Signal Design for Multi-Way Relay Channels

by

Shaham Sharifian
B.Sc., Sharif University of Technology, 2008
M.Sc., Chalmers University of Technology, 2010

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ABSTRACT

Today’s communication systems are in need of spectrally efficient and high throughput techniques more than ever because of high data rate applications and the scarcity and expense of bandwidth. To cope with increased data rate demands, more base stations are needed which is not cost and energy efficient in cellular networks. It has been shown that wireless relay networks can provide higher network throughput and increase power efficiency with low complexity and cost. Furthermore, network resources can be utilized more efficiently by using network coding in relay networks.

A wireless relay network in which multiple nodes exchange information with the help of relay node(s) is called a multi-way relay channel (MWRC). MWRCs are expected to be an integral part of next generation wireless standards. The main focus of this dissertation is the investigation of transmission schemes in an MWRC to improve the throughput and error performance. An MWRC with full data exchange is assumed in which a half-duplex relay station (RS) is the enabler of communication.

One of the challenges with signal demodulation in MWRCs is the existence of ambiguous points in the received constellation. The first part of this dissertation investigates a transmission scheme for full data exchange in MWRC that benefits from these points and improves its throughput by 33% compared to traditional relaying.

Then an MWRC is considered where a RS assists multiple nodes to exchange messages. A different approach is taken to avoid ambiguous points in the superposition of user symbols at the relay. This can be achieved by employing complex field network coding (CFNC) which results in full data exchange in two communication phases. CFNC may lead to small Euclidean distances between constellation points, resulting in poor error performance. To improve this performance, the optimal user precoding values are derived such that the power efficiency of the relay constellation is highest when channel state information is available at the users. The error performance of each user is then analyzed and compared with other relaying schemes.

Finally, focusing on the uplink of multi-way relay systems, the performance of an MWRC is studied in which users can employ arbitrary modulation schemes and the links between the users and the relay have different gains, e.g. Rayleigh fading. Analytical expressions for the exact average pairwise error probability of these MWRCs are derived. The probability density function (PDF) and the mean of the minimum Euclidean distance of the relay constellation are closely approximated, and a tight upper bound on the symbol error probability is developed.
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<th>Definition</th>
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<tr>
<td>AF</td>
<td>Amplify and Forward</td>
</tr>
<tr>
<td>ANC</td>
<td>Analog Network Coding</td>
</tr>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>BC</td>
<td>Broadcast</td>
</tr>
<tr>
<td>BPSK</td>
<td>Binary Phase Shift Keying</td>
</tr>
<tr>
<td>CDF</td>
<td>Cumulative Distribution Function</td>
</tr>
<tr>
<td>CDMA</td>
<td>Code Division Multiple Access</td>
</tr>
<tr>
<td>CF</td>
<td>Compress and Forward</td>
</tr>
<tr>
<td>CFNC</td>
<td>Complex Field Network Coding</td>
</tr>
<tr>
<td>CPF</td>
<td>Compute and Forward</td>
</tr>
<tr>
<td>CRS</td>
<td>Cognitive Radio System</td>
</tr>
<tr>
<td>CSI</td>
<td>Channel State Information</td>
</tr>
<tr>
<td>CSIT</td>
<td>Channel State Information at the Transmitter</td>
</tr>
<tr>
<td>CU</td>
<td>Channel Use</td>
</tr>
<tr>
<td>DF</td>
<td>Decode and Forward</td>
</tr>
<tr>
<td>DNC</td>
<td>Digital Network Coding</td>
</tr>
<tr>
<td>EHF</td>
<td>Extremely High Frequency</td>
</tr>
<tr>
<td>FDF</td>
<td>Functional Decode and Forward</td>
</tr>
<tr>
<td>FFNC</td>
<td>Finite Field Network Coding</td>
</tr>
<tr>
<td>GF</td>
<td>Galois Field</td>
</tr>
<tr>
<td>GFNC</td>
<td>Galois Field Network Coding</td>
</tr>
<tr>
<td>ICT</td>
<td>Information Communications Technology</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
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<td>---------</td>
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<tr>
<td>IFNC</td>
<td>Infinite Field Network Coding</td>
</tr>
<tr>
<td>MAC</td>
<td>Multiple Access</td>
</tr>
<tr>
<td>MAP</td>
<td>Maximum a Posteriori Probability</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multiple Input Multiple Output</td>
</tr>
<tr>
<td>ML</td>
<td>Maximum Likelihood</td>
</tr>
<tr>
<td>MWRC</td>
<td>Multi-Way Relay Channel</td>
</tr>
<tr>
<td>NC</td>
<td>Network Coding</td>
</tr>
<tr>
<td>OWRC</td>
<td>One-Way Relay Channel</td>
</tr>
<tr>
<td>P2P</td>
<td>Peer-to-Peer</td>
</tr>
<tr>
<td>PAM</td>
<td>Pulse Amplitude Modulation</td>
</tr>
<tr>
<td>PANC</td>
<td>Physical-layer Algebraic Network Coding</td>
</tr>
<tr>
<td>PDF</td>
<td>Probability Density Function</td>
</tr>
<tr>
<td>PEP</td>
<td>Pairwise Error Probability</td>
</tr>
<tr>
<td>PNC</td>
<td>Physical-layer Network Coding</td>
</tr>
<tr>
<td>PR</td>
<td>Plain Routing</td>
</tr>
<tr>
<td>PSD</td>
<td>Power Spectral Density</td>
</tr>
<tr>
<td>QAM</td>
<td>Quadrature Amplitude Modulation</td>
</tr>
<tr>
<td>QPSK</td>
<td>Quadrature Phase Shift Keying</td>
</tr>
<tr>
<td>RS</td>
<td>Relay Station</td>
</tr>
<tr>
<td>rv</td>
<td>Random Variable</td>
</tr>
<tr>
<td>SEP</td>
<td>Symbol Error Probability</td>
</tr>
<tr>
<td>SHF</td>
<td>Super High Frequency</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
</tr>
</tbody>
</table>
TDMA  Time Division Multiple Access

