. Those portions of the $T$-diagram corresponding to vees and loners, as well as bidimers whose source tiles are both on the boundary (\(\begin{array}{c} \text{V} \\ \text{I} \end{array}\)), are easy to recover. The black ray diagrams in Figure 2.8 show their recovery. Imagine filling in the remaining red ray diagrams, whose ends look like \(\begin{array}{c} \text{H} \\ \text{H} \end{array}\), by following them naïvely, backwards from their endings to the boundary. The ends of the four ray diagrams emanating from a bidimer or vortex will always form exactly one of the four patterns illustrated in Figure 2.9; in each case, it is straightforward to recover the position and type of source. This proves (ii).

Part (ii) follows from (i), because we can use a ternary encoding for the perimeter squares.

![Figure 2.8](image)

Figure 2.8: The same covering as in Figure 2.1 with only the boundary tiles showing. Ray diagrams emanating from sources on the boundary are in black and otherwise, they are drawn naïvely in red, to be matched with a candidate source from Figure 2.9.

Claim (iii) is true provided that Lemma 2.1 only needs to be applied to $O(c)$ pairs of adjacent ray diagrams. Each ray diagram bounds exactly two regions, each of which is bounded by at most three other ray diagrams. Thus, a ray diagram is adjacent to at most six others. Let $G = (V, E)$ be a graph whose vertex set corresponds to the set of ray diagrams. There is an edge between a pair of vertices whenever the corresponding ray diagrams are adjacent. The maximum degree of $G$ can never be more than 6 (the highest achievable degree is actually
(a) Clockwise and counterclockwise vortices. (b) Horizontal and vertical bidimers.

Figure 2.9: The four types of vortices and bidimers are recoverable from the ends of their ray diagrams, at the boundary of the grid. Extending the ray diagrams naively, backwards from the boundary, we form one of the two patterns in the red overlay. One occurs only for bidimers and the other for vortices. Successively placing tiles, working from the ends of the rays towards the central configuration, we also find the orientation of the source, as shown in the figure.

4, as in Figure 2.1, so the number of adjacencies, \(|E|\), and hence applications of Lemma 2.1 is linear in the number of rays, \(|V|\), which is at most four times the number of features, which is in \(O(c)\). This proves (iii).
Chapter 3

Enumerating tatami coverings with \( m \) monominoes

Let \( T(r, c, m) \) be the number of tatami coverings of a rectangular grid with \( r \) rows, \( c \) columns, and \( m \) monominoes. Also, \( T(r, c) \) will denote the sum

\[
T(r, c) = \sum_{m \geq 0} T(r, c, m).
\]

The enumeration of tatami coverings of rectangles, with respect to \( r, c, \) and \( m \) is complete for square grids, and partially complete for maximum \( m \). This chapter contains all of these results. Values of \( T(r, c, m) \) for all grids up to \( 14 \times 14 \) are given in Tables A.1–A.18. Table A.17 is \( T(r, c) \), the aggregate of these, and Tables A.18 and A.19 count coverings of the square, and coverings with maximum monominoes, respectively.

The following definition is used throughout this dissertation.

**Definition 3.1** (Diagonal). Let \( T \) be a tatami covering of the \( r \times c \) grid. A diagonal, \( D \), of \( T \) is a contiguous sequence of like-aligned dominoes whose centers lie on a line with slope \( \pm 1 \). The sequence must begin with a domino with its long edge on the boundary; the final domino touches an adjacent boundary and shares an edge with a monomino, which is also considered to be part of the diagonal (see Figures 3.2(a) and 3.1).

A diagonal flip of \( D \) consists of removing it from \( T \), reflecting horizontally, rotating by \( \frac{\pi}{2} \) radians, and placing it back onto the grid squares that were vacated.

There are three things to note about the diagonal flip:

- a flipped diagonal is a diagonal;
• the operation preserves the tatami restriction; and,

• it changes the orientation of the dominoes that it contains, and maps the monomino to the other extreme of the diagonal.

![Figure 3.1: A diagonal flip in a 9 × 9 vertical bond.](image)

### 3.1 Maximum number of monominoes

We now give necessary conditions for $T(r, c, m)$ to be non-zero.

**Theorem 3.2.** If $T(r, c, m) > 0$, then $m$ has the same parity as $rc$ and

$$m \leq \max(r + 1, c + 1).$$

**Proof.** Let $r, c$ and $m$ be such that $T(r, c, m) > 0$ and let $d$ be the number of grid squares covered by dominoes in an $r \times c$ tatami covering so that $m = rc - d$. Since $d$ is even, $m$ must have the same parity as $rc$.

We assume that $r \leq c$, and prove that $m \leq c + 1$. The proof proceeds in two steps. First, we will show that a monomino on a vertical boundary of any covering can be *mapped* to the top or bottom, without altering the position of any other monomino. Then we can restrict our attention to coverings where all monominoes appear on the top or bottom boundaries, or in the interior. Secondly, we will show that there can be at most $c + 1$ monominoes on the combined horizontal boundaries.

Let $T$ be a tatami covering of the $r \times c$ grid with a monomino, $\mu$, on the left boundary, touching neither the bottom nor the top boundary. The monomino $\mu$ is (a), part of a vee or a loner, or is (b), surrounded by horizontal dominoes. If $\mu$ is part of a diagonal, it can be mapped to the top or bottom boundary via a diagonal flip. In case (a) a diagonal clearly exists since it is a source and its ray will hit a horizontal boundary because $r \leq c$. 
(a) A diagonal flip. (b) The case for vees. (c) Figure 3.2: If both diagonals are blocked, then \( c < r \). The covering is at least this tall and at most this wide.

If \( \mu \) is surrounded by horizontal dominoes, then we argue by contradiction. Suppose neither the upward, nor the downward diagonal exists, then they must each be impeded by a distinct ray. Such rays have this horizontal region to the left so the upper one is directed SE and the lower NE and they meet the right boundary (before intersecting). Referring to Figure 3.2(c),

\[
\alpha + \beta + j = \gamma + \delta + 1 \leq r
\]

\[
\leq c \leq c' = \alpha + \gamma = \beta + \delta,
\]

where \( j \) is some odd number. Thus \( \alpha + \beta + j \leq \alpha + \gamma \) implying that \( \beta < \gamma \). On the other hand,

\[
\gamma + \delta + 1 = r \leq c \leq c' = \beta + \delta
\]

implies that \( \gamma < \beta \), which is a contradiction. Therefore at least one of the diagonals exists and the monomino can be mapped to a horizontal boundary via a diagonal flip.

We may now assume that there are no monominoes strictly on the vertical boundaries of the covering, and therefore all monominoes are either in the top or bottom rows or in vortices. Let \( v \) be the number of vortices. Encode the bottom and top rows of the covering by length \( c \) binary sequences \( Q \) and \( P \), respectively. In the sequences, 1s represent monominoes and 0s represent squares covered by dominoes.

Summarizing the argument below, we first find a bijective correspondence between occurrences of 11 in \( Q \) and 00 in \( P \), and vice versa. The 11s are replaced by 101 and 00s are replaced by 0. Second, we find \( v \) occurrences of 000, where \( v \) is the number of vortices, in each (updated) sequence. Replace each 000 with 00. If \( v \) is the number of vortices, then the total length of the updated sequences is \( 2c - 4v \), and total number of 1s is at most \( c - 2v + 1 \). Adding the monominoes in
the vortex gives the desired upper bound of $c - v + 1 \leq c + 1$.

A 11 in $Q$ is a vee in the top row; the vee has a region of horizontal dominoes directly below it. This region of horizontal bond must reach the bottom row somewhere, otherwise, by an argument similar to one given previously, we would have $c < r$ (see Figure 3.3(a)). Therefore, there must be a 00 in $P$ unique to these 1s in $Q$. One of these 0s is used to separate the 1s (see Figure 3.3(b)). The updated sequences contain no 11, but the total number of 1s remains unchanged.

Each vortex generates rays which reach the top and bottom boundaries, since $r \leq c$, and the dominoes on either side of the rays induce a 000 in $P$ and another 000 in $Q$ (see Figure 3.3(a)). (Although not used in this proof, note that the comments above also apply to bidimers.) Removing a 00 from each triple yields a pair of sequences whose combined length is $2c - 4v$, neither of which contains a 11 (see Figure 3.3(b)). Thus the total number of 1s is at most $\lceil |P|/2 \rceil + \lceil |Q|/2 \rceil$, which is at most $c - 2v + 1$. Adding back the $v$ vortex monominoes, we conclude that there are at most $c - v + 1$ monominoes in total, which finishes the proof.

Note that, to achieve the bound of $c + 1$, we must have $v = 0$, and that the maximum is achieved by a vertical bond.

![Figure 3.3](a)

![Figure 3.3](b)

Figure 3.3: Each vortex and vee is associated with segments of monomino-free grid squares shown in purple. (a) Segments associated with vortices have length at least three. Those associated with vees have at least two 0s. (b) The two types of updates to sequences $P$ and $Q$. The upper sequences are before the updates and the lower are after updates. The symbol $\times$ represents a deletion from the sequence.

The converse of Theorem 3.2 is false; for example, Alhazov et al. (see [1]) show that $T(9, 13, 1) = 0$. We now state a couple of consequences of Theorem 3.2.
Corollary 3.3. The following three statements are true for tatami coverings of an \( r \times c \) grid with \( r \leq c \).

(i) The maximum possible number of monominoes is \( c + 1 \) if \( r \) is even and \( c \) is odd; otherwise it is \( c \). There is a tatami covering achieving this maximum.

(ii) A tatami covering with the maximum number of monominoes has no vortices.

(iii) A tatami covering with the maximum number of monominoes has no bidimers.

Proof. (i) That this is the correct maximum value can be inferred from Theorem 3.2. A covering consisting only of vertical bond achieves it, for example.

(ii) This was noted at the end of the proof of Theorem 3.2.

(iii) We can again use the same sort of reasoning that was used for vortices in Theorem 3.2, but there is no need to “add back” the monominoes, since a bidimer does not contain one.

3.2 Tatami coverings of square grids

We have closed form formulae for all of \( T(n,n,m) \), which are stated below.

\[
T(n,n,m) = \begin{cases} 
m2^m + (m + 1)2^{m+1} & \text{if } m < n \text{ and } 2|n^2 - m \quad (3.1) \\
n2^{n-1} & \text{if } n = m \quad (3.2) \\
0 & \text{otherwise}. \quad (3.3)
\end{cases}
\]

Equation (3.3) was shown in Theorem 3.2 and the other two equations are results of the present chapter. Theorem 3.6 gives Equation (3.2) and Theorem 3.11 gives Equation (3.1).

3.2.1 Tatami coverings of the square, with maximum monominoes

In this section, we show by induction that \( T(n,n,n) = n2^{n-1} \). The result is re-derived later, from Theorem 4.2 to exhibit a natural partition of coverings into \( n \) classes of size \( 2^{n-1} \).

The following lemma and corollary provide a convenient characterisation of \( n \times n \) tatami coverings with \( n \) monominoes, in terms of diagonal flips (see Definition 3.1).
Lemma 3.4 (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]). For each \( n \times n \) covering with \( n \) monominoes, a bond can be obtained via a finite sequence of diagonal flips in which each monomino moves at most once. The original covering is obtained by applying the same sequence of diagonal flips to this bond, in reverse order.

Proof. There are several proofs of this, but the following idea in [12], by Jennifer Woodcock, is particularly succinct.

Let \( T \) be the \( T \)-diagram of an \( n \times n \) covering with \( n \) monominoes, and let \( A(T) \) be the area of a maximal connected bond which contains the centre of \( T \), called the central bond (see Figure 3.5). We may assume that \( T \) has at least one feature diagram, and by Corollary 3.3 it is either a loner or a vee. Let \( \rho \) be a ray diagram that passes nearest to the centre of \( T \). This ray diagram is the boundary between the dominoes of the central bond, and a diagonal with dominoes of the other orientation. Flipping the diagonal yields a \( T \)-diagram, \( T' \) with \( A(T') > A(T) \), and its monomino becomes part of the central bond. Repeating this process we obtain a bond on the \( n \times n \) grid using a finite number of diagonal flips, and no monomino is moved twice (see Figure 3.6).

Corollary 3.5 (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]). Every \( n \times n \) covering with \( n \) monominoes has exactly two corner monominoes and they are in adjacent corners.

Proof. The bond contains such corner monominoes, and in the proof of Lemma 3.4 no corner monomino is moved because the containing diagonal is part of the central bond. Since a bond on the \( n \times n \) grid with \( n \) monominoes has two monominoes in adjacent corners, so must every other \( n \times n \) covering with \( n \) monominoes.

Corollary 3.5 shows that the four rotations by \( \pi/2 \) radians of any \( n \times n \) covering with \( n \) monominoes are distinct.
Figure 3.5: In the covering $T$ from Lemma 3.4, the ray diagram $\rho$ separates the central bond from a diagonal, shown with green dominoes. Flipping the diagonal adds to the central bond, which guarantees a finite number of flips and protects the diagonal’s monomino from being moved again.

Figure 3.6: A sequence of 5 diagonal flips, shown in blue, beginning with a bond, results in this covering. Flipped monominoes are coloured red.

**Theorem 3.6** (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]). The number of $n \times n$ coverings with $n$ monominoes, $T(n,n,n)$, is $n2^{n-1}$.

**Proof.** Let $T_n$ be the tatami coverings counted by $T(n,n,n)$ which have monominoes in their top corners, and let $s(n) = T(n,n,n)/4$. We will show that

$$s(n) = 2^{n-2} + 4s(n-2),$$

where $s(1) = \frac{1}{4}$ and $s(2) = 1$. (3.4)

The solution to (3.4) is $s(n) = n2^{n-3}$.

If $n$ is even, then the bond in $T_n$ is a horizontal bond, with monominoes occurring on its left and right boundaries. Coverings are obtained by flipping diagonals containing these monominoes. Each monomino besides those in the top corners, is contained in exactly two diagonals. Lemma 3.4 yields the following refinement of our vocabulary. We say a monomino (or a diagonal that contains it) is flipped if it is moved from its original position in the bond. The containing diagonal can be specified with a direction — up or down, when $n$ is even (see Figure 3.7).
Call the lower leftmost monomino $w$, and the lower rightmost monomino $e$. If either of these is flipped up, then there remain exactly $n - 3$ diagonals that can be flipped, and additionally, the flips can be made independently (see Figure 3.8). This accounts for the $2^{n-2}$ in Equation (3.4). Each of the remaining four cases, where neither $e$ nor $w$ is flipped up, is described by a bijective correspondence between this subset of $T_n$ and $T_{n-2}$.

We associate the monominoes of an $(n - 2) \times (n - 2)$ covering with those of an $n \times n$ covering as follows. Draw the (unique) bond in $T_{n-2}$, rotated by $\pi$ radians, in the centre of the bond of $T_n$. Pairs of monominoes in this drawing which share an edge are associated, and naturally, their upward and downward diagonals are associated as well (see Figure 3.9). Note that $w$ and $e$ are associated with the “fixed” corner monominoes of the smaller covering.

Note that a left-side monomino and a right-side monomino cannot both be flipped downward (or upward) in an $n \times n$ covering, if the total number of tiles in the two diagonals is greater than $n$ (see Figure 3.10).

Let $T$ and $T'$ be the unique bond in $T_n$ and $T_{n-2}$, respectively. Let $\alpha$ and $\beta$ be diagonals in $T$, associated with diagonals $\alpha'$ and $\beta'$ in $T'$, respectively, and note that $\alpha'$ has one less tile than $\alpha$, and $\beta'$ has one less tile than $\beta$. Therefore, $\alpha'$ and $\beta'$
Figure 3.9: Pairs of monominoes in the respective superimposed coverings are associated if they share an edge. Their respective up and down diagonals are also associated.

Figure 3.10: In each covering, the two magenta diagonals cannot both be flipped because they intersect.

can be flipped in $T'$ if and only if $\alpha$ and $\beta$ can both be flipped in $T$. Thus, for each configuration of flipped diagonals in $T$, the associated diagonals can be flipped in $T'$, and vice versa (see Figure 3.11).

Figure 3.11: A pair of coverings in the bijection. Note that flips of the red monominoes, $e$ and $w$, are irrelevant.

This bijection implies that for each of the 4 cases where neither $e$ nor $w$ is flipped up, there are exactly $s(n - 2)$ coverings in $T_n$, which accounts for the second term in (3.4).

The odd case is very similar, and is described in [12].
3.2.2 Non-maximal tatami coverings of the square

Let $T_{n,m}$ be the set of $n \times n$ tatami coverings with $m$ monominoes. To find $|T_{n,m}|$, we use a setup similar to that of Theorem 3.6. Lemmas 3.7, 3.8 and 3.9 on the composition of such coverings seem apparent from Figures 3.12–3.14, but we prove them in general.

We show that any covering in $T_{n,m}$ with $m < n$ has exactly one bidimer or vortex and we show that $m$ uniquely determines the shortest distance from this source to the boundary. Such a feature determines all tiles in the covering except a number of diagonals that can be flipped independently. Proving the result becomes a matter of counting the number of allowable positions for the bidimer or vortex, each of which contributes a power of 2 to the total count.

For example, the $20 \times 20$ covering in Figure 3.12 has a vertical bidimer which forces the placement of the green and blue tiles, while the remaining diagonals are coloured in alternating grey and magenta. There are eight such diagonals, so there are $2^8$ coverings of the $20 \times 20$ grid, with a vertical bidimer in the position shown. Each of these $2^8$ coverings has exactly 10 monominoes.

By Corollary 3.3, a covering in $T_{n,m}$ with $m < n$, must have at least one bidimer or vortex, which we will call $f$. Let $(x_f, y_f)$ be the centre of $f$ in the cartesian plane, where the bottom left of the $n \times n$ covering is at the origin.

Define $X_f$ as the lines $(x - x_f) + (y - y_f) = 0$ and $(x - x_f) - (y - y_f) = 0$,
bounded by the grid boundary, which form an X through \((x_f, y_f)\) (see Figures 3.12–3.14).

Without loss of generality, we may re-orient a covering, by rotating and reflecting, so that \(y_f \leq x_f \leq n/2\). The upper arms of \(X_f\) intersect the left and right boundaries of the grid, while the lower arms intersect the bottom. In this range, the \textit{distance} from this feature to the boundary of the grid is \(y_f\).

**Lemma 3.7.** Let \(T \in T_{n,m}\), with \(m < n\). Then \(T\) has exactly one bidimer or vortex, \(f\), but not both, and no vees.

**Proof.** If \(a\) is another bidimer or vortex in the covering, then \(X_a\) intersects \(X_f\), which contradicts the fact that feature diagrams do not intersect. A vee has the same rays as a bidimer, by replacing the adjacent monominoes with a domino, so the only features that may appear, besides \(f\), are loners.

Let \(T_f\) be the covering with \(f\) as its only feature. This corresponds to the (unique) bond of \(T_{n,m}\), used in Theorem 3.6.

**Lemma 3.8.** The covering \(T_f\) can be obtained from any covering containing the feature \(f\) via a finite sequence of diagonal flips in which each monomino moves at most once. Reversing this sequence gives the original covering.

**Proof.** This is a simple modification of the proof of Lemma 3.4.

**Lemma 3.9.** All the diagonals in \(T_f\) can be flipped independently.
Figure 3.14: Counter clockwise vortex. Note that $x_f$ and $y_f$ are not integers.

Proof. We show that no two diagonals intersect, either at a monomino or a domino.

Let $\alpha$ and $\beta$ be the monominoes in distinct diagonals, possibly with $\alpha = \beta$, which intersect. Let $L_\alpha$ and $L_\beta$ be the longest line segments contained in the respective diagonals. Note that their slopes are in $\{1, -1\}$. Their slopes must differ, so we may assume that $L_\alpha$ has slope 1, and $L_\beta$ has slope $-1$. The line segments $L_\alpha$ and $L_\beta$ intersect (inside the grid), and therefore at least one of them intersects with $X_f$, which makes the corresponding diagonal unflippable. Therefore distinct diagonals may not intersect.

The crux of the argument in Theorem 3.11 is this:

Lemma 3.10. If $T_f \in T_{n,m}$, with $y_f \leq x_f \leq n/2$, then $m = n - 2y_f$ if $f$ is a bidimer, and $m = n - y_f + 1$ if $f$ is a vortex. The number of (flippable) diagonals in $T_f$ is

$$\begin{cases} n - 2y_f - 2, & \text{if } y_f < x_f; \\ n - 2y_f - 1, & \text{if } y_f = x_f < n/2; \text{and} \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Let $\text{tl}$, $\text{tr}$, $\text{bl}$, and $\text{br}$ be the segments of the boundary of the grid delimited by $X_f$, as shown in grey in Figure 3.15(a). We count the number of monominoes and diagonals in $T_f$ by measuring the lengths of $\text{tl}$, $\text{tr}$, $\text{bl}$, and $\text{br}$, minus the tiles
in the rays of $f$, with the different cases illustrated in Figures 3.15(b)–3.15(e).

![Diagram](image)

(a) The range $y_f \leq x_f \leq n/2$ is green.

(b) The range $y_f < x_f$.

(c) The range $y_f = x_f = n/2$.

(d) The range $y_f + x_f < n$.

Figure 3.15: The segments tl, tr, bl, and br. When tl and br have positive length, one of their combined tiles is part of a ray of $f$, shown in green or blue. Similarly for tr and bl combined.

Figures 3.15(b)–3.15(e) show (in general) that if tl and br are non-zero, then together they contain exactly one grid square covered by the ray of $f$. The same is true of tr and bl.

Because of the above, non-zero tl and br differ in parity, so their sum is odd, and therefore they must contain exactly one corner monomino (which is not in a diagonal). Similarly for tr and bl.

We tabulate the numbers of monominoes and diagonals in tl, tr, bl and br for different $(x_f, y_f)$ in Table 3.1 and add pairs of rows to prove each case of the theorem statement.

<table>
<thead>
<tr>
<th>$(x_f, y_f)$</th>
<th>Positions</th>
<th>Monominoes</th>
<th>Diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_f = x_f$</td>
<td>bl and tr</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y_f &lt; x_f$</td>
<td>bl and tr</td>
<td>$x_f - y_f$</td>
<td>$x_f - y_f - 1$</td>
</tr>
<tr>
<td>$y_f = x_f = n/2$</td>
<td>tl and br</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y_f + x_f &lt; n$</td>
<td>tl and br</td>
<td>$n - x_f - y_f$</td>
<td>$n - x_f - y_f - 1$</td>
</tr>
</tbody>
</table>

Table 3.1: The numbers of monominoes and diagonals in tl, tr, bl and br for different $(x_f, y_f)$.

If $y_f < x_f$, then $m = x_f - y_f + n - x_f - y_f = n - 2y_f$ if $f$ is a bidimer, and
\[m = n - 2y_f + 1\] if \(f\) is a vortex. The number of diagonals is \(n - 2y_f - 2\), as required for this case.

If \(y_f = x_f < n/2\), then \(m = 0 + n - y_f - y_f = n - 2y_f\) if \(f\) is a bidimer, and \(n - 2y_f + 1\) if \(f\) is a vortex. The number of diagonals is \(n - 2y_f - 1\).

If \(y_f = x_f = n/2\), then \(m = 0\) if \(f\) is a bidimer, and \(m = 1\) if it is a vortex. The number of diagonals is 0.

We sum up the number of positions which give a particular \(m\) in Lemma 3.10, to count all such coverings.

**Theorem 3.11.** If \(n\) and \(m\) have the same parity, and \(m < n\), then \(T(n, n, m) = m2^m + (m + 1)2^{m+1}\).

**Proof.** We count the number of coverings \(T_f\) in \(T_{n,m}\) for each bidimer or vortex, \(f\), satisfying the conditions of Lemma 3.10, and for each of these we count the coverings obtainable via a set of (independent) diagonal flips.

If \(f\) is a bidimer, then \(m = n - 2k\), where \(k\) is the shortest distance from \((x_f, y_f)\) to the boundary of the grid. These positions where \(f\) can be centred are

\[(k,k), (k,k+1), (k,k+2), \ldots (k,n-k), (k+1,n-k), \ldots,\]
\[(n-k,n-k), (n-k,n-k-1), \ldots (n-k,k), (n-k-1,k), \ldots (k+1,k),\]

of which there are \(4(n-k-k)\). We apply Lemma 3.10 by making the necessary rotations and reflections, so that when \(m > 0\), the four positions \((k,k), (k,n-k), (n-k,n-k)\) and \((n-k,k)\), have \(m-1\) diagonals, while the remaining \(4m-4\) positions have \(m-2\) diagonals.

The same logic applies when \(f\) is a vortex and \(m > 1\), and the results are in Table 3.2.

By Lemmas 3.8 and 3.9, all of the coverings which contain \(f\) can be obtained from \(T_f\) by making a set of independent flips. Thus, the number of these is \(2^{d(f)}\), where \(d(f)\) is the number of diagonals in \(T_f\).

Lemma 3.7 tells us that there is no other way to obtain a covering in \(T_{n,m}\), so we conclude by summing the \(2^{d(f)}\)s for each \(f\) such that \(T_f \in T_{n,m}\). Each term in the following sum comes from a row of Table 3.2, in the same respective order, and similarly, the three factors in each sum term come from the last three columns.
<table>
<thead>
<tr>
<th>Type</th>
<th>Feature</th>
<th>Positions</th>
<th>Diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_f = y_f$</td>
<td>h and v bidimers</td>
<td>4</td>
<td>$m - 1$</td>
</tr>
<tr>
<td>$x_f = y_f$</td>
<td>cc and c vortices</td>
<td>4</td>
<td>$m - 2$</td>
</tr>
<tr>
<td>$x_f &lt; y_f$</td>
<td>h and v bidimers</td>
<td>$4(m - 1)$</td>
<td>$m - 2$</td>
</tr>
<tr>
<td>$x_f &lt; y_f$</td>
<td>cc and c vortices</td>
<td>$4(m - 2)$</td>
<td>$m - 3$</td>
</tr>
</tbody>
</table>

Table 3.2: Horizontal, vertical, counterclockwise and clockwise are abbreviated as h, v, cc and c, respectively. We assume $m > 0$ if $f$ is a bidimer, and $m > 1$ if $f$ is a vortex.

That is, there are $m2^m$ coverings with vortices and $(m+1)2^{m+1}$ with bidimers. \(\square\)

This completes the enumeration of $n \times n$ tatami coverings with $m$ monominoes. Summing over $T(n,n,m)$ over all $m$ yields a nice formula.

**Corollary 3.12.** The number of $n \times n$ tatami coverings is $2^{n-1}(3n - 4) + 2$.

**Proof.** By Corollary 3.3, we have that $T(n,n,m) = 0$ when $m > n$. Let $T(n,n) = \sum_{m \geq 0} T(n,n,m)$, so that

$$T(n,n) = n2^{n-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} \left((n - 2i)2^{n-2i} + (n - 2i + 1)2^{n-2i+1}\right),$$

and notice that the sum simplifies to

$$T(n,n) = n2^{n-1} + \sum_{i=1}^{n-1} i2^i.$$

Now we use the fact that $2^k + 2^{k+1} + \cdots + 2^{n-1} = 2^n - 1 - 2^k + 1 = 2^n - 2^k$ to
rearrange the sum.

\[ T(n, n) = n2^{n-1} + \sum_{i=1}^{n-1} i2^i = n2^{n-1} + (n - 1)2^n - \sum_{i=1}^{n-1} 2^i = 2^{n-1}(3n - 4) + 2 \]

By cross referencing the result of Corollary 3.12 with the On-Line Encyclopedia of Integer Sequences, a surprising correspondence with integer compositions of \( n \) is apparent (see A027992 in [27]).

**Proposition 3.13** (Erickson, Schurch, 2011, [14]). *The number of \( n \times n \) tatami coverings is equal to the sum of the squares of all parts in all compositions of \( n \).*

**Proof.** See [14].

3.3 Tatami coverings of proper rectangles

Compared with tatami coverings of the square, little attention has been paid to the counts for \( T(r, c, m) \) when \( r < c \). Similar techniques may be applied, however, by considering diagonal flips and other feature diagrams of the T-diagram. Rectangular coverings up to \( 14 \times 14 \) have been counted by brute force and the counts are listed in Tables A.1–A.17 and A.19.

Calculating non-trivial \( T(r, c, m) \), when \( r < c \), is an open problem for all fixed \( m \), with \( m > 1 \), however, significant progress has been made by Jennifer Woodcock on counting those with maximum monominoes. Conjecture 3.14 agrees with the counts produced by brute force, at least up to \( (r, c) = (12, 13) \) (see Table A.19).

**Conjecture 3.14** (Woodcock, 2010, (private communication)). *Let \( T_{\text{max}}(r, c) \) denote the number of \( r \times c \) tatami coverings with the maximum possible number of monominoes.*
For $r \geq 3$,

$$T_{\text{max}}(r, c) = \begin{cases} 
2^{r-4}(r+4)(r+2) & \text{if } c > 2r+1 \text{ and } r \text{ and } c \text{ are both even} \\
2^{r-4}(r+3)^2 & \text{if } c > 2r+1 \text{ and } r \text{ is odd} \\
2^{r-6}(r+2)^2 & \text{if } c > 2r+1 \text{ and } r \text{ is even and } c \text{ is odd} \\
2^{r-4}(3r-c+4)(c-r+2) & \text{if } r \equiv c \mod 2 \text{ and } r+1 \leq c \leq 2r+1 \\
2^{r-4}(3r-c+4)(c-r+2) + 2^{r-4} & \text{if } r \text{ is odd and } c \text{ is even and } r+2 \leq c \leq 2r \\
2^{r-6}(29r+17) & \text{if } r \text{ is odd and } c = r+1 \\
2^{r-6}(3r-c+4)(c-r+2) + 2^{r-6}(2r-2c-3) & \text{if } r \text{ is even and } c \text{ is odd and } r+1 \leq c \leq 2r+1 \\
r2^{r-1} & \text{if } r = c.
\end{cases}$$

(3.5)

The case where $m = 0$ has the historical distinction of motivating the present research. Exercise 7.1.4.215 in “The Art of Computer Programming”, by Don Knuth ([19]), asks for a generating function that counts the number of domino-only tatami coverings of fixed height. Ruskey and Woodcock, using ideas from a decomposition due to Hickerson ([17]), provide the following solution.

**Theorem 3.15** (Ruskey, Woodcock, 2009, [26]). Let $T(r, c, 0)$ be the number of tatami coverings of the $r \times c$ grid with 0 monominoes. For fixed-height grids, these are counted by the rational generating function,

$$\sum_{c \geq 0} T(r, c, 0)z^c = \begin{cases} 
1 & \text{for } r = 0 \\
\frac{1-z^2}{1-z^2} & \text{for } r = 1 \\
\frac{1+z^2}{1-z^2} & \text{for } r = 2 \\
\frac{1+z^{r-1}+z^{r+1}}{1-z^{r-1}-z^{r+1}} & \text{for } r \text{ odd, } 3 \leq r \leq c \\
\frac{(1+z)(1+z^{r-2}+z^r)}{1-z^{r-1}-z^{r+1}} & \text{for } r \text{ even, } 4 \leq r \leq c.
\end{cases}$$

**Proof.** See [26].

The case where $m = 1$ is given by Alhazov, Iwamoto, and Morita, in [1], as a rational generating function.

**Theorem 3.16** (Artiom Alhazov, Kenichi Morita, Chuzo Iwamoto, 2009, [1]). The number of coverings of the $r \times c$ grid, with $r \leq c$, and exactly 1 monomino, is “generated”
by

\[ A_r(z) + \sum_{r \leq c} T(r, c, 1) z^c = \begin{cases} \frac{2}{(1-2z)^2} & \text{for } r = 1, \\ \frac{2z + 6z^3 - 4z^5 - 4z^7}{(1 - z^2 - z^4)^2} & \text{for } r = 3, \\ \frac{2z + 4z^r - 2 + 6z^r - 4z^{2r-3} - 8z^{2r-1} - 4z^{2r+1}}{(1 - z^{r-1} - z^{r+1})^2} & \text{for odd } r \geq 5, \end{cases} \]

where \( A_r(z) \) is a polynomial of degree at most \( r - 1 \).

Recall that \( T(r, c) = \sum_{m \in \mathbb{Z}} T(r, c, m) \). As it turns out, the generating functions for \( T(r, c) \) are also rational functions, for fixed \( r \). This result, due to Mark Schurch, is restated here, but we give a new proof using the transfer matrix method (see [28]). For each fixed \( r \), an application of the transfer matrix method produces the generating function \( T_r(\lambda) \), defined in Theorem 3.17.

**Theorem 3.17** (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]). The generating function

\[ T_r(\lambda) = \sum_{c \geq 0} T(r, c) \lambda^c, \]

is a rational function, where \( T(r, c) \) is defined above.

**Proof.** See [28], Section 4.7, pg. 241 on the transfer matrix method, for undefined terms.

Let \( D = (V, E) \) be a digraph. The vertex set, \( V \), comprises all coverings of the \( r \times 1 \) grid, by vertical dominoes, \( \Box \), monominoes, \( \blacklozenge \), and the left square of horizontal dominoes, \( \square \) (see Figure 3.16). The number of coverings, \( |V| \), is equal to \( p(r) \), where \( p(n) = p(n-2) + 2p(n-1) \), with initial conditions \( \{ p(1) = 2, p(2) = 5 \} \) (see A000129 in [27]).

![Figure 3.16: The r x 1 coverings by vertical dominoes, and two types of “monomino”, shown for the case of r = 3.](image-url)
Figure 3.17: A horizontal domino can only be extended by a monomino.

1. If a square of vertex $i$ is covered by $\square$, the corresponding square of vertex $j$ must be covered by $\square$ (see Figure 3.17).

2. If two adjacent squares of vertex $i$ are covered by distinct tiles, and neither of them are $\square$, then the corresponding squares of $j$ are covered by a $\square$.

Let $A$ be the adjacency matrix of $D$. Theorem 4.7.2 of [28] says that the number of length-$c$ walks in $D$ that begin at vertex $i$ and end at vertex $j$ is the coefficient of $\lambda^c$ in

$$F_{ij}(D, \lambda) = \frac{\det(I - \lambda A : j, i)}{\det(I - \lambda A)}. \quad (3.6)$$

We find that $F_{ij}(D, \lambda) = (I - \lambda A)^{-1}_{ij}$, which is a rational function.

Further to the above proof, the last column in a covering cannot contain $\square$. For example, with $r = 3$ the last column may only be vertex 1, 5, or 11. The rational function we want is

$$T_r(\lambda) = 1 + \lambda \left( \sum_{1 \leq i \leq p(r)} \sum_{j \in V_c} F_{ij}(D, \lambda) \right),$$

where $V_c$ is the subset of vertices that can be the last column of an $r \times c$ covering. The result is shifted by $\lambda$ because we want to count columns of the $r \times c$ grid, rather than grid-lines.

The fractal nature of the adjacency matrices, for various $r$, is unsurprising, it is worth comparing its density with that of its non-tatami counterpart (compare Figure 3.18 with Figure 3.19).

Remark that the degree of the numerator in Equation (3.6) satisfies

$$\deg((-1)^{i+j} \det(I - \lambda A)) < p(r),$$
Figure 3.18: The $2378 \times 2378$ adjacency matrix for $r = 9$, with the tatami constraint. Compare its density, approximately 0.002, with that of Figure 3.19.

since the dimensions of $A$ are $p(r) \times p(r)$. Therefore, the coefficients of

$(-1)^{i+j} \det(I - \lambda A : j, i)$

can be determined from the first $p(r)$ coefficients of the series expansion. This is faster than finding the inverse mentioned in the proof of Theorem 3.17, if the initial values are computed efficiently. Note that they can be calculated (inefficiently) with $(A^n)_{ij}$, since this is the number of length-$n$ walks from $i$ to $j$, by Theorem 4.7.1 of [28]. See Algorithm 1 for this method, and see http://alejandroerickson.com/tatami for an implementation in the Maple programming language.

**Algorithm 1** Calculate $T_r(\lambda)$ without using $(I - \lambda A)^{-1}$.

**Require:**

$g(\lambda) \leftarrow \det(I - \lambda A)$

$\text{endWalks} \leftarrow \{v \in V(D) : v \text{ has no } ' \square' \}$

$T(\lambda) \leftarrow |\text{endWalks}| + \sum_{c=1}^{p(r)} \left( \sum_{1 \leq i \leq p(r)} (A^c)_{ij} \right) \lambda^c$ \hspace{1cm} $\triangleright$ Initial conditions.

$f(\lambda) \leftarrow T(\lambda)g(\lambda) \mod \lambda^{p(r)}$

$T(\lambda) \leftarrow 1 + \lambda \frac{f(\lambda)}{g(\lambda)}$ \hspace{1cm} $\triangleright$ Shift by 1 and add the $r \times 0$ covering.
Let $L(\lambda), P(\lambda),$ and $Q(\lambda)$ be polynomials such that $T_r(\lambda) = L(\lambda) + P(\lambda)/Q(\lambda)$, where $Q(0) = 1$ and $\deg(P(\lambda)) < \deg(Q(\lambda))$ (the parameter $r$ is understood). Table 3.3 contains the coefficients of $Q(\lambda)$ up to $r = 11$, and Table 3.4 gives $L(\lambda)$ and $P(\lambda)$ up to $r = 10$. The first 14 coefficients of each $T_r(\lambda)$ are in Table A.17 for $1 \leq r \leq 14$.