TR   Traditional Relaying

TWRC  Two-Way Relay Channel
ACKNOWLEDGEMENTS

I owe many thanks to my supervisor, Prof. T. Aaron Gulliver for his great guidance and vision during my study. He was always encouraging and motivating me and helped a lot in all stages of my Ph.D. program. I really appreciate his availability and helpfulness in discussions and for all the time he took reviewing my publication drafts with great care and also for his constructive and detailed comments. I am sure fulfilling this research project would have been impossible without his support and the fruitful discussions that we had.

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Last, but not least, I thank my wife, Parto, for her patience and wholehearted love and for always being by my side in spite of all the challenges of her own graduate studies, and my parents and brothers (Sharif and Hessam) for their unconditional support and love and teaching me good values in life. Without their support none of my accomplishments would have been achievable.
DEDICATION

In memory of my brother, Hessam.
Chapter 1

Introduction

Modern communication systems are in need of bandwidth-efficient techniques more than ever because of bandwidth-hungry and high data rate applications such as audio and video streaming, multi-player games, multimedia messages, and telephone services. For operators and service providers it is always desirable to have as much bandwidth as desired to provide high data rate services to their customers. However, limited operational range, scarcity, and expensiveness of bandwidth make this difficult to achieve.

With the rapid development and deployment of future wireless generations, stable network resources including reasonable link reliability, power efficient transmission, and ubiquitous coverage for wireless broadband services are required. The transmit power is usually strictly limited to avoid interference between radio signals and to save energy. The received signal power is dependent on different factors such as the carrier frequency and the distance between the source and destination. Due to the open nature of radio connection, the received signal suffers from attenuation due to path loss effects and signal shadowing when obstacles are located between the source and the destination, resulting in smaller coverage. A larger coverage area implies the communication distance between the source and destination can be farther.

One way to deal with the above requirements and use the available spectrum more effectively is to employ signal processing techniques and medium access control protocols such as using high-order modulation schemes, channel coding, transmit diversity, and scheduling [1]. Providing more bandwidth access can also help the demand for high data rates and broadband services. This can be done in different ways. One way is to take advantage of frequency bands which are not currently utilized properly. The frequency bands which are dedicated for mobile applications
are the super high frequency (SHF) band (from 3 GHz to 30 GHz) and extremely high frequency (EHF) band (from 30 GHz to 300 GHz). The current problem in properly making use of these frequency bands is that signals at these frequencies attenuate much faster over distance. This means a lower received signal power at the destination, and therefore a lower communication data rate. Thus, SHF and EHF are currently employed for very short-range communications. There is some ongoing research to employ SHF and EHF bands in cellular systems [2].