Salient patterns in these coefficients are summarized in Conjectures 3.18 and 3.19. Note that Conjecture 3.18 implies $Q(\lambda)$ is a self-reciprocal polynomial for $r \equiv 2 \pmod{4}$. Interestingly, the corresponding generating functions for $m = 0, 1$, from Theorems 3.16 and 3.15 do not have a similar self-reciprocal property.

**Conjecture 3.18.** Let $L(\lambda), P(\lambda),$ and $Q(\lambda)$ be as defined above, where $P(\lambda)$ and $Q(\lambda)$ are relatively prime polynomials, $\deg(Q(\lambda)) = n$, and $r \geq 1$. Then,

$$Q(\lambda) = \begin{cases} 
-\lambda^n Q\left( \frac{1}{\lambda} \right), & \text{if } r \equiv 0 \pmod{4}, \\
-\lambda^n Q\left( \frac{-1}{\lambda} \right), & \text{if } r \equiv 1 \pmod{4}, \\
\lambda^n Q\left( \frac{1}{\lambda} \right), & \text{if } r \equiv 2 \pmod{4}, \\
\lambda^n Q\left( \frac{-1}{\lambda} \right), & \text{if } r \equiv 3 \pmod{4}.
\end{cases}$$
A mod 4 pattern also seems to occur in the degrees of the denominators of $T_r(\lambda)$. The rigid structure we encounter in tatami tilings prompts us to infer this pattern upon all values as well.

**Conjecture 3.19.** Let $Q(\lambda)$ be as defined above. Then,

\[
\deg(Q(\lambda)) = \begin{cases} 
8m^2 + 2m + 1, & \text{if } r \equiv 0 \pmod{4}, \\
8m^2 + 4m + 2, & \text{if } r \equiv 1 \pmod{4}, \\
8m^2 + 10m + 4, & \text{if } r \equiv 2 \pmod{4}, \\
8m^2 + 8m + 6, & \text{if } r \equiv 3 \pmod{4}.
\end{cases}
\]

We conclude this section by noting that we have not been able to devise a uniform presentation of the generating function of $T_r(\lambda)$ similar to what was done in [26].

It would also be interesting to consider the generating function $T_r(x, y, \lambda)$, in which the coefficient of $x^h y^v \lambda^m$ is the number of coverings with $h$ vertical dominoes, $v$ horizontal dominoes, and $m$ monominoes.

Coefficients of $Q(\lambda)$ are ordered from left to right by ascending degree, and then folded like these arrows: $\rightarrow$ for $r \leq 3$, $\leftrightarrow$ for $r = 4, 5, 6, 7, 9$, and $\rightarrow$ for $r = 8, 10, 11$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$q$</th>
<th>$Q(\lambda)$</th>
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<td>4</td>
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<td>6</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td>1, -1, -5, 5, 13, -13, -27, 27, 48, -48, -83, 81, 125, -120, -160</td>
</tr>
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</tr>
<tr>
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<td>-34, -83, 89, 156, -165, -199, 210, 202, -206, -185, 193, 154</td>
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<td>1, -1, -5, 5, 13, -13, -27, 27, 48, -48, -83, 81, 125, -120, -160</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1, 1, -5, 5, 13, 13, -27, -27, 48, 48, -83, -81, 125, 120, -160</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-34, 83, 89, -156, -165, 199, 210, -202, -206, 185, 193, -154</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-34, -83, 89, 156, -165, -199, 210, 202, -206, -185, 193, 154</td>
</tr>
</tbody>
</table>

Table 3.3: Coefficients of denominators, $Q(\lambda)$, where $q = \deg(Q(\lambda))$. The ordering reflects the patterns in Conjecture 3.18.
<table>
<thead>
<tr>
<th>$r$</th>
<th>$l$</th>
<th>$p$</th>
<th>$L(\lambda)$</th>
<th>$P(\lambda)$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1,0,2,-1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>-3,1</td>
<td></td>
<td>4,-2,2,6,-10,2</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>-13,3,3,2</td>
<td>14,-12,10,0,10,-104,114,-80,34,12,-2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td>-28,10,17,10,3</td>
<td>29,-31,24,60,-97,61,-196,83,31,-84,96,-1,13,-8</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>21</td>
<td>-90,27,33,32,18,10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$P(\lambda)$

| 7   | 6   | 21  | -169,71,139,76,54,42,17 |

$P(\lambda)$

| 8   | 36  | -505,176,251,178,71,138,98,48 |

$P(\lambda)$

| 9   | 41  | -897,425,956,408,128,126,250,224,88 |

$P(\lambda)$

| 10  | 55  | -2593,999,1736,946,-40,18,186,674,504,224 |

$P(\lambda)$

Table 3.4: Coefficients of $L(\lambda)$ and $P(\lambda)$ in ascending order of degree, where $l = \deg(L(\lambda))$ and $p = \deg(P(\lambda))$. For $r \geq 5$, the coefficients of $P(\lambda)$ are displayed in the next row.
Chapter 4

Square grids, maximum monominoes, v vertical dominoes

Recall that $T_n$ is the set of monomino-domino tatami coverings of the $n \times n$ grid with the maximum number, $n$, of monominoes, oriented so that they have a monomino in each of the top left and top right corners. On the basis of some computer investigations, Don Knuth discovered that the generating polynomial for small tatami coverings of $T_n$, with respect to the number of vertical dominoes they contain, is a product of cyclotomic polynomials and a mainly mysterious, irreducible polynomial (private communication, December 2010). Knuth’s discovery and our own observations motivated Conjecture 4 in [12], which is presented here as Equation (4.2). In this chapter we generalise and prove Knuth’s cyclotomic factors, and determine some important properties of the mysterious polynomial.

Let $H(n,k)$ be the number of coverings in $T_n$ with exactly $k$ horizontal dominoes, and let $V(n,k)$ be the number with exactly $k$ vertical dominoes. Let $S_n(z) = \prod_{i=1}^{n} (1 + z^i)$. We prove that the polynomial

$$VH_n(z) := 2 \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} S_{n-i-2}(z)S_i(z)z^{n-i-1} + \left(S_{\lfloor \frac{n-3}{2} \rfloor}(z)\right)^2,$$

(4.1)

is equal to $\sum_{k \geq 0} H(n,k)z^k$ for odd $n$ and is equal to $\sum_{k \geq 0} V(n,k)z^k$ for even $n$. The key step is to show the relationship between diagonal flips, defined in Chapter 3 and the number of subsets of $\{1,2,\ldots,n\}$ whose elements sum to a given $k$. 
Knuth’s observation generalises to

$$VH_n(z) = P_n(z) \prod_{j \geq 1} S_{\left\lfloor \frac{n-2}{2^j} \right\rfloor}(z), \quad (4.2)$$

where $P_n(z)$ is the “mysterious” polynomial. We prove Equation (4.2) in Theorem 4.6.

The remaining factors of Equation (4.2) are of the form $S_k(z)$, where $k$ is a binary right shift of $n - 2$, and the complete factorisation of these is known in general. The $i$th cyclotomic polynomial, $\Phi_i(z)$, is defined as $\prod_{\omega \in \Omega}(z - \omega)$, where $\Omega$ is the set of $i$th primitive roots of unity. Lemma 5 in [12] states that

$$S_k(z) = \prod_{j=1}^k (\phi_{2j}(z))^{\left\lfloor \frac{k+j}{2} \right\rfloor}, \quad (4.3)$$

and cyclotomic polynomials are known to be irreducible. Thus $VH_n(z)$ can apparently be factored completely as

$$VH_n(z) = P_n(z) \prod_{j \geq 1} \Phi_{2j}(z)^{\left\lfloor \frac{n-2}{2^j} \right\rfloor}. \quad (4.4)$$

We have verified the irreducibility of $P_n(z)$ for $1 < n < 200$ (the degree of $P_{199}(z)$ is 13022 and its largest coefficient has 55 digits), and thus we hope that Equation (4.4) is the complete factorisation of $VH_n(z)$ for all $n \geq 2$.

The class of polynomials, $P_n(z)$, has some compelling properties, some of which are proven, others which are empirical. For example, we observe in Conjecture 4.9 that the alternating sums of $P_n(z)$ are the coefficients of the ordinary generating function

$$\sum_{n \geq 2} P_n(-1)z^{n-2} = \frac{(1 + z)(1 - 2z)}{(1 - 2z^2)\sqrt{1 - 4z^2}},$$

for $1 < n < 200$. If the conjecture is true, then $P_{2(n+1)}(-1) = \binom{2n}{n}$. Furthermore $P_n(-1)$ is equal to the sum of the absolute values of the coefficients of $P_n(z)$, only for $n \geq 20$. This second fact is surprising, considering the way $P_n(z)$ is derived – why $n \geq 20$?

The complex roots of $P_n(z)$ appear to cluster neatly around the unit circle, and form convergent sequences as $n \rightarrow \infty$ (see Figures 4.5–4.6).
Theoretical progress on $P_n(z)$ comprises Theorem 4.7 and Theorem 4.8. The former states that $\deg(P_n(z)) = \sum_{k=1}^{n-2} Od(k)$, where $Od(n)$ is the largest odd divisor of $n$. We prove in Theorem 4.8 that for all $n \geq 2$, the sum of the coefficients of $P_n(z)$ is equal to $n2^{\nu(n-2)-1}$, where $\nu(n)$ is the number of 1-bits in the binary representation of $n$.

Once again, we employ the diagonal flip (see Definition 3.1). The added observation that a diagonal flip changes the orientation of some dominoes, enables us to further exploit it. The crux of the argument uses the partition of $T_n$ from Theorem 2 of [14] which reveals diagonal flips each with 1, 2, . . . , $k$ dominoes, respectively, that can be flipped independently. We use this to express $VH_n(z)$ in terms of $S_k(z)$, the generating polynomial for the number of subsets of $\{1, 2, \ldots, k\}$ whose elements sum to $i$.

### 4.1 Representing a covering as a string

We describe binary and ternary string representations for $n \times n$ coverings with $n$ monominoes. Recall that each monomino, besides the two corner monominoes, is in exactly two diagonals in the bond, and in a given covering a monomino is flipped in one of these diagonals, or it is unflipped. A ternary symbol for each monomino indicates which of the three possible states it assumes. Each covering is described by a unique string of these ternary symbols, called trits, represented in the same order as the following indexed labelling (see caption at Figure 4.1(c)).

Monominoes and their diagonals are labelled as shown in Figure 4.1, such that the index, $i$, of a monomino is equal to the length of one of its diagonals, and $n - i - 1$ is the length of the other. This relationship between diagonal length and index is helpful in proving Lemma 4.1.

The ternary string representing the $10 \times 10$ covering in Figure 4.1(c) is $s = (0, 1, -1, 0, 0, 1, -1)$, where $s_i = 1$ if the $i$th monomino is flipped upward, $s_i = -1$ if it is flipped downward, and $s_i = 0$ if it is unflipped.

The ternary symbol is theoretically significant because it unites pairs of diagonals associated with the same monomino, but computer hardware prefers bit-strings. Therefore, we define the binary analogue where each trit, $s_i$, is replaced
by two bits, \( p \). They are related by

\[
 s_i = -1 \iff p_i = 10, \tag{4.5}
\]
\[
 s_i = 0 \iff p_i = 00; \quad \text{and,} \tag{4.6}
\]
\[
 s_i = 1 \iff p_i = 01. \tag{4.7}
\]

We use \( l_i^\uparrow, l_i^\downarrow, r_i^\uparrow, r_i^\downarrow, t_i^\rightarrow, t_i^\leftarrow, b_i^\rightarrow, b_i^\leftarrow \) to denote the diagonals that the monominoes \( l_i, r_i, t_i, b_i \) can be flipped on. Naturally, \( l_i \) and \( r_i \) can only be (diagonally) flipped up or down, whilst \( t_i \) and \( b_i \) can only be flipped left or right.

Let \( d_n(a) \) be the number of dominoes in the diagonal \( a \), also called the length or size of the diagonal. It is a function of the index and direction of \( a \):

\[
d_n(a) = \begin{cases} 
    i, & \text{if } a \in \{l_i^\uparrow, r_i^\uparrow, t_i^\rightarrow, b_i^\rightarrow\}; \\
    n - i - 1, & \text{if } a \in \{l_i^\downarrow, r_i^\downarrow, t_i^\leftarrow, b_i^\leftarrow\}.
\end{cases}
\]

Flipped diagonals which intersect are called conflicting, and can occur as one of two types (see Figure 4.2).

**Type 1** A pair of diagonals with monominoes originating on the same boundary are flipped toward one another (e.g. \( t_i^\rightarrow, t_j^\leftarrow \) for some \( i < j \)).
Figure 4.2: Example of, (a), a Type 1 conflict, and, (b), a Type 2 conflict.

**Type 2** A pair of diagonals with monominoes originating on opposite boundaries are flipped in the same direction (e.g. \((l_i^\uparrow, r_j^\uparrow)\)) and their combined length is at least \(n\) (see Table 4.1).

<table>
<thead>
<tr>
<th>Pair</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_i^\downarrow, r_j^\downarrow)</td>
<td>(j \leq i - 1)</td>
</tr>
<tr>
<td>(l_i^\uparrow, r_j^\uparrow)</td>
<td>(i \leq j - 1)</td>
</tr>
<tr>
<td>(t_i^\leftarrow, b_j^\leftarrow)</td>
<td>(n \leq j + i)</td>
</tr>
<tr>
<td>(t_i^\rightarrow, b_j^\rightarrow)</td>
<td>(i + j \leq n - 2)</td>
</tr>
</tbody>
</table>

Table 4.1: Conditions for Type 2 conflicts.

Lastly, if \(a\) is a diagonal containing a given monomino, let \(\bar{a}\) be the monomino’s other diagonal.

### 4.1.1 A partition of \(T_n\)

Let \(T_n(a) \subseteq T_n\), where \(a\) is a diagonal such that \(d_n(a) \geq d_n(\bar{a})\), be defined as the collection of coverings in \(T_n\) in which \(a\) is the longest flipped diagonal; for each flipped diagonal \(b\), distinct from \(a\), we have \(d_n(b) < d_n(a)\).

Let \(T_n(\emptyset)\) be the set of coverings in which no monomino is flipped on its longest diagonal. Note the distinction between a monomino flipped on its longest diagonal, and the longest flipped diagonal in the whole covering.

The sets \(T_n(\emptyset)\) and \(T_n(a)\), for each diagonal \(a\) defined above, are a partition of \(T_n\).
4.2 Enumerating coverings in $T_n$ with $k$ vertical dominoes

Let $S(s,k)$ be the number of subsets of $\{1,2,\ldots,s\}$ whose sum is $k$. The number of coverings with $k$ vertical (or horizontal) dominoes is expressible in terms of this function by making independent flips of diagonals whose lengths are some subset $\{1,2,\ldots,s\}$. We identify these sets of diagonals in the proof of Lemma 4.1.

**Lemma 4.1.** Let $V(n,k)$ and $H(n,k)$ be the number of coverings in $T_n$ with exactly $k$ vertical and horizontal dominoes, respectively. If $n$ is even, then $V(n,k)$ is equal to $H(n,k)$.

$$ VH(n,k) := 2 \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \sum_{k_1+k_2=k} S(n-i-2,k_1)S(i-1,k_2) \right) $$

$$ + \sum_{k_1+k_2=k} S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_1 \right) S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_2 \right). $$

(4.8a)

(4.8b)

When $n$ is odd, $V(n,k)$ is equal to $H(n,k)$.

**Proof.** Each outer sum term of (4.8a) adds the coverings for $T_n(a)$, for some diagonal $a$, and the term (4.8b) counts those in $T_n(\emptyset)$.

**Case $n$ even:** The bond covering in $T_n$ consists only of horizontal dominoes, and flipping the diagonal $a$ contributes $d_n(a)$ vertical dominoes. Diagonals $l_i^\uparrow$ and $r_i^\uparrow$ have even length, for all $i$, while $l_i^\downarrow$ and $r_i^\downarrow$ have odd length. We use this fact to find sets of diagonals which have lengths $1,2,\ldots,s$, for some $s \in \mathbb{N}$, by combining allowable diagonals in opposite corners, for each $T_n(a)$. Table 4.2 shows the lengths of the longest allowable diagonals in each corner for each $T_n(a)$, and from this we can find the required sets of diagonals. For example, the allowable diagonals in $T_n(l_i^\uparrow)$ are shown in Figure 4.3(a) (for $(n,i) = (18,5)$) and their respective lengths are

$$ l_1^\uparrow, l_3^\uparrow, \ldots, l_{i-2}^\uparrow, 1,3,\ldots,i-2, $$

$$ l_{i+2}^\uparrow, l_{i+4}^\uparrow, \ldots, l_{n-3}^\uparrow, n-i-3,n-i-5,\ldots,2, $$

$$ r_i^\downarrow, r_{i+3}^\downarrow, \ldots, r_{n-2}^\downarrow, n-i-2,n-i-4,\ldots,1, $$

$$ r_2^\downarrow, r_4^\downarrow, \ldots, r_{i-1}^\downarrow, 2,4,\ldots,i-1. $$

We have $d_n(l_i^\uparrow) = n-i-1$, so we are interested in the number of combinations of the above independently flippable diagonals with exactly $k - (n-i-1)$ vertical
Figure 4.3: Allowable diagonals shown in alternating grey and white, (a), for $T_n(l_i)$, where $(n, i) = (18, 5)$, and (b), for $T_{18}(\emptyset)$.

dominoes. That number is

$$\sum_{k_1 + k_2 = \lfloor \frac{n-1}{2} \rfloor} S(n - i - 2, k_1) S(i - 1, k_2).$$

The indices of the diagonals $l_i^\rightarrow$ for which $d_n(l_i^\rightarrow) \geq d_n(l_i^\downarrow)$ and $r_i^\rightarrow$ for which $d_n(r_i^\rightarrow) \geq d_n(r_i^\downarrow)$, range from 1 to $\lfloor \frac{n-1}{2} \rfloor$, as required for (4.8a).

Now suppose $a = \emptyset$. If $i$ is the largest index such that $d_n(l_i^\downarrow) < d_n(l_i^\rightarrow)$ and $j$ is the largest index such that $d_n(r_j^\downarrow) < d_n(r_j^\rightarrow)$, then $\max(i, j) = \lfloor \frac{n-2}{2} \rfloor$ and $|i - j| = 1$. The allowable diagonals in $T_n(\emptyset)$ and their respective sizes are shown in the table below (see Figure 4.3(b)).

<table>
<thead>
<tr>
<th>$l_i^\downarrow$</th>
<th>$l_i^\rightarrow$</th>
<th>$r_j^\downarrow$</th>
<th>$r_j^\rightarrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_i^\downarrow$</td>
<td>$i$</td>
<td>$r_j^\downarrow$</td>
<td>$j$</td>
</tr>
<tr>
<td>$l_i^\downarrow$</td>
<td>$n - i - 3$</td>
<td>$r_j^\downarrow$</td>
<td>$n - j - 3$</td>
</tr>
<tr>
<td>$l_i^\downarrow$</td>
<td>$n - i - 5$</td>
<td>$r_j^\downarrow$</td>
<td>$n - j - 5$</td>
</tr>
</tbody>
</table>

Choosing subsets of the independently flippable diagonals with $k$ vertical dominoes contributes the term

$$\sum_{k_1 + k_2 = k} S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_1 \right) S \left( n - \left\lfloor \frac{n-2}{2} \right\rfloor - 1, k_2 \right),$$
Table 4.2: The longest allowable diagonals in each of four corners for each $T_n(a)$. Entries are calculated using the parity of $i$ and $j$, the avoidance of conflicts, and the requirement that $a$ be the longest diagonal in $T_n(a)$. Recall that conflict Type 2 occurs between diagonals $a$ and $b$ iff $d_n(a) + d_n(b) \geq n$.

and since $n - (\lfloor(n-2)/2\rfloor - 1) - 3 = \lfloor(n-2)/2\rfloor$, this is equal to (4.8b) for even $n$.

**Case $n$ odd:** The bond covering is a vertical bond with $\lfloor(n-2)/2\rfloor$ monominoes at the top (besides the two that are fixed) and $\lceil(n-2)/2\rceil$ non-fixed monominoes along the bottom boundary. When diagonal $a$ is flipped, $d_n(a)$ horizontal dominoes are added to the covering, instead of vertical dominoes. Hence we argue for $H(n,k)$ rather than $V(n,k)$.

Now $t_j^-$ and $t_j^+$ have even length, and $b_j^-$ and $b_j^+$ have odd length (see Table 4.2). For example, the allowable diagonals in $T_n(t_i^-)$ are shown in Figure 4.4(a) (for $(n,i) = (17,6)$), and their respective lengths are

\[
\begin{align*}
& t_1^-, t_3^-, \ldots, t_{i-2}^- & 1, 3, \ldots, i - 2, \\
& t_{i+2}^-, t_{i+4}^-, \ldots, t_{n-3}^- & n - i - 3, n - i - 5, \ldots, 2, \\
& b_i^-, b_3^-, \ldots, b_{n-i-2}^- & 1, 3, \ldots, n - i - 2, \\
& b_{n-i}^-, b_{n-i+2}^-, \ldots, b_{n-2}^- & i - 1, i - 3, \ldots, 1.
\end{align*}
\]
Once again $d_n(t_i \rightarrow) = n - i - 1$, so we are interested in the number of combinations of the above independently flippable diagonals with exactly $k - (n - i - 1)$ horizontal dominoes. As before, that number is

$$
\sum_{k_1 + k_2 = k - (n - i - 1)} S(n - i - 2, k_1) S(i - 1, k_2).
$$

Now suppose $a = \emptyset$, then if $i$ is the largest index such that $d_n(t_i \rightarrow) < d_n(t_i \leftarrow)$ and $j$ is the largest index such that $d_n(b_j \rightarrow) < d_n(b_j \leftarrow)$ then $\max(i, j) = \lfloor \frac{n - 2}{2} \rfloor$ and $|i - j| = 1$. The allowable leftward diagonals in $T_n(\emptyset)$ and their respective sizes are given in the table below.

$$
t_2 \leftarrow, t_4 \leftarrow, \ldots, t_j \leftarrow, \quad 2, 4, \ldots, j,
$$
$$
b_1 \leftarrow, b_3 \leftarrow, \ldots, b_i \leftarrow, \quad 1, 3, \ldots, i
$$

and by horizontal symmetry, the rightward diagonals have the same lengths. We conclude that the coverings with $k$ horizontal dominoes of $T_n(\emptyset)$ is also generated by (4.8b) when $n$ is odd.

The terms $\mathcal{VH}(n,k) z^k$ can be summed over $k$ to obtain the generating polynomial $T(n,z)$ (same as $\mathcal{VH}_n(z)$), mentioned in Conjecture 4 of [12].
Theorem 4.2. Let \( V_H(n) = \sum_{k \geq 0} V_H(n, k)z^k \). We have

\[
V_H(n)(z) := 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{n-i-2}(z)S_{i-1}(z)z^{n-i-1} + \left( S_{\lfloor \frac{n-2}{2} \rfloor}(z) \right)^2,
\]

(4.9)

where \( S_n(z) = \sum_{k \in \mathbb{Z}} S(n, k)z^k \). This “generates” \( V(n, k) \) for even \( n \), and \( H(n, k) \) for odd \( n \).

Proof. The details in this proof are used to prove Theorem 3.6 again. We obtain \( V_H(n)(z) \) by simplifying the sum \( \sum_{k \in \mathbb{Z}} V_H(n, k)z^k \) to

\[
\sum_{k \in \mathbb{Z}} \left( 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k_1+k_2=k} S(n-i-2, k_1)S(i-1, k_2) \right) z^k
+ \sum_{k \in \mathbb{Z}} \left( \sum_{k_1+k_2=k} S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_1 \right) S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_2 \right) \right) z^k.
\]

A small adjustment gives

\[
2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \left( z^{n-i-1} \sum_{k \in \mathbb{Z}} \left( \sum_{k_1+k_2=k} S(n-i-2, k_1)z^{k_1}S(i-1, k_2)z^{k_2} \right) \right) z^k
+ \sum_{k \in \mathbb{Z}} \left( \sum_{k_1+k_2=k} S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_1 \right) z^{k_1}S \left( \left\lfloor \frac{n-2}{2} \right\rfloor, k_2 \right) z^{k_2} \right) z^k,
\]

(4.10)

(4.11)

Recall that for each \( k \) and \( i \), the term \( S(n-i-2, k_1)S(i-1, k_2) \) counts some of the coverings in \( T_n(a) \), for some diagonal, \( a \). Here, \( d(a) = n - i - 1 \), and all of the coverings that are counted have exactly \( k \) vertical (or horizontal, if \( n \) is odd) dominoes. If \( i \) is fixed, the sum of these over all \( k \) is \( |T_n(a)| \). We easily obtain Equation (4.9) from Equations (4.10) and (4.11).

Theorem 3.6 follows from the proof of Theorem 4.2 in a way that provides a natural partition of \( n2^{n-1} \) into equivalence classes, which solves an implied question by Don Knuth, noted in [13].

Proof of Theorem 3.6 The comments in the above proof reveal that \( S_{n-i-2}(1)S_{i-1}(1) = |T_n(a)| \), for some diagonal \( a \) with \( d(a) = n - i - 1 \). Thus we have \( 2 \lfloor \frac{n-1}{2} \rfloor \) sets of coverings from \( |T_n(a)| = 2^{n-3} \), and one more from the last
term, which gives $|T_n(\emptyset)| = 2^2\lfloor \frac{n-2}{2} \rfloor$. Even and odd cases appear again when these floors are evaluated.

If $n$ is odd there are $n-1$ classes of type $T_n(a)$, and the last one is obtained from $|T_n(\emptyset)| = 2^{n-3}$. This is the partition we desire, since we are only considering coverings of a single orientation.

If $n$ is even there are only $n-2$ classes of type $T_n(a)$, but $|T_n(\emptyset)| = 2^{n-2}$. We arbitrarily divide $|T_n(\emptyset)|$ into halves to obtain the last two classes of size $2^{n-3}$ that are required. 

The degree of $V_H_n(z)$ is $\frac{n^2-n}{2} - (n-1)$, because this is the largest number of vertical dominoes possible in a covering of $T_n$, for even $n$ (and horizontal dominoes for odd $n$). For example, the covering with all $l_i$ flipped up and all $r_i$ flipped down contains exactly $n-1$ horizontal dominoes.

The coefficients of $V_H_n(z)$ are listed in Table 4.3 up to $n = 10$, and the following conjecture is true at least up to $n = 20$. If $Q(z)$ is a polynomial, then write $\langle z^k \rangle Q(z)$ to denote the coefficient of $z^k$.

**Conjecture 4.3.**

(a) For $k \leq n-2$, we have $\langle z^k \rangle V_H_n(z) = \langle z^k \rangle \prod_{m \geq 0} (1 + z^m)^2$, the number of partitions of $k$ into distinct parts with two types of each part (see A022567 in [27]).

(b) For $0 \leq k < n-3$, we have

$$\langle z^{\deg(V_H_n(z)) - k} \rangle V_H_n(z) = 2 \langle z^k \rangle \prod_{m \geq 0} (1 + z^m),$$

twice the number of partitions of $k$ into distinct parts (see A000009 in [27]).

Rotating a covering of $T_n$ by $\pi/2$ radians interchanges vertical and horizontal dominoes, and this transformation can be applied to the generating polynomial $V_H_n(z)$ to obtain the polynomial $V_H_n(z^{-1})z^{n(n-1)/2}$. Thus we can easily derive the bivariate generating polynomial $R_n(x,y)$, whose coefficient of $x^v y^h$ is the number of tatami coverings with exactly $v$ vertical dominoes and $h$ horizontal dominoes.

Our remarks along with some small algebraic manipulations prove the following corollary.

**Corollary 4.4.** Let $R_n(x,y)$ be as defined above. We have

$$R_n(x,y) = 2V_H_n(xy^{-1})y^{\frac{n^2-n}{2}} + 2V_H_n(x^{-1}y)x^{\frac{n^2-n}{2}}.$$  

(4.12)
Table 4.3: Table of coefficients of $\mathcal{V}_H(z)$ for $2 \leq n \leq 10$. The $(n,k)$th entry represents the number of coverings of $T_n$ with $k$ vertical dominoes when $n$ is even, and $k$ horizontal dominoes when $n$ is odd. See Table A.20 for larger values of $n$.

We list some basic properties of $R_n(x,y)$.

- The degree of $R_n(x,1)$ as well as the degree of every term in $R_n(x,y)$ is $\frac{n^2 - 2}{2}$;
- coefficients of $R_n(x,1)$ are given in Table A.21 for small $n$;
- the polynomial $R_n(x,y)$ can be recovered from $R_n(x,1)$, and the latter is the generating polynomial for the set of all $n \times n$ coverings with $n$ monominoes with exactly $v$ vertical dominoes (or $h$ horizontal dominoes);
- the polynomial $R_n(x,1)$ is self reciprocal because of interchangeability of vertical and horizontal dominoes; and finally,
- the polynomial $R_n(x,1)$ has similar properties to those listed for $\mathcal{V}_H(z)$ in Conjecture 4.3, in the sense that for some increasing integer function $f$, we have $\langle x^k \rangle R_n(x,1) = \langle x^k \rangle R_{n+1}(x,1)$, whenever $k < f(n + 1)$.

If there is an even number of dominoes, which is the case when $(n^2 - n)/4$ is an integer, then $\langle x^k y^k \rangle R_n(x,y) = 4 \langle z^k \rangle \mathcal{V}_H(z)$, where $k = (n^2 - n)/4$. Rotating the covering maps $k$ vertical dominoes to $k$ horizontal dominoes, and vice versa. The coverings counted by these coefficients are called balanced tatami coverings, appropriately named by Knuth (private communication), because the number of vertical and horizontal dominoes are equal. Here is $\langle z^k \rangle \mathcal{V}_H(z)$ for $2 \leq n \leq 56$: 0, 0, 2, 2, 0, 0, 10, 20, 0, 0, 114, 210, 0, 0, 1322, 2460, 0, 0, 16428, 31122, 0, 0, 214660, 410378, 0, 0, 2897424, 5575682, 0, 0, 40046134, 77445152, 0, 0, 563527294, 1093987598, 0, 0, 8042361426, 15660579168, 0, 0, 116083167058, 226608224226, 0, 0, 1691193906828, 3308255447206, 0, 0, 24830916046462, 48658330768786, 0, 0, 366990100477712, (see
49

A182107 in \cite{27}). Note that this is perhaps better viewed as four sequences, one for each $0 \leq j < 4$ such that $n \equiv j \pmod{4}$.

\section{A mysterious factor of $VH_n(z)$}

In this section we prove that the generating polynomial $VH_n(z)$ has (very nearly) the factorisation conjectured in \cite{12}. We use the following lemma.

\begin{lemma}
For all $x \geq 0$,
\begin{equation}
\lceil x \rceil = \sum_{k \geq 1} \left\lfloor \frac{x}{2^k} + \frac{1}{2} \right\rfloor.
\end{equation}
\end{lemma}

\begin{proof}
Let $n = \lfloor x \rfloor$ and apply strong induction on $n$. Clearly Equation (4.13) holds for the base case, when $n = 0$. Suppose it holds for $0, 1, \ldots, n - 1$, then
\begin{align*}
\sum_{k \geq 1} \left\lfloor \frac{x}{2^k} + \frac{1}{2} \right\rfloor &= \left\lfloor \frac{x}{2} + \frac{1}{2} \right\rfloor + \sum_{k \geq 1} \left\lfloor \frac{x}{2^k} + \frac{1}{2} \right\rfloor \\
&= \left\lfloor \frac{x}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{x}{2} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{\lfloor x \rfloor}{2} \right\rfloor = \lfloor x \rfloor.
\end{align*}
Two applications of Equation (3.11) in \cite{15} yield the penultimate equation and the final equation follows by considering the parity of $\lfloor x \rfloor$, or by using (3.26) in \cite{15} with $m = 2$ and $x' = x/2$.
\end{proof}

\begin{theorem}
The generating polynomial $VH_n(z)$ has the factorisation
\begin{equation}
VH_n(z) = P_n(z)D_n(z)
\end{equation}
where $P_n(z)$ is a polynomial and
\begin{equation}
D_n(z) = \prod_{j \geq 1} S_{\lfloor \frac{n-2}{2j} \rfloor}(z).
\end{equation}
\end{theorem}

\begin{proof}
We prove that $D_n(z)$ divides $VH_n(z)$ by using the factorisation of $S_n(z)$ into cyclotomic polynomials (\cite{12}, Lemma 5),
\begin{equation}
S_n(z) = \prod_{j \geq 1} \Phi_{2j}(z)^{\lfloor \frac{n+j}{2j} \rfloor},
\end{equation}

and showing that the power of $\Phi_i(z)$ is greater in each term of $\mathcal{V}^H_n(z)$ than it is in $D_n(z)$.

The power of $\Phi_{2j}(z)$ in $D_n(z)$ is obtained by substituting Equation (4.15) into Equation (4.14):

$$D_n(z) = \prod_{i \geq 1} S_{\left\lfloor \frac{n-2}{2i} \right\rfloor}(z) = \prod_{i \geq 1, j \geq 1} \Phi_{2j}(z) \sum_{i \geq 1, j \geq 1} \left\lfloor \frac{n-2}{2i} \right\rfloor + j \right\rfloor = \prod_{j \geq 1} \Phi_{2j}(z).$$

We simplify $D_n(z)$ to

$$D_n(z) = \prod_{j \geq 1} \Phi_{2j}(z) \left\lfloor \frac{n-2}{2j} \right\rfloor$$

by applying Lemma 4.5 and with Equation (3.11) in [15].

Expanding the second term of $\mathcal{V}^H_n(z)$ gives

$$\left( S_{\frac{n-2}{2}}(z) \right)^2 = \prod_{j \geq 1} \Phi_{2j}(z)^2 \left\lfloor \frac{n-2}{2j} \right\rfloor,$$

which is divisible by $D_n(z)$ since

$$\left\lfloor \frac{n-2}{2j} \right\rfloor \leq 2 \left\lfloor \frac{n-2}{2j} + j \right\rfloor$$

for all $j \geq 1$ and positive even integers $n$.

The other terms in $V_n(z)$ are of the form

$$S_{n-k-2}(z)S_{k-1}(z)z^d = \left( \prod_{j > 0} \Phi_{2j}(z) \left\lfloor \frac{(n-k-2)+j}{2j} \right\rfloor \right) \left( \prod_{j > 0} \Phi_{2j}(z) \left\lfloor \frac{(k-1)+j}{2j} \right\rfloor \right) z^d$$

for each $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ where $d$ is the appropriate power of $z$. These terms are
all divisible by $D_n(z)$ if the exponents in Equation (4.16) satisfy

$$\frac{n - 2}{2j} \leq \left\lfloor \frac{k - 1}{2j} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{n - k - 2}{2j} + \frac{1}{2} \right\rfloor. \quad (4.17)$$

Let $r_1$ and $r_2$ be integers such that $0 \leq r_i < 2j$ and $\frac{k - 1}{2j} = \left\lfloor \frac{k - 1}{2j} \right\rfloor + \frac{r_1}{2j}$ and $\frac{n - 2}{2j} = \left\lfloor \frac{n - 2}{2j} \right\rfloor + \frac{r_2}{2j}$. We eliminate occurrences of $\left\lfloor \frac{k - 1}{2j} \right\rfloor$ and $\left\lfloor \frac{n - 2}{2j} \right\rfloor$ from Inequality (4.17) since they are integers and can be removed from floors, and rewrite the inequality as

$$0 \leq \left\lfloor \frac{r_1}{2j} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{r_2 - r_1 - 1}{2j} + \frac{1}{2} \right\rfloor. \quad (4.18)$$

It is straightforward to show that if the second term is $-1$, then the first term is equal to 1.

Therefore, $D_n(z)$ divides each and every term of $\mathcal{VH}_n(z)$. \hfill \Box

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Table 4.4: Table of coefficients of $P_n(z)$ for $3 \leq n \leq 11$. See Table A.22 for larger values of $n$.

Our computer investigations show that $P_n(z)$ is irreducible for $1 < n < 200$, and we know the complete factorisation of $S_k(z)$, for each positive integer $k$. We suspect, therefore, that the complete factorisation is

$$\mathcal{VH}_n(z) = P_n(z) \prod_{j \geq 1} \Phi_{2j}(z) \left\lfloor \frac{n - 2}{2j} \right\rfloor. \quad (4.19)$$

The factor $P_n(z)$ is somewhat more mysterious than $D_n(z)$; e.g., we have no formula to express it besides $\mathcal{VH}_n(z)/D_n(z)$. Take $P_{11}(z)$ for example, which is equal to $1 - 1z^1 + 1z^2 + 0z^3 + 1z^4 + 1z^5 + 1z^6 + 2z^7 + 2z^8 + 4z^9 - 8z^{10} + 10z^{11} - 4z^{12} + 10z^{13} - 8z^{14} + 8z^{15} - 8z^{16} + 10z^{17} - 10z^{18} + 12z^{19} - 8z^{20} + 10z^{21} - 12z^{22} + 10z^{23} - 6z^{24} + 6z^{25} - 6z^{26} + 6z^{27} - 4z^{28} + 4z^{29} - 4z^{30} + 2z^{31}$. The coefficients are almost all non-zero, a great many of them are even, they alternate in sign for a
long stretch, the central coefficients are larger than the ones at the tails, and the polynomial is irreducible.