Spectrum measurements of municipal areas in different locations around the world have shown that only about 13% of the dedicated bandwidth is utilized and the spectrum is not in use for most (87%) of the time [3, 4]. In order to utilize the idle spectrum, dynamic channel assignment schemes for users were suggested which led to innovative new products and services [5–7]. Later, cognitive radio systems (CRS) based on the idea of dynamic channel assignment, received significant attention from the research community and is still an ongoing research field [8–10].

Another way to cope with the increased demand for dynamic and huge network resources is employing more base stations, but this is not a cost and energy efficient solution. From both the environmental and economic points of view, high power consumption in the network is not desirable. Currently, 2% of the world total carbon emission is from information communications technology (ICT) industries which is 3% of the global energy consumption [11].

An alternative solution being employed in next generation cellular systems is to deploy low-cost relay stations in the cells [12]. The function of the relay nodes is to compensate for the signal attenuation and relay the data signal from the source to the destination. Distributed placement of relays within a cell reduces the propagation losses between the relay and the user nodes, resulting in larger link data rates. The reuse efficiency that comes from multiple simultaneous transmissions within the cell from different relays to users leads to capacity gains. Furthermore, each relay creates additional signal paths which results in spatial diversity. Diversity is desirable as it increases the reliability of the wireless communications system. Since spatial diversity in a relay network is obtained via cooperation between terminals, it is referred to as cooperative diversity [13, 14]. In general, wireless relay networks have been shown to provide diversity gains and higher network throughput, extend coverage area, improve detection reliability and system capacity, and increase power and spectral efficiency with low complexity and cost [14–18].

There are different relaying schemes for relay networks. For example, the relay
may be non-regenerative and simply amplify and forward the received signal, or it may be regenerative and perform decoding and re-encoding of the received signal before transmitting it. The use of relays is predicted for future wireless and mobile broadband radio [19]. In fact, the use of relays has been considered in the IEEE 802.16j standard [20]. The effect of relays on the coverage area in cellular communications has been studied in [21]. It was reported that with relays, there is a 6% and 7% improvement in coverage area in metropolitan and indoor areas, respectively, compared to only deploying base stations, for the same cost. Similar improvements in the downlink capacity can also be achieved by employing relays as reported in [22].

In the realm of cooperative communication, one-way relay channels (OWRCs), where one user transmits data to the other user unidirectionally [14], and two-way relay channels (TWRCs), where two users transmit data to each other bidirectionally, have been studied extensively [23–25].

Although the aforementioned techniques can provide more network resources or utilize them more efficiently, more advanced techniques are needed with the growing need for network resources. One technique which was originally proposed to maximize the throughput of lossless wireline networks in multicast scenarios is network coding (NC) [26]. It has been shown that NC can achieve throughput gains which are not achievable with traditional data forwarding schemes [26–28]. Wireless network resources can also be utilized more efficiently with network coding in conjunction with cooperative communications. Digital network coding (DNC) [29–31], analog network coding (ANC) [32, 33], and physical-layer network coding (PNC) [34–39] in TWRCs have recently attracted significant interest in the wireless communications research community. These network coding protocols are explained in more detail in Chapter 2. It has been shown that applying network coding at the relay can increase the throughput of a TWRC by 100% compared to conventional relaying [34]. Other aspects of TWRCs such as multi-antenna regenerative TWRCs with one bidirectional node pair have also been studied [40–43].

### 1.1 Multi-Way Relaying

Consider a wireless relay network with multiple user nodes in which each user has its own message and wants to decode the messages of the other users. If no direct link is available between these users, they can exchange information with the help of one or more relay nodes. Such a wireless network is called a multi-way relay
channel (MWRC) [44]. Unlike OWRCs where the source nodes are different from the destination nodes, in an MWRC each user node is both a source and destination node. In an MWRC, the relay does not treat the user messages individually. In fact, network coding deals with user messages in a smarter way to increase power or spectral efficiency.

An MWRC has two extreme cases, namely full data exchange, in which every user wants to transmit its message to all other users, and pairwise data exchange, consisting of multiple two-way relay channels [44]. Pairwise data exchange in an MWRC or multiuser two-way relay channels can be achieved by employing two-way relaying with more than one bidirectional user pair where the RS serves users to exchange messages with their pre-assigned partners. This scheme was considered in [45–47] with a multi-antenna regenerative RS, and in [48, 49] with a multi-antenna non-regenerative RS. The separation of bidirectional transmitting pairs was done spatially via beamforming at the RS. The same scheme with a single-antenna regenerative RS, in which the separation of bidirectional transmitting pairs is done via code division multiple access (CDMA) and frequency or time division multiple access (F/TDMA) was considered in [50] and [51], respectively.