The degree of $P_{11}(z)$ is $\deg(P_{11}(z)) = \deg(VH_{11}(z)) - \deg(D_{11}(z))$, both of which are easily calculated since $\deg(VH_n(z)) = \frac{n^2-n}{2} - (n-1)$, and $\deg(S_n(z)) = \binom{n+1}{2}$ gives a sum for the degree of $D_n(z)$ (see proof of Theorem 4.7 below). In general $\deg(P_n(z))$ is equal to the sum of the sequence of largest odd divisors of the numbers $1, 2, \ldots, n-2$, which is a sequence with some nice properties (see A135013 in [27]).

**Theorem 4.7.** For each $n \geq 2$,

$$\deg(P_n(z)) = \sum_{k=1}^{n-2} Od(k),$$

where $Od(k)$ is the largest odd divisor of $k$.

**Proof.** The degree of $D_n(z)$ is the sum of the degrees of its factors, given in Equation (4.14), so we can write

$$\deg(P_n(z)) = \deg(VH_n(z)) - \deg(D_n(z))$$

$$= \binom{n-1}{2} - \sum_{k \geq 1} \left( \left\lfloor \frac{n-2}{2k} \right\rfloor + 1 \right)$$

since $\deg(S_n(z)) = \binom{n+1}{2}$.

The proof that $\sum_{k=1}^{n} Od(k) = \deg(P_{n+2}(z))$ is by induction, and the base case, where $n = 0$, is easily verified. Let $p_n = \deg(P_n(z))$, for $n \geq 2$, to abbreviate the notation. It remains for us to show that $p_{n+3} - p_{n+2} = Od(n+1)$.

Let $n'01^a$ be the binary representation of $n$, so that $(n+1)2 = n'10^a$, and let $[A] = 1$ if the statement $A$ is true, and $[A] = 0$ otherwise. Observe that

$$\left\lfloor \frac{n+1}{2^k} \right\rfloor = [k \leq a] + \left\lfloor \frac{n}{2^k} \right\rfloor$$

which we use to simplify

$$\sum_{k \geq 1} \left( \left\lfloor \frac{n+1}{2^k} \right\rfloor + 1 \right) - \left( \left\lfloor \frac{n}{2^k} \right\rfloor + 1 \right)$$
and write
\[ p_{n+3} - p_{n+2} = (n + 1) - \sum_{k=1}^{\alpha} \left( \left\lfloor \frac{n}{2^k} \right\rfloor + 1 \right). \]

Using Equation (4.22) and the fact that \((n + 1)/2^k\) is an integer for \(1 \leq k \leq \alpha\), we write
\[ p_{n+3} - p_{n+2} = (n + 1) - \sum_{k=1}^{\alpha} \left( \left\lfloor \frac{n + 1}{2^k} + \frac{1}{2} \right\rfloor \right), \]
and then express this as the remaining sum terms in Equation (4.13)
\[ p_{n+3} - p_{n+2} = \sum_{k \geq \alpha + 1} \left( \left\lfloor \frac{n + 1}{2^k} + \frac{1}{2} \right\rfloor \right) \]
\[ = \sum_{k \geq \alpha + 1} \left( \left\lfloor \frac{n + 1}{2^k} + \frac{1}{2} \right\rfloor \right). \]

Applying Equation (4.13) again, we have \(p_{n+3} - p_{n+2} = (n + 1)/2^\alpha\), which is equal to \(Od(n + 1)\), as required.

In addition to finding \(\deg(P_{11}(z))\), we can evaluate at \(z = 1\) with \(P_{11}(1) = VH_{11}(1)/D_{11}(1) = 22\), a ratio which is also easy to calculate in general because \(VH_n(1)\) and \(D_n(1)\) have well understood combinatorial interpretations. It also leads to an interesting sequence, whose derivation for all \(n\) is given below.

**Theorem 4.8.** The sum of the coefficients of \(P_n(z)\) is equal to \(n^{2^v(n-2)-1}\), where \(v(n)\) is the number of 1s in the binary representation of \(n\).

**Proof.** The sum of the coefficients of \(P_n(z)\) is equal to \(P_n(1)\), which is expressible as \(VH_n(1)/D_n(1)\). The numerator evaluates to \(n^{2^n-3}\), since this is the number of coverings in \(T_{2^n}\), and the denominator is evaluated as described below.

It is well known that \(\Phi_k(1) = p\) if \(k\) is a non-zero power of a prime \(p\) and \(\Phi_k(1) = 1\) if \(k\) is divisible by two distinct primes (see [20], p.74). We can evaluate \(D_n(1)\) using Equation (4.16),
\[ D_n(1) = \prod_{i \geq 1} \Phi_{2i}(1) \left\lfloor \frac{n-2}{2^i} \right\rfloor = 2^{\Sigma_{i \geq 1} \left\lfloor \frac{n-2}{2^i} \right\rfloor}, \]
by ignoring the factors for which \(2i\) is not a power of 2. Apply Equation (4.24)
in \[15\] to obtain \(D_n(1) = 2^{n-2-v(n-2)}\). Thus
\[
P_n(1) = n2^{n-3-(n-2)+v(n-2)} = n2^{v(n-2)-1}.
\]

We have verified that \(P_n(z)\) is irreducible over the integers for \(1 < n < 200\), but we do not understand its structure well enough to prove it for all \(n\). We state below some of the observable structure which has also been verified for \(1 < n < 200\), as Conjecture 4.9, and we plot some complex roots for odd \(n\) up to 67 in Figure 4.5.

**Conjecture 4.9.**

(a) If \(k \geq 1\) and \(n \equiv 2 \pmod{2^k}\), then \(\langle z^i \rangle P_n(z) = \langle z^i \rangle P_{n+j}(z)\) for \(i \leq \frac{n-2}{2^k-1}\) and \(j \leq 2^k\).

(b) When \(n\) is odd, \(P_n(z)\) has exactly one real root \(\alpha_n\), with \(-1 < \alpha_n \leq -0.5\), and \(\{\alpha_n\}_{n \text{ odd}}\) is a monotonically decreasing sequence.

(c) When \(n\) is even, \(P_n(z)\) has no real root.

(d) The polynomial \(P_n(z)\) is irreducible over the integers for \(n \geq 2\).

(e) The alternating sums of coefficients are given by the generating function
\[
\sum_{n\geq 2} P_n(-1)z^{n-2} = \frac{(1 + z)(1 - 2z)}{(1 - 2z^2)\sqrt{1 - 4z^2}}.
\]

(f) For even \(n\), the sum of the absolute values of coefficients of \(P_n(z)\) is equal to \(P_n(-1)\) when \(n \geq 20\).

The right hand side of Equation (4.23) is the sum of two generating functions, with odd and even powered terms, respectively. The sequence of coefficients of the odd power terms is \(-\sum_{i=0}^{k} 2^{k-i} \binom{2i}{i}\), for \(k \geq 0\) (see A082590 in [27]), and that of the even power terms is \(\frac{(2k)}{2}\), for \(k \geq 1\) (see A000984 in [27]). The first few numbers \(P_n(-1)\), starting with \(n = 2\), are: 1, -1, 2, -4, 6, -14, 20, -48, 70, -166, 252, -584, 924, -2092, 3432, -7616, 12870, -28102, 48620, -104824, 184756, -394404, 705432, -1494240, 2704156, -5692636, 10400600.

Conjecture 4.9(f) compares the above sequence with the sum of the absolute values of the coefficients of \(P_n(z)\). The first few of these are listed, also starting
with \( n = 2 \): 1, 3, 4, 10, 10, 22, 28, 64, 76, 180, 260, 606, 932, 2124, 3440, 7666, 12872, 28178, 48620, 104946, 184756, 394638, 705432, 1494600, 2704156, 5693376, 10400600.
Figure 4.5: The complex zeros of $P_n(z)$ for odd $n$, where $3 \leq n \leq 67$. Darker and smaller points are used for larger $n$. 
Figure 4.6: The complex zeros of $P_n(z)$ for even $n$, where $4 \leq n \leq 68$. Darker and smaller points are used for larger $n$. 
Chapter 5

Combinatorial generation of tatami coverings

Our efforts to enumerate tatami coverings reveal algorithms for generating them. We show how to list all of the coverings in $T_n$, as well as those with $k$ vertical dominoes, by using the partition in Section 4.1.1. We also generate equivalence classes of coverings of two-way infinitely wide, horizontal strip, in Section 5.3. All of these algorithms run in constant amortised time per covering.

5.1 Coverings of the $n \times n$ grid with $n$ monominoes

In this section we describe two combinatorial algorithms for generating all elements of $T_n$. Each algorithm is a natural extension of our proof of Lemma 4.1, and we use the notation described in Section 4.1.

Firstly, consider the alternate proof of Theorem 3.6 given on page 46 (the original proof leads to another, recursive algorithm). Each class, $T_n(a)$, of the partition contains $k$ independently flippable diagonals (see Section 4.1.1 and Figures 4.3-4.4), and therefore generating the coverings of $T_n(a)$ is equivalent to generating all subsets of $k$, for which there exist various constant amortised time (CAT) algorithms. Table 4.2 contains constant-time calculations which provide the required set of flippable diagonals. Recall, for example, the flippable diagonals for $T_{18}(l_5^4)$, which are given on page 42. We conclude that there is a CAT algorithm which generates all $n \times n$ tatami coverings with $n$ monominoes.
5.1.1 Gray code

Consider a Gray code whose operation is the diagonal flip. We provide a Gray code for $T_n$.

Using the ternary representation given in Section 4.1, flipping a diagonal is equivalent to incrementing or decrementing a symbol. Let $A, B \in T_n$, with ternary representations $A = a_1a_2\ldots a_{n-2}$ and $B = b_1b_2\ldots b_{n-2}$, and define the distance, $H$, between $A$ and $B$ as

$$H(A, B) = \sum_{i=1}^{n-2} |a_i - b_i|.$$ 

This is consistent with the Hamming distance between binary representations of $A$ and $B$ (see Section 4.1). Successive coverings, $A$ and $B$, in our Gray code, satisfy $H(A, B) = 1$.

Recall that each class $T_n(a)$ contains a set, $A$, of independently flippable diagonals and therefore generating the coverings of $T_n(a)$ is equivalent to generating all subsets of $A$. For the sake of argument we use the binary reflected Gray code, defined in terms of listing all subsets of $\{a_1, a_2, \ldots , a_n\}$.

**Definition 5.1** (Binary Reflected Gray Code (BRGC) [16]). Let $G_n$ be the BRGC listing of the subsets of $\{a_1, a_2, \ldots , a_n\}$, where the $a_i$ are distinct, but otherwise arbitrary. Then $G_n$ satisfies $G_1 = (\emptyset, \{a_1\})$, and $G_n = (g_1, g_2, \ldots , g_k, \{a_n\} \cup g_k, \{a_n\} \cup g_{k-1}, \ldots , \{a_n\} \cup g_1), \emptyset g_n$ satisfies $G_n = (g_1, g_2, \ldots , g_k, \{a_n\} \cup g_k, \{a_n\} \cup g_{k-1}, \ldots , \{a_n\} \cup g_1), \emptyset g_n = G_{n-1}$.

Let $T \in T_n$ and let $\{x_1, x_2, \ldots , x_k\}$ be the set of flipped diagonals in $T$. For convenience write $T = \{x_1, x_2, \ldots , x_k\}$. Let $A$ be the set, $\{a_1, a_2, \ldots , a_{|A|}\}$, of flippable diagonals in $T_n(a)$, where $a$ is a diagonal or $a = \emptyset$. Let $\emptyset \cup g_i = g_i$, and let $a \cup G_{|A|} = (\{a\} \cup g_1, \{a\} \cup g_2, \ldots , \{a\} \cup g_{2^{|A|}}).$ The list $a \cup G_{|A|}$ is a Gray code for $T_n(a)$.

A Gray code for $T_n$ is obtained by concatenating the set of Gray codes, $a \cup G_{|A|}$, for each $T_n(a)$. For even $n$, this list of equivalence classes begins with

$$T_n(l_1^1), T_n(r_2^1), T_n(l_3^1), T_n(r_4^1), \ldots , T_n(a^{(1)}),$$

where $a^{(1)} \in \{l_1^+, r_2^+\}$, and all of the classes of the form $T_n(l_i^+)$ and $T_n(r_j^+)$ are listed. The next class is $T_n(\emptyset)$, followed by

$$T_n(r_{n-2}^+), T_n(l_{n-3}^+), T_n(r_{n-4}^+), T_n(l_{n-5}^+), \ldots , T_n(a^{(2)}),$$

(5.2)
where \(a^{(2)} \in \{r_i^+, l_j^-\}\), and all of the classes of the form \(T_n(r_i^+)\) and \(T_n(l_j^-)\) are listed.

Notice that \(r_{i+1}^+\) is flippable in \(T_n(l_{i+1}^-)\), and \(l_{i+1}^-\) is flippable in \(T_n(r_i^+)\). Let \(A\) be the set, \(\{a_1, a_2, \ldots, a_{|A|}\}\), of flippable diagonals in \(T_n(a)\), letting \(a_{|A|} = r_{i+1}^+\) if \(a = l_{i}^-\), and \(a_{|A|} = l_{i+1}^-\) if \(a = r_i^+\), so that the last covering listed in \(T_n(a)\) is \(\{a, a_{|A|}\}\). This is one flip away from \(\{a, a_{|A|}\}\), which is the first element listed in the next equivalence class. If \(a = a^{(1)}\) it is \(T_n(\emptyset)\), and otherwise it is \(T_n(a_{|A|})\).

The respective equivalence classes in (5.1) and (5.2) are symmetric, so an analogous argument holds for the latter. The class that separates them, \(T_n(\emptyset)\), remains to be discussed.

If \(a = a^{(1)}\), then \(T_n(a_{|A|})\) is undefined, but instead \(\{a_{|A|}\} \in T_n(\emptyset)\). Let \(B = \{b_1, b_2, \ldots, b_{|B|}\}\) be the flippable diagonals for \(T_n(\emptyset)\), and set \(b_{|B|} = a_{|A|}\). List \(C_{|B|}\) in reverse order so that the last covering is \(\emptyset\). Now (5.2) can be listed, and all of the interfaces between equivalence classes satisfy the Gray code requirement that they differ by exactly one diagonal.

If \(n\) is odd, then the equivalence classes are ordered

\[
T_n(b_1^{-}), T_n(t_{n-3}^-), T_n(b_3^{-}), T_n(t_{n-5}^-), \ldots, T_n(a^{(1)}),
T_n(\emptyset),
T_n(b_{n-2}^-), T_n(t_{2}^-), T_n(b_{n-4}^-), T_n(t_{4}^-), \ldots, T_n(a^{(2)}),
\]

with definitions for \(a^{(1)}\) and \(a^{(2)}\) similar to those above (see Figure 4.4). The argument for even \(n\) is easily adapted for odd \(n\).

**Theorem 5.2.** There exists a Gray code for listing the elements of \(T_n\) such that successive coverings differ by exactly one diagonal flip. The list can be created in constant amortised time per covering.

**Proof.** The preceding description applies to the binary representation for coverings of \(T_n\), so that each class \(T_n(a)\) can be generated with Algorithm 1 of [3] in constant amortised time. There are \(n - 1\) or \(n\) calls to Algorithm 1, and for each one, the input is a set of \(n - 3\) or \(n - 2\) flippable diagonals. Each of these diagonals is determined in constant time by using Table 4.2. This requires \(O(n^2)\) preprocessing operations, but \(|T_n| = n2^{n-3}\), so our algorithm is CAT.

**Remark 5.3.** Successive classes \(T_n(a)\) and \(T_n(b)\) have a symmetric difference of exactly two flippable diagonals, provided that \(a \neq \emptyset\).
Two problems remain open in this discussion. Is there a Gray code for $T_n$ whose first and last elements also differ by 1 diagonal flip? A positive answer would provide a cyclic Gray code for all $n \times n$ tatami coverings with $n$ monominoes.

5.2 Coverings in $T_n$ with $v$ vertical dominoes.

In this section, we present the CAT procedure genVH$(n,k)$, which generates the coverings counted by $VH(n,k)$ (example output is shown in Figure 5.1 and also at http://alejandroerickson.com/tatami/). Let $S(n,k)$ denote the set of subsets of \{1,2,\ldots,n\} whose elements sum to $k$; thus $|S(n,k)| = S(n,k)$. The procedure follows naturally from the sums in Equation (4.8a-4.8b), since each term $S(a,i)S(b,j)$ counts some set of ‘/’ and ‘\’-oriented diagonals. The sets $S(a,i) \times S(b,j)$ are generated in constant amortised time (CAT) by a modification of $C4$ from \[30\] (see Listing 5.1). Our modified algorithm, modC, is invoked for each sum term of Equation (4.8a-4.8b). Procedure modC is CAT for the same reasons that $C4$ is CAT.

Figure 5.1: The coverings of $T_8$ with exactly 7 vertical dominoes. This is the output of genVH$(8,7)$ printed in the order the coverings are generated (as one would naturally read text).

There is one subtlety involved in exploiting the CATness of $C4$. Invoking $C4(a,i)$ requires $\Omega(a)$ preprocessing steps if its input list is recreated for each call, but $C4(a,i)$ may not produce so many combinations for small $a$ and large $i$. The result is that we may make many calls to modC that require too much preprocessing, but this is dealt with, as follows: a top level call to $C4(i,j)$ in \[30\] takes the list $[i+1,2,3,\ldots,i+1]$, which requires $i+1$ steps to create, however, $C4(i,j)$ also
Listing 5.1: Python code for a modified version of C4 from [30] to compute $S(a,i) \times S(b,j)$. Global variables aiSet and bjSet are the lists representing $S(a,i)$ and $S(b,i)$, respectively

```python
def modC(a,i,b,j,comp,isFirst):
    global aiSet,bjSet
    if( a == 0):
        if( isFirst):
            modC(b,j,0,0,False,False)
        else:
            Output(aiSet,bjSet)
    else:
        if(isFirst):
            L = aiSet
        else:
            L = bjSet
        if( i > a*(a+1)/2 ):
            i = a*(a+1)/2 - i; comp = not comp
        if( i<a ):
            if(comp):
                L[a] = L[0]; L[0] = i+1
                modC( i, i, b, j, comp, isFirst)
                L[0] = L[a]; L[a] = a+1
            else:
                modC( i, i, b, j, comp, isFirst)
        else:
            L[a] = L[0]; L[0] = a
            if(comp):
                modC( a-1, i, b, j, comp, isFirst)
                L[0] = L[a]; L[a] = a+1
                modC( a-1, i-a, b, j, comp, isFirst)
            else:
                modC( a-1, i-a, b, j, comp, isFirst)
                L[0] = L[a]; L[a] = a+1
                modC( a-1, i, b, j, comp, isFirst)
```
concludes with the same list (see Listing 5.1). Let $A$ and $B$ be the largest integers for which $\text{modC}$ is called to compute $S(a, i) \times S(b, i)$. We set $\text{aiSet} = [1, 2, 3, \ldots, A]$ and $\text{biSet} = [1, 2, 3, \ldots, B]$, and by setting $\text{aiSet}[0] = a + 1$ and $\text{biSet}[0] = b + 1$, we initialise for each call to $\text{modC}$ with exactly two operations.