Full data exchange in MWRC can be achieved by simultaneous transmission of multiple users or pairwise transmission of users. In pairwise transmission based MWRCs with binary signaling [52, 53], a pair of users form a TWRC and transmit their data simultaneously in each time-slot of the multiple access phase and the relay receives the sum of signals. In the broadcast phase, the relay sends the decoded message and all users receive and store it. At the end of all user pair transmissions, the users retrieve messages from the other users by subtracting their own information. To ensure unique decodability in pairwise MWRC, each user pair needs to have at least one common user with the following and the preceding user pair. Hence, a pairwise MWRC with $K$ users can be considered as $K - 1$ TWRC transmissions with the first $K - 1$ time-slots in multiple access phase and the next $K - 1$ time-slots in broadcast phase. In order to decode every message correctly, each user has to decode all user pair messages correctly. In pairwise MWRC, if a user experiences poor channel conditions, it may lead to incorrect detection for other users. It was shown in [54] that with this approach the error propagation severely degrades the performance with both DF and AF relaying, especially with a large number of users. A joint decoding mechanism based on belief propagation was proposed in [55] which reduces these errors. Full data exchange in MWRC with traditional relaying, or with
multiple bidirectional node pairs employing binary signaling, becomes more spectrally inefficient as the network size increases. As will be discussed in more detail in Chapters 2 and 4, complex field network coding (CFNC) [56] only requires two time-slots to exchange messages between users in an MWRC, and it can employ higher order modulation schemes than binary signaling. To achieve this, CFNC uses a precoding vector to separate the different combinations of user symbols in the signal space, thereby making them distinguishable at the relay.

Different relaying protocols for MWRC have also been considered in the literature which include amplify and forward (AF) [53], decode and forward (DF) [44, 57], compress and forward (CF) [44], compute and forward (CPF) [58, 59], and functional decode and forward (FDF) [52, 60] relaying protocols. In [44], the achievable rate regions for AF, DF, and CF strategies were derived for both high and low SNR regimes in an MWRC with full-duplex user nodes and a relay. The rate region of a three-user FDF MWRC in binary symmetric channels was studied in [61]. Each user retrieves the messages from all other users by receiving functions of message pairs from the relay and decoding them sequentially, which was shown to achieve the common rate capacity upper bound. In [52] it was shown that FDF pairwise MWRC with binary linear codes can theoretically achieve the common rate which is the minimum of the maximum achievable information rates of all users. These relaying protocols (AF, DF, CF, CPF, and FDF) are explained in more detail in Chapter 2. In [53] closed-form formulas are derived for the end-to-end signal to noise ratio (SNR) and its distribution function in pairwise MWRCs with AF relaying. Further, the outage probability of an AF MWRC has been obtained assuming error free successive interference cancellation. Pairing schemes have been proposed in the literature for MWRCs with FDF relaying [52] and AF relaying [53]. The optimal pairing order that maximizes the achievable sum rate of DF MWRCs has also been studied [57].

MWRC is a recent field of research in wireless communications and is in its early stages of development. MWRC applications include teleconferencing between multiple users, multi-player gaming, wireless peer-to-peer (P2P) communications between user nodes, weather station communications via a satellite [61], and information exchange among multiple sensors with a single access point in a wireless sensor network. In all of these cases, each node has a message for, and wants to receive the messages from, all other nodes [62].

Another example of an MWRC given in [62] considers members of an emergency response team, each equipped with a wireless device. The requirement is to transmit
and receive data to and from all other responders at a disaster site (e.g. a site where an earthquake or a volcanic eruption has happened), where both wired and wireless traditional communications infrastructure are inoperable. In this situation, installing a relay station (RS) enables emergency team members to communicate via multi-way relaying. For example, a wireless multi-way communication between emergency response personnel who are located near different stations will result in better coordination between them.

Because of their potential importance, designing MWRCs with a focus on their throughput and performance is considered in this dissertation. A summary of the contributions as well as the organization of the dissertation is provided below.

1.2 Contributions and Dissertation Organization

In Chapter 2 the theoretical background of wireless communications is given. This helps to explain the need for cooperation between users in wireless networks. Then the concept of cooperative communication and different types of cooperative relay networks are described in more detail. Next, a brief review of different relaying protocols and network coding techniques employed in MWRCs is provided.