**Theorem 5.4.** The coverings in $T_n$ with exactly $k$ vertical dominoes can be exhaustively generated in constant amortised time.

**Proof.** The outer procedure does a constant amount of preprocessing steps per call to $\text{modC}$. This subroutine is CAT, so the outer procedure is also CAT. □

### 5.3 Finite tatami coverings of the infinite strip

A **strip** of height $r$ is a two-way infinitely wide integer grid of constant height $r$. The T-diagrams defined for rectangular grids also apply to the strip because the strip has no inside corners. The only difference is that there are no vertical boundaries. A **finite monomino-domino tatami strip covering** is a monomino-domino tatami covering of the strip with a finite number of T-diagram feature diagrams. We refer to these as strip coverings, as we do not consider any other type.

Two T-diagram feature diagrams are **isomorphic** if the respective sets of line segments they comprise are horizontal translations of each other. Two strip coverings are isomorphic if their respective feature diagrams, listed from left to right, are isomorphic.

Strip coverings, up to isomorphism, encapsulate some of the combinatorial properties of rectangle coverings without so many of the geometric details that arise when packing feature diagrams into a rectangle. In fact, we only consider Lemma 2.1(H1). On the other hand, the T-diagram of a strip covering can be bounded by two vertical lines, thereby converting it to a rectangular T-diagram. In this section we enumerate and generate coverings with $k$-features, of the height $r$ strip.

**Theorem 5.5.** If $R(r, n)$ is the number of non-isomorphic strip coverings with exactly $n$ features, then it satisfies the system of homogeneous linear recurrence relations,

\begin{align*}
V_r(n) & = 4(r - 1)V_r(n - 1) + 2H_r(n - 1), \text{ where } V_r(0) = 1, V_r(1) = 4r - 2; \quad (5.3) \\
H_r(n) & = 2V_r(n - 1), \text{ where } H_r(0) = 1; \quad (5.4) \\
R(r, n) & = V_r(n) + H_r(n). \quad (5.5)
\end{align*}
Proof. Recall that a T-diagram partitions the strip into regions, covered by vertical or horizontal bond. Let $V_r(n)$ and $H_r(n)$ be the number of non-isomorphic strip coverings whose leftmost regions are vertical and horizontal bond, respectively. The number of non-isomorphic features on the height $r$ strip are as follows:

**Bidimers:** There are $r - 1$ vertical, and $r - 1$ possible horizontal bidimers.

**Vortices:** There are $r - 2$ clockwise, and $r - 2$ possible counterclockwise vortices.

**Vees:** There is 1 vee on the top boundary and 1 vee on the bottom.

**Loners:** There are four loners, ↗, ↘, ↙, and ↖. The first two occur on the bottom boundary, and the latter on top boundary.

All of the bidimers, vortices and vees have vertical bond to their left and right. The ↘ and ↗ loners have horizontal and vertical bond to their left and right, respectively, while the ↖ and ↙ loners have vertical and horizontal bond to their left and right, respectively.

The bond coverings of the strip are either a horizontal or vertical bond. These are counted by the initial conditions $H_r(0) = V_r(0) = 1$. If the leftmost region of the covering is horizontal bond, then the leftmost feature must be a ↘ or ↗ loner. The region to the left of the remaining features is a vertical bond, so $H_r(n) = 2V_r(n - 1)$.

The total number of features with vertical bond on their left side is $(r - 1) + (r - 1) + (r - 2) + (r - 2) + 1 + 1 + 1 + 1$, so this gives $V_r(1) = 4r - 2$. Exactly two of these features, namely ↘ and ↖ loners, have horizontal bond on their right, so $V_r(n) = 4(r - 1)V_r(n - 1) + 2H_r(n - 1)$.

Thus $R(r, n) = V_r(n) + H_r(n)$, as required. \qed

The following Maple commands give the generating function in Corollary [5.6]

eqn1 := V(n) = 4*(r-1)*V(n-1) + 2*H(n-1);
init1 := V(0) = 1, V(1) = 4*r-2;
eqn2 := H(n) = 2*V(n-1);
init2 := H(0) = 1, H(1)=2;
eqn3 := R(n) = V(n) + H(n);
init3 := R(0) = 2, R(1) = 4*r;
soln1 := rsolve({eqn1,init1,eqn2,init2,eqn3,init3},
{R(n),V(n),H(n)},’genfunc’(z));
Corollary 5.6. The generating function

\[ R_r(z) = \sum_{k \geq 0} R(r, k)z^k, \tag{5.6} \]

satisfies

\[ R_r(z) = \frac{-2(2r - 4)z + 2}{-4z^2 - (4r - 4)z + 1}. \tag{5.7} \]

The first few values for \( 3 \leq r \leq 5 \) are given below.

<table>
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<th>( r \backslash n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
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<td>2</td>
<td>12</td>
<td>104</td>
<td>880</td>
<td>7456</td>
<td>63168</td>
<td>535168</td>
<td>4534016</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>16</td>
<td>200</td>
<td>2464</td>
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<td>374272</td>
<td>4612736</td>
<td>56849920</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>20</td>
<td>328</td>
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<td>86560</td>
<td>1406272</td>
<td>22846592</td>
<td>371170560</td>
</tr>
</tbody>
</table>

An application of the Rational Expansion Theorem from [15] shows that the growth rate is approximately \((4r)^n\) (see page 340 of [15]). Let \( R(z) = P(z)/Q(z) \), with \( P(z) = -2(2r - 4)z + 2 \), and \( Q(z) = -4z^2 - (4r - 4)z + 1 \). Applying the quadratic formula, we have \( Q(z) = (1 - \rho_1 z)(1 - \rho_2 z) \), where \( \psi = \sqrt{(r - 1)^2 + 1} \),

\[ \rho_1 = \frac{-2}{r - 1 + \psi} \quad \text{and} \quad \rho_2 = \frac{-2}{r - 1 - \psi}. \]

The Rational Expansion Theorem says that \( \langle z^n \rangle R(z) = a_1 \rho_1^n + a_2 \rho_2^n \), where the \( a_i \)s are (known) constants. Our goal is to find the growth rate of \( \langle z^n \rangle R(z) \), for large \( r \), which is dominated by \( \rho_2 \), since \( |\rho_1| < 1 \) for \( r \geq 2 \). We show that \( \rho_2 \approx 4r \); equivalently, we show that

\[ \lim_{r \to \infty} r(\psi - (r - 1)) = 1/2. \tag{5.8} \]

Let \( 1/t = r \), so that

\[ \lim_{r \to \infty} r(\psi - (r - 1)) = \lim_{t \to 0} \frac{\sqrt{(1/t - 1)^2 + 1} - 1/t + 1}{t} \]

\[ = \lim_{t \to 0} \frac{\sqrt{(t - 1)^2 + t^2} - 1 + t}{t^2}, \]

and notice that this is the indeterminate form \( 0/0 \). Two applications of L'Hopital’s
rule yield Equation (5.8). The values of \( r(\psi - (r - 1)) \) are

\[
0.828427124, 0.708203931, 0.64911064, 0.61552813, 0.59411708, 0.57933771,
\]

for \( 2 \leq r \leq 7 \). Therefore for larger \( r \), \( \langle z^n \rangle R(z) = a_1 \rho_1^n + a_2 \rho_2^n \approx a_2 (4r)^n \).

**Corollary 5.7.** There exists a CAT algorithm for generating non-isomorphic, height \( r \) strip coverings.

**Proof.** There are \( 4r \) possible non-isomorphic features in height \( r \) strip coverings, each of which can be expressed uniquely as an element of \( \{0, 1, \ldots, 4r - 1\} \). The recurrence relations in the proof of Theorem 5.5 describe a tree whose internal nodes are at least of degree 2, and whose leaves all represent output. The recursive algorithm which naturally arises from Theorem 5.5 iterates through the features that can be added, given the bond of the leftmost region. After adding each feature to the covering, using its unique symbol, the algorithm recurses. There is a constant number of operations per call and a constant number of calls per leaf. Therefore the algorithm is CAT, since there are more leaves than internal nodes.

Comparing these results on non-isomorphic strip coverings to Theorem 3.17, we have a closed form formula for the generating function \( R_r(z) \), while our methods for computing \( T_r(z) \) become cost prohibitive at around \( r = 15 \). In addition, the methods of this section yield a straightforward CAT algorithm, whereas the same cannot be said of Theorem 3.17.

Monomino-domino fixed-height coverings are a natural extension of domino fixed-height coverings, proposed by Knuth in \[19\], and discussed in \[26\]. It is worth mentioning, therefore, that Theorem 5.5 might offer an improvement to Theorem 3.17 by considering things such as non-isomorphic strip coverings whose T-diagrams are bounded on the left and right, and strip coverings with minimal distance between features. The desired respective positions for adjacent pairs of features can be tabulated, for example, in a \( 4r \times 4r \) matrix.

Enumerating isomorphic strip coverings which fit in a bounded portion of the strip, perhaps is equivalent to counting a type of integer partition. That is, the total amount of space between features is a constant, while the placement of a feature can be shifted horizontally by an even number of grid squares, if it is unhindered by a neighbouring feature or a vertical boundary.
Finally, some work must be done to determine what can happen at the vertical boundaries.
Chapter 6

Domino Tatami Covering is NP-complete

In this chapter we consider domino-only tatami coverings of rectilinear regions, which we define as a finite subset of the integer grid. We say a rectilinear region, $R$, is covered by dominoes if it is covered exactly by non-overlapping dominoes. We describe a polynomial reduction from the NP-complete problem planar 3SAT to Domino Tatami Covering ($\text{DTC}$, see Definition 6.1). The gadgets used in the reduction were discovered with the help of a SAT-solver.

As a consequence it is therefore NP-complete to decide whether there is a perfect matching of a graph that meets every 4-cycle, even if the graph is restricted to be an induced subgraph, $D$, of the grid-graph. This abstraction provides a way of comparing $\text{DTC}$ to its non-tatami counterpart. Since $D$ is bipartite, deciding whether or not it admits a perfect matching is equivalent to a maximum flow problem, for which there are various polynomial algorithms.

**Definition 6.1** (Domino Tatami Covering ($\text{DTC}$)).

**INSTANCE:** A rectilinear region, $R$, represented as $n$ grid squares.

**QUESTION:** Can $R$ be covered exactly by non-overlapping dominoes such that no four of them meet at any one point?

**Theorem 6.2.** Domino tatami covering is NP-complete

There are some previous complexity results about tilings and domino coverings. Historically, perhaps the first concerned colour-constrained coverings, such as those of Wang tiles. It is well known, for example, that covering the $k \times k$ grid...
with Wang tiles is NP-complete (see [21]). On the other hand tatami does not appear to be a special case of these, nor of similar colour restrictions on dominoes (e.g. [2, 31]).

A more closely related mathematical context is found, instead, among the graph matching problems discussed by Churchley, Huang, and Zhu, in [4]. In their paper, an \textit{H-transverse matching} of a graph $G$, is a matching $M$, such that $G - M$ has no subgraph $H$. In a tatami covering of the rectilinear grid, $G$ is a finite induced subgraph of the infinite grid-graph, $H$ is a 4-cycle, and we require a perfect matching of the edges. In fact, if the matching is not required to be perfect, the problem is polynomial.

There is no comprehensive structure theorem for tatami coverings of rectilinear grids, but evidently much of the structure is still there, as is illustrated in Figure [6.1]. In contrast with other tatami-related results, however, we make no attempt to characterise this structure. Instead, our reduction relies on the interactions between coverings of a few specific regions that are discovered using a SAT-solver.

SAT-solvers have been applied to a broad range of industrial and mathematical problems in the last decade. Our reduction from planar 3SAT uses Minisat (see [7]) to help automate gadget generation, as was also done by Ruepp and Holzer (see [25]). It is easy to see that instances of other locally restricted covering problems can be expressed as satisfiability formulae, which suggests that SAT-solvers may provide a methodological applicability in hardness reductions involving those problems.
6.1 Preliminaries

Let $\phi$ be a CNF formula, with variables $U$, and clauses $C$. The formula is planar if there exists a planar graph $G(\phi)$ with vertex set $U \cup C$ and edges $\{u, c\} \in E$, where one of the literals $u$ or $\overline{u}$ is in the clause $c$. When the clauses contain at most three literals, $\phi$ is an instance of planar 3SAT ($P3\text{SAT}$), which is NP-complete (see [22]).

We construct an instance of $DTC$ which emulates a given instance, $\phi$, of $P3\text{SAT}$, by replacing the vertices and edges of $G(\phi)$ with a rectilinear region, $R(\phi)$, that can be tatami-covered with dominoes if and only if $\phi$ is satisfiable. Let $n = |U \cup C|$. In Section 6.3 we show that $R(\phi)$ can be created in $O(n)$ time, and that it fits in an $O(n) \times O(n)$ grid, by using Rosenstiehl and Tarjan’s algorithm (see [24]).

6.2 Gadgets

In this section we describe wire, NOT gates, and AND gates, which form the required gadgets. The functionality of our gadgets depends on the coverings of a certain $8 \times 8$ sub-grid.

Lemma 6.3. Let $R$ be a rectilinear grid, with an $8 \times 8$ sub-grid, $S$. If a domino crosses the boundary of $S$ in a domino tatami covering of $R$, then at least one corner of $S$ is also covered by a domino that crosses its boundary.

Proof. Suppose $R$ is covered by dominoes, and consider those dominoes which cover $S$. Such a cover may not be exact, in the sense that a domino may cross the boundary of $S$. If we consider all such dominoes to be monominoes within $S$, we obtain a monomino-domino covering of $S$. This covering inherits the tatami restriction from the covering of $R$, so it is one of the $8 \times 8$ monomino-domino coverings enumerated in [14] (and/or [12]).

The proof of Lemma 3.10, paragraph 3, states that there is a monomino in at least one corner of $S$ if $0 < m < n$; Corollary 3.5 states that there is a monomino in at least one corner of $S$ if $m = n$. This monomino corresponds to a domino which crosses the boundary in a corner of $S$, as required.

The rectilinear region $R(\phi)$ incorporates a network of $8 \times 8$ squares, whose centres reside on a $16Z \times 16Z$ grid, and whose corners form part of the boundary of $R(\phi)$. Lemma 6.3 implies that no domino may cross their boundaries, and thus
Figure 6.2: All monomino-domino tatami coverings of the square have at least one
monomino in their corners (see [12, 14]). The squares in \( R(\phi) \) have isolate corners,
so these must be covered in exactly one of the two ways given by Exercise 7.1.4.215
in [19], shown in the left and right-hand cross-hatched squares in Figure 6.3(a).

each one must be covered in one of the two ways shown in Figure 6.3(a). (For
proofs see [26] and Exercise 215, Section 7.1.4 in [19]).

The coverings of these squares are related to each other by connecting regions.
The part of an \( 8 \times 8 \) square which borders on a connector may be covered either
by two tiles, denoted by \( F \) to signify “false”, or three tiles, denoted by \( T \) to signify
“true” (see Figure 6.3(a)). Note that the covering of a square is not \( T \) or \( F \) by
itself, because connectors below and beside it would meet the square at differing
interfaces.

A connector, which imposes a relationship between the coverings of a set of
\( 8 \times 8 \) squares, is verified by showing that it can be covered if and only if the
relationship is satisfied. The connectors we describe were generated with SAT-
solvers, but they are simple enough that we can verify them by hand, as is done
below.

**NOT gate.** The NOT gate interfaces with two \( 8 \times 8 \) squares (see Figure 6.3(a)), and
can be covered if and only if these squares are covered with differing configura-
tions.

**Wire gadget.** Wire transmits \( T \) or \( F \) through a sequence of squares (see Fig-
ure 6.4(a)). A turn may incorporate a NOT gate in order to maintain the same
configuration (see Figure 6.4(b)).

**AND gate.** The AND gate interfaces with two \( 8 \times 8 \) input squares, and one output
square (see Figure 6.5). It can be covered with dominoes if and only if the output
Figure 6.3: **NOT** gate can be covered if and only if the input differs from the output. Numbered tiles indicate the (non-unique) ordering in which their placement is forced. Red dotted lines indicate how domino coverings are impeded in (d) and (e).

Figure 6.4: Wire gadget.

value is the **AND** of the inputs (see Figs. 6.6 and 6.7).

**Variable gadget.** We use a vertical segment of wire. The variable gadget is set to T or F by choosing the appropriate covering of one of its $8 \times 8$ squares. Its value (or its negation) is propagated to clause gadgets via horizontal wire gadgets, representing edges.

**Clause gadget.** The clause gadget is a circuit for $\neg((a \land (\bar{b} \land c)))$, or the equivalent with fewer inputs, ending in a configuration that can be covered if and only if the
output signal of the circuit is \( T \). To satisfy the layout requirements, the inputs to the clause are vertically translated by wire (see Figure 6.8).
Figure 6.8: A three input clause gadget from the circuit \( \neg(a \land (\bar{b} \land \bar{c})) \). Vertical wire translates horizontal inputs without changing the signal. The end of the clause is coverable if and only if its signal is \( T \).

6.3 Layout

Let \( G(\phi) \) be a planar embedding of the Boolean 3CNF formula \( \phi \), using Rosenstiehl and Tarjan’s (see [24]) algorithm, so that each vertex is represented by a vertical line segment, and each edge is represented by a horizontal line segment. All parts lie on integer grid lines, inside of a \( O(n) \times O(n) \) grid, where \( n = |U \cup C| \), and the embedding is found in \( O(n) \) time.

There exists a constant \( K \), which is the same for any planar 3CNF formula, such that \( G(\phi) \) can be scaled to fit on the \( nK \times nK \) grid, and its parts replaced by the gadgets described above. Each gadget has a constant number of grid squares, which ensures that \( R(\phi) \) has \( O(n^2) \) grid squares altogether.

The variable gadget is connected to edges by branches. The layout of \( G(\phi) \) prevents conflicts between edges meeting the variable gadget on the same side, while two edges can meet the left and right sides of the variable gadget without interfering with each other. The inputs of the clause gadget are symmetric, so there are no conflicts when connecting these to horizontal edges (see Figure 6.8(a)).