The dissertation contributions are focused on proposing transmission scenarios for MWRCs and improving their throughput and performance along with analyzing their error rate for different channels including AWGN and Rayleigh fading. Some ideas for future research directions are also suggested.

One of the challenges with signal demodulation in MWRCs is the existence of ambiguous point(s) in the received constellation at the relay. This means the superposition of two or more transmitted symbols by users can result in the same constellation point at the relay which makes it impossible for the relay to correctly demodulate the received signal.

In Chapter 3, a new transmission scheme for full data exchange in an MWRC with binary phase shift keying (BPSK) modulation is designed to benefit from ambiguous point(s). The proposed transmission algorithm exploits the common knowledge available to all users and provides a throughput gain over plain routing. This shows that physical-layer network coding can also be beneficial in systems with more than two user nodes. Besides having low complexity, the proposed algorithm can easily be scaled to higher number of users. It can also be employed with quadrature phase shift keying (QPSK) modulation, which provides the same throughput gain as BPSK.
Furthermore, the power efficiency and error performance of the proposed scheme are compared with plain routing to show there is a trade-off between power and spectral efficiencies in selecting a relaying scheme.

In Chapter 4, a multi-way relay channel in which clusters of users perform full data exchange has been considered. Here, a different approach is taken to avoid the ambiguous point(s) in the superposition of symbols from different users of a cluster which are simultaneously received at the relay. This makes it possible to uniquely recover each user data in only two time-slots for each cluster, and can be achieved by assigning suitable precoding values to the user symbols and performing network coding over the complex field, which is known as complex field network coding. In CFNC, some precoding values may lead to small Euclidean distances between constellation points at the relay, resulting in poor performance. The users are assumed to employ quadrature amplitude modulation (QAM) and precoding values are designed such that reception of a rectangular QAM constellation is ensured at the relay. Then a general problem in which each user employs an arbitrary PAM or rectangular QAM constellation is considered and the optimal precoding values are derived such that the power efficiency of the relay QAM constellation is highest. The proposed design has the flexibility to accommodate users and allow them to join or leave the network as necessary. Moreover, the error performance of each user in such an MWRC is analyzed and compared with the proposed PNC relaying and plain routing. Then it is shown that by employing CFNC and exploiting user self-information, the size of the constellation for the downlink broadcast transmission can be decreased which leads to both downlink and uplink error performance improvement.

Focusing on the uplink of multi-way relay systems, the performance of an MWRC, where the links between the users and relay have different SNRs, is studied in Chapter 5. For instance, in a Rayleigh fading environment, users most likely have channels with different gains resulting in different received SNRs. Analytical expressions for exact average pairwise error probability of such MWRCs are derived. Then the probability density function and mean of the minimum Euclidean distance of the relay constellation are closely approximated. Furthermore, it is shown through analysis and simulations that this approximation is always a lower bound to the actual value. By exploiting the approximation of minimum Euclidean distance, a tight upper bound on the symbol error probability is developed using a nearest-neighbor approximation.

The conclusion of this dissertation along with some future research directions are presented in Chapter 6.
1.3 List of Publications

The results in this dissertation have been presented in the following publications and are listed here for ease of reference.

Published results of Chapter 3:


Published results of Chapter 4:


Published results of Chapter 5:

Chapter 2

Background

2.1 Fading in Wireless Communications

There are multiple key distinctions between wireless and wireline communications. Wireless channels are time-varying due to the relative motion of the transmitter and receiver, and the surrounding environment. Furthermore, in wireless communications the interference of different signals being transmitted over a wireless medium can degrade the performance. Another distinctive feature of wireless channels is the variation in the received signal strength due to channel impairments which is termed as fading. The radio channel responsible for creating the fading is known as fading channel. The variations in signal strength due to fading happen in both long term average value and short term fluctuations. Therefore, fading can be classified into two categories, namely large scale fading and small scale fading which are both explained below. These differences make the wireless channels more complex and highly unpredictable in nature, both in time and frequency domains, and limit their performance.

2.1.1 Large Scale Fading

Large scale fading means that the average received signal power varies over a relatively large distance compared to the wavelength of the signal. This kind of fading is the result of signal power attenuation over distance due to path loss and shadowing.