Example. The planar Boolean formula from Figure 1 in [22] gives the DTC instance in Figure 6.9.
6.4 SAT-solver

The search for logical gates required fast testing of small DTC instances. We reduced DTC to SAT in order to use the SAT-solver, Minisat (see [7]), and efficiently test candidate regions connecting $8 \times 8$ squares while satisfying the conditions of the desired gate. The DTC solver was also allowed to make certain decisions about the region, rather than simply testing regions generated by another program. See Section A.2 for the computer script described below.

Our search algorithm requires the following inputs:

- an $r \times c$ rectangle of grid squares, partitioned into pairwise disjoint sets $K, X, A, C$; and,

- a set of partial (good) coverings, $G$, and partial (bad) coverings, $B$, of $C$.

The output, $R$, is the region $A' \cup K$, where $A' \subseteq A$, which satisfies the following constraints.

(g) There exist coverings of $R$ which form partial tatami domino coverings with each element of $G$.

(b) There exists no covering of $R$ which forms a partial tatami domino covering with an element of $B$.

The outer loop of the search algorithm calls the SAT-solver to find a region that satisfies all elements of $G$, and avoids a list of forbidden regions, which is initially empty. Upon finding such a region, the inner loop checks whether the region satisfies any element of $B$. The search succeeds when (g) and (b) are both satisfied, and fails if the outer loop’s SAT instance has no satisfying assignment.
The search space grows very quickly for several reasons, not least of which is the fact that $2^{160}$ regions are possible within the $20 \times 8$ rectangle occupied by our AND gate (if corners are allowed to meet one another). In addition, the list of forbidden regions, $L$, becomes too large for the SAT-solver to handle efficiently.

We used two heuristics on the inputs to obtain a feasible search. The first was searching for a smaller AND gate, which we modified to fit the placement of the $8 \times 8$ squares. The second was choosing forbidden squares, $X$, and required squares, $K$, to reduce the number of trivially useless regions that are tested.

### 6.4.1 DTC as a Boolean formula

The SAT instances used above are modifications of a formula which is satisfiable if and only if a given region has a domino tatami covering.

Let $R$ be the region we want to cover, and consider the graph whose vertices are the grid squares of $R$, and whose edges connect vertices of adjacent grid squares. Let $H$ be the set of horizontal edges and let $V$ be the set of vertical edges. The variables of the SAT instance are $H \cup V$, and those variables set to true in a satisfying assignment are the dominoes in the covering. The clauses are as follows, where $h, h' \in H$ and $v, v' \in V$.

1. Ensure a matching: For each pair of incident horizontal edges $(h, h')$, require the clause $\bar{h} \lor \bar{h'}$, and similarly for $(v, v'), (h, v)$.

2. Ensure the matching is perfect: For each set of edges $\{h, h', v, v'\}$, which are incident to a vertex, require the clause $h \lor h' \lor v \lor v'$.

3. Enforce the tatami restriction: For each 4-cycle, $hvh'v'$, require the clause $h \lor h' \lor v \lor v'$.

### 6.5 Lozenge Tatami Covering

There are other locally constrained covering problems that are easily represented as Boolean formulae. Some of these are obviously polynomial, such as monomino-domino tatami covering, but others may be NP-complete. SAT-solvers can sometimes be used in such problems to create elaborate gadgets, which may help find a hardness reduction.

An example problem, whose computational complexity is open, is Lozenge-only Tatami Covering (see Definition 1.1). This problem is the decision about
whether or not a finite sub-grid of the triangular lattice can be covered with lozenges, such that no 5 lozenges meet at any point. A structure similar to that of tatami coverings occurs for this constraint (see Figure 6.10).

![Image of triangle-lozenge tatami covering.](image)

Figure 6.10: A triangle-lozenge tatami covering.

Our main question about DTC is the complexity of the case where the region is simply connected (no holes). It seems likely that the problem is still NP-complete, but a completely new approach will be required (see Section 7.1).

Secondarily, we are interested in $H$-transverse perfect matchings for $H$ and $G$ other than $C_4$ and grid-graphs. Are there other $H$-transverse perfect matchings of interest which induce a tatami-like global structure in the containing graph?

Another variant, mildly advocated by Don Knuth (personal communication), concerns inner corners of the coverings, such as occurs at the upper left in the letter $T$ in Figure 6.1. If corners such as these, where a $+$ occurs, are forbidden but corners such as the upper right one in the $I$ are allowed (a $\perp$ shape or one of its rotations), then the nature of tatami coverings changes. The complexity of such coverings is unknown.
Chapter 7

Open problems

7.1 Structure and complexity

The tatami structure opens up many questions, and we have answered a representative subset of these. Open problems related to the tatami restriction are discussed here. The first one concerns the global structure of general rectilinear regions.

**Problem 7.1.** Given a rectilinear region, $R$, what are the minimal partial coverings of $R$ which determine a unique monomino-domino tatami covering of $R$?

Clearly, the four features in Figures 2.2(a)-2.4 force the propagation of rays, just as they do in rectangular coverings, but some care needs to be taken at inside corners. For example, a ray may begin at an inside corner. Given a characterisation of such minimal partial coverings, we would like to establish the analogue of Lemma 2.2; is every covering of the region, $R$, uniquely determined by the tiles on its boundary?

Another consideration is the variant mentioned in Section 6.5. Rather than forbidding four tiles from meeting, we might forbid the edges of tiles to form a $+$-shape together; let this be called the $+$-tatami restriction. Tatami and $+$-tatami are the same for coverings of the rectangle, but the alternative definition forbids three tiles to meet at an inside corner. In this case a ray cannot meet an inside corner, which causes the tatami structure to break down somewhat. On the other hand, the results of Chapter 6 do not make explicit use of this. The SAT-solver searches could easily be modified to avoid $+$-shapes in the covering. Given any tatami problem on the rectilinear grid, we might ask the same problem with the $+$-tatami variant, so we choose $\mathcal{DTC}$ to represent this class of problems.
Problem 7.2. Is the +--tatami variant of DTC NP-complete?

Another variant of DTC is its restriction to simply connected regions. There appears to be no way to simulate planar 3SAT in a region with no holes. We might start by imagining an $n$-tentacled cephalopod-shaped region and having to find coverings of the tentacles which do not conflict in the mantle. Can the shape of the mantle be chosen to encode some NP-complete problem whose solution lies in the coverings of the tentacles? Whatever the case may be, this strengthening of DTC deserves a number.

Problem 7.3. Is DTC NP-complete, even when the region $R$ is simply connected?

Constructive solutions to certain instances of DTC appear to be determined by the locations of bidimers in the region. These ×-shaped features motivate a loosely related continuous problem, proposed by Frank Ruskey. Consider the following configuration of a pair of orthogonal line segments, intersecting at a point, $(x, y)$, strictly inside an arbitrary rectilinear region, $R$: a water strider at the point $(x, y)$ is a pair of orthogonal line segments, which intersect at $(x, y)$ (see Figure 7.1). Their lengths are maximal, without touching the boundary of $R$, and their slopes are ±1.

The fact that water striders are open sets, and $(x, y)$ cannot be on the boundary is only critical to the analogy with bidimers, but this may or may not be more interesting than the natural alternatives.

Definition 7.4 (Water Strider Problem, 2011, [12]).

INSTANCE: A rectilinear region, $R$, with $n$ segments, and vertices in $\mathbb{R}^2$.

QUESTION: Is there a configuration of at most $k$ water striders, such that no two water striders intersect, and no more water striders can be added?

A modification of the problem provides a stricter adherence to the analogue with DTC, and it is given below. A set of non-intersecting water striders in $R$ partitions $R$ into sub-regions. Each water strider bounds exactly four sub-regions, which are naturally above, below, and to the sides of its centre. In the modified Water Strider Problem, sub-regions to the left of each water strider may not have horizontal boundaries and the regions above and below it may not have vertical boundaries (of non-zero length).

Returning to the realm of the discrete, another natural problem on rectilinear grids, proposed by Matt DeVos (personal communication), is as follows.
Definition 7.5 (Partial Tatami Covering, PTC).

**INSTANCE:** A rectilinear region $R$, made up of $n$ segments, and a partial covering of $R$, with $k$ tiles.

**QUESTION:** Is this partial covering part of a tatami covering of $R$?

The point of interest, of course, is the computational complexity of this, not to mention its $+\text{-tatami}$ variant.

Problem 7.6.

(a) Is PTC NP-complete?

(b) Is PTC polynomial when $n = 4$?

The following decision problem on rectangular grids, proposed by Martin Matamala (private communication), is named Tomoku, after the puzzle game in [8] (see Figure 7.3). Without the tatami condition this decision problem is NP-complete (Theorem 4, [6]).

Definition 7.7 (Tomoku).

**INSTANCE:** An $r \times c$ rectangle, and for each row and each column, three integers indicating the number of grid squares covered by vertical dominoes, horizontal dominoes, and monominoes, respectively.
**QUESTION:** Does there exist a covering of the $r \times c$ rectangle with these row and column projections?

**Problem 7.8.** Is Tomoku NP-complete?

It is known that there are pairs of coverings with the same row and column projection. For example, an $n \times n$ covering with a central clockwise vortex gives the same row and column projections as a counterclockwise vortex.

Two instances are represented graphically in Figure 7.3, reprinted from [8]. Here we draw the tiles contained in each row and column, which is why monominoes appear twice. Tatami coverings do exist for these row and column projections, and finding them is an entertaining diversion.

The solution can be drawn with a pencil, but backtracking, and hence erasing is common. A mechanical realisation of the tatami restriction which improves game play is described in Section A.3.

### 7.2 Enumeration

The starting point for enumerating tatami coverings is characterising $T(r,c,m)$ for all parameters.
Figure 7.3: Instances of Tomoku, reprinted from [8]. The 5 × 12 puzzle is quite challenging.

**Problem 7.9.** Find $T(r, c, m)$ when $m > 1$, and $m$ is not maximum, and $m$ has the same parity as $rc$.

Conjecture 3 from [12] seems to simplify Problem 7.9, so it is included below.

**Conjecture 7.10** (Erickson, Ruskey, Schurch, Woodcock, 2011, [12]). For all $d \geq 0$ and $m \geq 1$ there is an $n_0$ such that, for all $n \geq n_0$,

$$T(n, n + d, m) = T(n_0, n_0 + d, m),$$

whenever $n(n + d)$ has the same parity as $m$.

Experimentally, it appears that the smallest $n_0$ is $m + d + 4$, if $d \geq 1$.

Conjecture 3.14 should be mentioned here, though we have little doubt that it gives us $T(r, c, m)$ when $m$ is maximum, and $r \leq c$.

Chapter 4 contains our most compelling enumeration problems, which concern the mysterious polynomial, $P(z)$. Of the ideas in Conjecture 4.9, we are most interested in proving the irreducibility of $P(z)$. Second to that is Conjecture 4.9[f]; why does this pattern begin at $n \geq 20$? The broader question, however, is why does $VH_n(z)$ factor this way at all?

**Problem 7.11.** Is there a natural geometric interpretation for the factorisation of $VH_n(z)$? Is there a closed form formula for $P(z)$?

Chapter 4 touches briefly on balanced coverings of the $n \times n$ grid with $n$ monominoes. Observe that a tatami covering of the $r \times c$ rectangle is a partition of the rectangle into $O(\max(r, c))$ regions of horizontal and vertical bond (and isolated grid squares), by Lemma 2.2, and these regions are bounded either by ray diagrams, or the rectangle’s boundary, which gives them a certain class of shape. Thus, a balanced tatami covering defines two collections of such shapes.
whose total areas are the same (omitting monominoes). Balanced coverings were first investigated by Don Knuth (private communication), perhaps for this or other reasons, but virtually nothing has been published on them.

**Problem 7.12.** How many balanced tatami coverings are there of the \( r \times c \) rectangle, with \( m \) monominoes.

### 7.3 Combinatorial Algorithms

Algorithms which use the diagonal flip are easily conceived. For example, Section 3.2.2 provides for making independent diagonal flips to produce the coverings of the \( n \times n \) rectangle, with a given bidimer or vortex. Coverings of rectangles with maximum monominoes are also characterised by their diagonal flips, by Lemma 3.3, and algorithms for generating various cases are in the details of Conjecture 3.14.

Foremost, however, is to generate all coverings of a given rectangle, perhaps with an analogue to our algorithm for strip coverings, in Section 5.3.

**Problem 7.13.** Generate all tatami coverings of the \( r \times c \) grid in constant amortised time.

Strip coverings ignore the precise horizontal location of a feature, and consider only the order in which the features appear, from left to right. In contrast, coverings of the rectangle are bounded, and their features have precise locations. Furthermore, the vertical boundaries introduce a minor complication, because (a) vees and loners may originate on vertical boundaries, and (b) a pair of rays which terminate at a vertical boundary may conflict if the covering is extended to a strip covering.

### 7.4 Triangular Tatami Coverings

Among the conceivable generalisations of tatami coverings, there is one which exhibits a global structure, akin to the one discussed in this dissertation (see Figure 7.8). A *triangle (tile)* in the triangular lattice is an equilateral triangle with unit side-length, and a *lozenge (tile)* is composed of two triangles, joined along one edge. A tile is *arranged on the lattice* if its vertices are at lattice points.
Definition 7.14 (Triangle-lozenge 5-tatami covering). A triangle-lozenge 5-tatami covering is an arrangement of triangles and lozenges, on the triangular lattice, in which no 5 tiles meet at any point. We shorten this to 5-tatami covering.

Enumerating 5-tatami coverings of hexagonal regions extends research on boxed plane partitions (see [5, 23]), just as monomino-domino tatami coverings of rectangular regions extend research on domino coverings. Our tatami problems on general rectilinear regions, on the other hand, can be asked for appropriately restricted finite regions of the triangular lattice; i.e. they may or may not be simply connected, or the outside corners of the region may or may not touch each other, etc. We describe the structure of triangle-lozenge 5-tatami coverings below.

We proceed as we did for monomino-domino tatami coverings, and reuse some of the vocabulary. Whenever two lozenges of distinct orientations share an edge, they propagate this same pattern, called a ray until it reaches the boundary (see Figure 7.4). A case analysis reveals that a ray may begin at exactly one of four types of features shown in Figures 7.4–7.7: a tridimer, vortex, loner, or vee.

Figure 7.4: A ray propagates itself until it reaches a boundary. Six orientations are possible.

Figure 7.5: When a ray begins with a lozenge (of a different orientation), the resulting feature is a tridimer.

Grids with no inside corners appear to be readily enumerable, and otherwise, we have the analogue to rectilinear regions and all of the related questions. Some questions need to be modified. For example the question of balanced coverings takes on a three-way symmetry, since there are three orientations for lozenges.
Figure 7.6: If the ray begins with a triangle and lozenge, then it is either a vortex, or one of two types of loner; a hiloner or a loloner.

Figure 7.7: When there are two triangles at the beginning of the ray, we have a vee, or two loloners.

An $a \times b \times c$ box is an equi-angular hexagon whose vertices are lattice points, with side lengths $a, b, c, a, b, c$. Let $T(a, b, c, m)$ be the 5-tatami coverings of the $a \times b \times c$ box with exactly $m$ triangles.

Problem 7.15. What is $T(a, b, c, m)$?

When $m = 0$, this counts the plane partitions of the $a \times b \times c$ box which satisfy the 5-tatami constraint. We are motivated to find some elegant “slice” of MacMahon’s formula (23, 29),

$$
\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2'}
$$

which counts box plane partitions in general.
Figure 7.8: This $16 \times 3 \times 3$ tiling has every possible feature, up to rotation and reflection, except a vee. The tridimer can be replaced by a vee by cutting its central red lozenge into two triangles.
Chapter 8

Final Remarks

Curiosity has been our primary motivation to explore the tatami restriction so deeply, and we have good reason to be curious about it. Not only has the idea been around since at least the 17th century, it is a natural restriction of a well studied problem — monomino-domino coverings — with a starkly simple description: no four tiles meet. It is remarkable that the tatami structure was not studied earlier, from a computer science and mathematics point of view.

Our characterisation of the tatami restriction reveals that monomino-domino tatami coverings are a rich and mathematically harmonious topic. The structure is at first surprising, yet predictable, and visually appealing.

Summarising, tatami coverings of an $r \times c$ rectangle are determined by $O(\max(r, c))$ sources. Of these, there are four types, up to reflection and rotation, and each consists of at most five tiles. Up to four rays emanate from each source, marking all of the boundaries between horizontal and vertical bond, forming a partition of the rectangle. Tatami coverings of rectilinear regions inherit much of the same structure, though some additional thought needs to be given to what can happen at inside corners. The most telling diagrams are Figures 2.1–2.4 which show the features and coverings of rectangles, and Figure 1.1 which shows a covering of a non-rectangular region.

We use the structure for various enumeration results on coverings of rectangles, which fall into several categories. The first concerns coverings with a certain number of monominoes; we enumerate coverings with a maximum or minimum number of monominoes, with a precise number of monominoes, and with any number of monominoes. The second concerns coverings with a certain number of vertical dominoes.

The latter of these is perhaps the most interesting result after the structure
itself, and is restated below. Recall that \( V(n,k) \) and \( H(n,k) \) are the number of coverings in \( T_n \) with \( k \) vertical and horizontal dominoes, respectively, and that 
\[
S_n(z) = \prod_{i=1}^{n}(1 + z^i).
\]
Theorem 4.2 states that \( V_H n(z) = \sum_{k \geq 0} V(n,k)z^k \) and 
\[
V_H n(z) = \sum_{k \geq 0} H(n,k)z^k
\]
for even and odd \( n \), respectively.

**Theorem 4.6** (Erickson, Ruskey, 2013, [11]). The generating polynomial \( V_H n(z) \) has the factorisation

\[
V_H n(z) = P_n(z)D_n(z)
\]

where \( P_n(z) \) is a polynomial and

\[
D_n(z) = \prod_{j \geq 1} S_{\lfloor \frac{n-2}{2j} \rfloor}(z).
\]

This result was first observed for small \( n \) by Don Knuth. Our general solution makes use of several results on tatami coverings, such as Theorem 3.2 on the maximum number of monominoes, and the diagonal flip characterisation. The recursive nature of the factorisation in Theorem 4.6 compels us to find a connection with the recursive nature of the coverings themselves, as well as with the family of polynomials, \( P_n(z) \) (see Section 7.2).

In answering some of the most natural questions about tatami coverings, we showed that the topic has considerable mathematical richness. We developed a tool set, namely the structure, for solving problems in this area, and these techniques can be applied to several of the open problems mentioned in Chapter 7.

It is the present author’s earnest hope that other computer scientists and mathematicians will become interested in tatami coverings, be it for the purest pursuit of knowledge, or some undiscovered application. They will not be disappointed.
Bibliography


Appendix A

Appendix

A.1 Tables

Tables A.1-A.19 were computed with a backtracking program that is ignorant of the tatami structure proven in Chapter 2. Periods (.) are used at parameters with no coverings due to Theorem 3.2.

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Table A.1: Number of tatami coverings of the $r \times c$ grid with 0 monominoes, and $r \leq c$. 
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Table A.3: Number of tatami coverings of the $r \times c$ grid with 2 monominoes, and $r \leq c$.

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Table A.5: Number of tatami coverings of the $r \times c$ grid with 4 monominoes, and $r \leq c$. 
### Table A.6: Number of tatami coverings of the $r \times c$ grid with 5 monominoes, and $r \leq c$.

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Table A.6: Number of tatami coverings of the $r \times c$ grid with 5 monominoes, and $r \leq c$. 
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Table A.7: Number of tatami coverings of the $r \times c$ grid with 6 monominoes, and $r \leq c$. 
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|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $r$          |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 1            |     |     |     |     |     |     |     |     |     |     |     |     |     |
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| 3            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 4            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 5            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 6            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 7            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 8            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 9            |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 10           |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 11           |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 12           |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 13           |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 14           |     |     |     |     |     |     |     |     |     |     |     |     |     |

Table A.8: Number of tatami coverings of the $r \times c$ grid with 7 monominoes, and $r \leq c$. 
Table A.9: Number of tatami coverings of the \( r \times c \) grid with 8 monominoes, and \( r \leq c \).
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Table A.10: Number of tatami coverings of the $r \times c$ grid with 9 monominoes, and $r \leq c$. 
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Table A.11: Number of tatami coverings of the \( r \times c \) grid with 10 monominoes, and \( r \leq c \).
Table A.12: Number of tatami coverings of the $r \times c$ grid with 11 monominoes, and $r \leq c$. 

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Table A.13: Number of tatami coverings of the $r \times c$ grid with 12 monominoes, and $r \leq c$. 
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Table A.14: Number of tatami coverings of the $r \times c$ grid with 13 monominoes, and $r \leq c$. 
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Table A.15: Number of tatami coverings of the $r \times c$ grid with 14 monominoes, and $r \leq c$. 
Table A.16: Number of tatami coverings of the $r \times c$ grid with 15 monominoes (and greater), with $r \leq c$.
Table A.17: Number of tatami coverings of the $r \times c$ grid with any number of monominoes, and $r \leq c$; i.e., the sums of Tables A.1-A.16.

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Table A.18: Number of tatami coverings of the $n \times n$ grid with $m$ monominoes. The last row appears to be A027992 in [27].
Table A.19: Number of tatami coverings of the $r \times c$ grid with the maximum number of monominoes and $r < c$ (see Conjecture 3.14).

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Table A.19: Number of tatami coverings of the $r \times c$ grid with the maximum number of monominoes and $r < c$ (see Conjecture 3.14).
Table A.20: Table of coefficients of $V_H(n(z)$ for $2 \leq n \leq 20$. The $(n,k)$th entry represents the number of coverings of $T_n$ with $k$ vertical dominoes when $n$ is even, and $k$ horizontal dominoes when $n$ is odd (continued on next page).
Table A.20: Table of coefficients of $V_H^n(z)$ (continued from previous page).
A.2 SAT-solver gadget search

We include the python script which calls MiniSat (see [7]), in Listing A.1.

Listing A.1: Python script which calls MiniSat to find gadgets for reduction in Chapter 6.

Disclaimer: This script serves its intended purpose, however, it has not been optimized in general, and may contains errors and bugs. Note also, that references to tilings are considered to be coverings, within this dissertation.

```python
import subprocess
import sys
import csv

# Input two file names (The second will be overwritten with output
# from minisat). minisat must be in the same directory as findgadget.py

f = open(sys.argv[1], 'r')
```
Table A.21: Table of coefficients of \( R_n(z,1) \) for \( 2 \leq n \leq 19 \). The \((n,k)\)th entry represents the number of coverings in all four rotations of \( T_n \) with \( k \) vertical (or horizontal, by rotational symmetry), dominoes (continued on next page).
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Table A.21: Table of coefficients of \( R_n(z, 1) \) (continued from previous page).
try:
    r = int(g[0])
    c = int(g[1])
except ValueError:
    quitError('Invalid input file')
goodClauses = [] # use this to store all the good clauses and do a # single write.

s = 0 # These record which squares are part of the # configurations. They have less restrictions on them to allow for # dominos which are not contained in the r x c grid.
Cregion = [[0]*c for i in range(r)] # These are the x_i variables for grid squares. There are r*c of them
I = [[0]*c for i in range(r)]
i = 0 # i, j indices for X
j = 0

C = int(g[r+2]) # Number of good configurations
badC = int(g[r+2+C*(r+1)]) # Number of bad configurations
nGoodVars = r*c + C*(r-1)*c + r*(c-1)
nBadVars = r*c + (r-1)*c + r*(c-1)
HH = []
VV = []
for i in range(badC): # store list of clauses describing bad configs for each k in badC
    badClauses = []
    for i in range(C): # one list of clauses for each bad configuration
        badClauses.append([])
        for i in range(r*c): # For convenience, we use r x c arrays to represent each of these.
            HH.append([[0]*c for i in range(r)])
            VV.append([[0]*c for i in range(r)])
for i in range(r*c): #if a literal is 0, we do not include it
    def clause(L):
        global goodClauses
        goodClauses.append([])
        for l in L:
            if l != 0:
                goodClauses[-1].append(l)
    def badclause(k,L):
        global badClauses
        # each clauses is a list of literals
        i = 0
        badClauses[k].append([])
        for l in L:
            if not l == 0:
                badClauses[k][-1].append(l)
            if i == 0:
                badClauses[k][-1].append(1)
        if (i == 0):
            quitError("adding empty clause to badClauses(\"+str(k)+\") + str(badClauses[k])")
#R is a ordered list of integers from +/-1 to +/-r*c
for i in range(len(R)):
    if R[i] > 0:
        s = s + '#'
    else:
        s = s + '.
        if R[i]%c == 0:
            s = s + '
    s = s + '
print s

# displays the kth tiling in the list of good configurations. Note that the bad configurations take k=0

def displayTiling(T,k):
    global r,c
    print T
    C = [[0]*(2*c+1) for i in range(r+1)]
    for i in range(r):
        for j in range(c-1):
            if(HH[k][i][j] in T):
                for i in range(r-1):
                    for j in range(c):
                        if(VV[k][i][j] in T):
    s = ''
    for i in range(r+1):
        for j in range(2*c+1):
            if(C[i][j] == 1):
                if(j%2 == 0):
                    s = s + '|'
                else:
                    s = s + '_'
            else:
                s = s + ' ';
        s = s + '

    print s

# input 'good ' or 'bad ', and an index
# output index of first line with that configuration

def confIndex(gb,i):
    if(gb == 'good '):
        if( not i < C):
            return -1
        else:
            return r+3+i*(r+1)
    elif (gb == 'bad '):
        if( not i < badC):
            return -1
        else:
            return r+3+( C+i )*( r+1)
    else:
        return -1

# read a file with a SAT assignment and return the assignment as a list.

def getSATAssignment(satoutFilename):
    g = open(satoutFilename ,'r')
    g.readline()# skip the line that says SAT
    # convert input to a list of integers
    csvg = csv.reader(g, delimiter= ' ')
    for row in csvg :
        csvg = row
        break
    csvg = map (int , csvg )
    g.close()
    return csvg

def quitError(msg):
sys.stdout.flush()
sys.stderr.write(msg)
sys.stderr.flush()
sysexit(1)

#this is an index that is incremented to label each variable.
#input region restrictions into good CNF
s=0

for a in g[2:r+2]:
    for aa in a:
        if(aa in ['X', '.', '#', 'C']):
            s = s + 1
X[i][j] = s
if(aa == 'X'): # disallow this square, or require it
clause([-X[i][j]])
if(aa == 'Y'):
clause([X[i][j]])
if(aa == 'C'):
    clause([X[i][j]])
Cregion[i][j] = 1
j = (j+1)%c
if(j == 0):
i = (i + 1)%r

# get good clauses for H and V requirements
for k in range(C):
i=0
j=0
for a in g[confIndex('good',k):confIndex('good',k)+r]:
    for aa in a:
        if(aa in ['.','<','>','A','V']): # there should be no # in
            # this part of the data

            if(j<c-1):
                s = s + 1
                HH[k][i][j] = s
            if(i<r-1):
                s = s + 1
                VV[k][i][j] = s
        if(aa == '<'): # require this matched horizontal edge (don't
            # need to record both <>)
            clause([HH[k][i][j]])
        if(aa == 'A'): # require this matched vertical edge
            clause([VV[k][i][j]])
            j = (j+1)%c
        if(j == 0):
i = (i+1)%r

badHH=HH[0] # these cannot be defined earlier
badVV=VV[0]

for k in range(badC):
i=0
j=0
for a in g[confIndex('bad',k):confIndex('bad',k)+r]:
    for aa in a:
        if(aa in ['.','<','>','A','V']):
            if(aa == '<'):
                badclause(k,[badHH[i][j]])
            if(aa == 'A'):
                badclause(k,[badVV[i][j]])
            j = (j+1)%c
        if(j == 0):
i = (i+1)%r

for k in range(badC):
    # Build a CNF with no region which forces the bad configuration.
    # Later we enforce a region and check if it is satisfiable. We can
    # use the same rules as we do below, because those are region
    # independent.
    for i in range(r):
        for j in range(c):
            # Rule 1 note that the (j<c-1) etc are redundant
            if(not (Cregion[i][j] == 1)):
                badclause(k,-X[i][j],(j<c-1)*badHH[i][j],(i<r-1)*badVV[i][j],(j>0)*badHH[i][j-1],
                            (i>0)*badVV[i-1][j]])
            # Rule 2. Uncovered squares must not be matched.
            if(j<c-1):
                badclause(k,[X[i][j],-badHH[i][j]])
            if(i<r-1):
                badclause(k,[X[i][j],-badVV[i][j]])
            if(j>0):
                badclause(k,[X[i][j],-badHH[i][j-1]])
            if(i>0):
                badclause(k,[X[i][j],-badVV[i-1][j]])
Rule 3. Tatami condition. Since this only applies to interior intersections, it also applies to Cregion. i.e at least one of the dimers will be contained among these four squares.

```python
if (i<r-1 and j<c-1):
    badclause(k,[-X[i][j],-X[i+1][j],-X[i][j+1],-X[i+1][j+1],badHH[i][j],badVV[i][j],badHH[i+1][j],badVV[i+1][j]])

# Rule 4. Also applies to Cregion. I never want to match adjacent edges. Perhaps redundant, but it simplifies the code.
if (i<r-1 and j<c-1):
    # _
    # |
    badclause(k,[-badHH[i][j],-badVV[i][j]])
    # |_
    # _|
    badclause(k,[-badHH[i][j],-badVV[i-1][j]])
    # _|
    # _|
    badclause(k,[-badHH[i][j-1],-badVV[i][j]])
    # _|
    # |
    badclause(k,[-badHH[i][j-1],-badVV[i-1][j]])

# print 'badClauses', badClauses

# CNF for tilings with good clauses
for k in range(C):
    for i in range(r):
        for j in range(c):
            # Rule 1 note that the (j<c-1) etc are redundant
            if (not (Cregion[i][j] == 1)):
                clause([-X[i][j],(j<c-1)*HH[k][i][j],(i<r-1)*VV[k][i][j],(j>0)*HH[k][i][j-1],(i>0)*VV[k][i-1][j]])

# Rule 2
if (j<j-1):
    clause([-X[i][j],-HH[k][i][j]])
if (i>i-1):
    clause([-X[i][j],-VV[k][i][j]])
if (j>0):
    clause([-X[i][j],-HH[k][i][j-1]])
if (i>0):
    clause([-X[i][j],-VV[k][i-1][j]])

# Rule 3
if (i<i-1 and j<j-1):
    clause([-X[i][j],-X[i+1][j],-X[i][j+1],-X[i+1][j+1],HH[k][i][j],VV[k][i][j],HH[k][i+1][j],VV[k][i+1][j]])

# Rule 4
if (i<i-1 and j<j-1):
    clause([-HH[k][i][j],-VV[k][i][j]])
if (i>i-1 and j<j-1):
    clause([-HH[k][i][j],-VV[k][i-1][j]])
if (i<i-1 and j>0):
    clause([-HH[k][i][j-1],-VV[k][i][j]])
if (i<i-1 and j>0):
    clause([-HH[k][i][j-1],-HH[k][i][j]])

if (j>j-1 and i<i-1):
    clause([-HH[k][i][j],-VV[k][i][j]])
if (j>j-1 and i<i-1):
    clause([-HH[k][i][j],-VV[k][i-1][j]])
if (i>i-1 and j>j):
    clause([-HH[k][i][j-1],-VV[k][i][j]])
if (i>i-1 and j>j):
    clause([-HH[k][i][j-1],-HH[k][i][j]])
```
if(i<r-1 and i>0):
    #
    # clause([-VV[k][i-1][j],-VV[k][i][j]])

satoutFilename = sys.argv[2][:-4] + 'out.txt'
satinFilename = sys.argv[2]
badsatinFilename = '__tmp.txt'
badsatoutFilename = '__tmpout.txt'

# DO A SINGLE WRITE FOR ALL GOOD CLAUSES
CNFstring = 'p cnf ' + str(nGoodVars) + ' ' + str(len(goodClauses)) + ' 
for _clause in goodClauses:
    for lit in _clause:
        CNFstring = CNFstring + ' ' + str(lit)
CNFstring = CNFstring + ' 0
f = open(satinFilename, 'w')
f.write(CNFstring)
f.close()

# make a single string for each set of bad clauses (to which we later
# append the region R)
badCNFstring = []
for k in range(badC):
    badCNFstring.append('p cnf ' + str(nBadVars) + ' ' + str(len(badClauses[k]) + r*c) + ' 
for _clause in badClauses[k]:
    if(_clause == []):
        quitError('empty clause ' + str(badClauses[k]))
    for lit in _clause:
        badCNFstring[k] = badCNFstring[k] + ' ' + str(lit)
badCNFstring[k] = badCNFstring[k] + ' 0
print "X",X
print "HH",HH
print "VV",VV
print "badHH",badHH
print "badVV",badVV
print "CNFstring",CNFstring
print "badClauses",badClauses
print "badCNFstring",badCNFstring
print "badCNFstring[0]",badCNFstring[0]

# we use this to negate the X_i literals of the region R.
neg = lambda x: -x

numRegions = 0 # count the number of regions we have tried
prevR = []
while(True):
    numRegions += 1
    sp = subprocess.Popen(['./minisat',satinFilename,satoutFilename],
    stdout=subprocess.PIPE)
    sp.wait()
    if(numRegions%100 == 0):
        print 'number of regions checked', numRegions
    g = getSATAssignment(satoutFilename)
    R = g[r*c] # the region output from last minisat of f
    if(prevR == []):
        quitError('error: two regions the same')
    if(numRegions%100 == 0):
        print R
displayRegion(R)
    print 'good configurations'
    for k in range(k):
        displayTiling(g,k)
    prevR = R
A.3  Tatami Maker: a combinatorially rich mechanical game board

The Tatami Maker game board forms a rectilinear grid that enforces the tatami restriction when it is covered by the accompanying tile pieces. Arbitrary rectilinear grids can be created by placing Tatami Maker’s modules alongside each other (see Figure A.1(a)). A simple mechanism is embedded at each grid line intersection, which obstructs the placement of a tile if the other three incident grid squares are covered (see Figure A.1(b)).

A tile piece covers one or two grid squares, and the underside of each of its four corners has the specially shaped foot shown in Figure A.2(b).
(a) Tatami Maker modules are placed alongside each other to create larger grids.

(b) The inside of Tatami Maker’s intersection mechanism. Each arm of the X-shape extends to a different grid square.

Figure A.1: Tatami Maker modules and mechanism.

interact with the mechanism in the game board by pushing an obstructing ball onto an unoccupied grid square, as well as guiding the tile correctly onto the grid.

Each mechanism occupies one grid intersection, which consists of one quadrant from each of four incident grid squares. A cavity in the game board contains a ball which may travel freely to any unoccupied quadrant. If a quadrant is occupied by a tile’s corner, then one of tile’s feet occupies the part of the cavity that is otherwise available to the ball. When three of the quadrants are occupied by tile corners, the ball is forced to occupy the remaining quadrant, thereby preventing a tile corner from being placed here.

A minimal game board module is one grid intersection. As a result, the dimensions of a module are given in terms of its intersections rather than its grid squares. Our prototype’s modules are $4 \times 4$, but they need not be square, or even rectangular.

The main design challenge is ensuring that the ball can be pushed by an incoming foot, unobstructed, to an available quadrant. We label the quadrants Q1, Q2, Q3, and Q4, in counterclockwise order. A critical case occurs when Q2 and Q4 are occupied by tile corners (and feet), and a tile foot placed in Q1 must push the ball to Q3. Intuitively, the ball must disturb the feet in Q2 and Q4 on the way to Q3, otherwise the midpoint of the intersection would accommodate the ball when all four quadrants are occupied. The mechanism is designed, therefore, so that the ball lifts the tiles in Q2 and Q4 as it passes to Q3, and so that it will not become stuck against another part of the mechanism before it arrives in Q3.
The availability of desktop 3D printers has lowered the cost of creating prototypes sufficiently enough that rough estimation and iterative trial and error was the most economical way of solving these design challenges. Tatami Maker was prototyped with a Solidoodle 2 printer; the printer is a 3D CNC machine that extrudes a filament of hot ABS plastic into a $15 \times 15 \times 15$ cm print area to create real-life versions of virtual 3D models. With each iteration of the prototype, changes to the shape of the cavity and the feet of the tile pieces were made to tease out the required behaviour of the mechanism.

Tatami Maker was presented in [9] at Bridges, 2013.
| In | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 2  | -2 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 3  | -3 |  4 |  5 |  6 |  7 |  8 |  9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| 4  | -4 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 5  | -5 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 6  | -6 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 7  | -7 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 8  | -8 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 9  | -9 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 10 | -10|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 11 | -11|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 12 | -12|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 13 | -13|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 14 | -14|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 15 | -15|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 16 | -16|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 17 | -17|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 18 | -18|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 19 | -19|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 20 | -20|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 21 | -21|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 22 | -22|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 23 | -23|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 24 | -24|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 25 | -25|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 26 | -26|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 27 | -27|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 28 | -28|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 29 | -29|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 30 | -30|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 31 | -31|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |

Table A.22: Table of coefficients of $P_n(z)$ for $2 \leq n \leq 20$. It is irreducible for $2 \leq n < 200$. 