Path loss is caused by dissipation of the power radiated by the transmitter. Path loss models generally assume that the loss is the same at a given transmit-receive distance. If a long period of time is considered, the average received signal power will
be a deterministic function of the distance. Variation due to path loss occurs over very large distances (100-1000 meters) [63]. The expression for the average path loss at a distance $d$ meters from the transmitter is given by

$$L = \frac{P_t}{\bar{P}_r} = \frac{d^\gamma}{K},$$

where $P_t$ is transmitted signal power, $\bar{P}_r$ is average received signal power, $\gamma$ is the path loss exponent, and $K$ is a unitless constant which depends on the antenna characteristics and the average channel attenuation. The path loss exponent is typically between two and six, depending on the environment. The constant $K$ is proportional to the transmit and receive antenna gains and square of the operating wavelength. From Eq. (2.1), the average received signal power, $\bar{P}_r$, can be expressed in decibels (dB) as

$$\bar{P}_r \text{ dBm} = P_t \text{ dBm} + K \text{ dB} - 10\gamma \log_{10}(d).$$

(2.2)

Shadowing is caused by large obstacles (such as hills or buildings) between the transmitter and receiver that attenuate signal power through absorption, reflection, diffraction, and scattering. Variation due to shadowing occurs over distances proportional to the length of the obstructing object (10-100 meters in outdoor environments and less in indoor environments) [63]. By averaging over a smaller time window, the received signal strength will be a random variable given by

$$P_r \text{ dBm} = \bar{P}_r \text{ dBm} + X_\sigma \text{ dB},$$

(2.3)

where $\bar{P}_r$ is from Eq. (2.2) and $X_\sigma$ is a zero-mean random variable. In the most common model for combined path loss and shadowing, $X_\sigma$ is considered as a zero-mean Gaussian random variable with variance $\sigma^2$. This shadowing model is known as log-normal shadowing.

From the discussion above, the parameters required to describe the large scale fading at distance $d$ from the transmitter include the constant $K$, the path loss exponent $\gamma$, and the variance $\sigma^2$.

### 2.1.2 Small Scale Fading

Small scale fading is due to the constructive and destructive interference of different multipath components between the transmitter and receiver [63]. These multipath
components are created due to reflection, refraction, and scattering of radio waves from various objects and surfaces. Here, the variation in the received signal power occurs over small distances, hence the name small scale fading. The rapidly varying received signal power in such wireless channels needs to be characterized using statistical models.

Fading channels are linear time-varying systems where the time variations are random [63]. Therefore, they can be modeled as a time-varying impulse response $h(\tau, t)$. If a single pulse is transmitted over a multipath channel, the received signal will appear as a pulse train, with each pulse corresponding to a distinct multipath component. The delay spread of a multipath channel, $T_m$, is defined as the time delay between the arrival of the first (shortest path) and the last (longest path) received signal components. Additionally, the coherence bandwidth of a multipath channel, $B_c$, is defined as the range of non-negative values of the Fourier transform of the multipath intensity profile and can be roughly approximated as $B_c \sim 1/T_m$.

If the delay spread of the channel is relatively large compared to the symbol duration in time (i.e., the inverse of the signal bandwidth), then the time spreading of the received signal can be significant which will lead to substantial signal distortion. In this case, the channel is called frequency-selective fading. However, if the delay spread of the channel, $T_m$, is significantly smaller compared to the inverse of the signal bandwidth $W$, then the coherence bandwidth of the channel, $B_c$, will be significantly larger than $W$. This implies that all the frequency components of the transmitted signal see the same effective channel. In this case, the channel is called narrowband fading or flat fading and there is no intersymbol interference between consecutive transmitted symbols.

In the medium having lots of scatters, according to the central limit theorem, the single pulse that represents the frequency flat fading channel can be modeled as a complex Gaussian random process $h$. In the absence of any dominant propagation along a line-of-sight between the transmitter and receiver, $h$ is a zero-mean complex Gaussian random process. Therefore, the absolute value of the channel gain, $|h|$, has an exponential distribution and the channel phase, $\angle h$, is uniformly distributed over $[0, 2\pi]$. This type of channel is called Rayleigh fading channel which has a Rayleigh distribution. In this dissertation, Rayleigh fading channel model is considered in Chapter 5.


2.2 Cooperative Relay Networks

In cooperative relay networks, multiple distributed user nodes cooperate with one another by relaying signals intended for other nodes. A node is considered as a relay whenever it acts as an intermediate repeater between the transmitter and receiver. In this way, each user node sacrifices some of its resources (e.g., bandwidth and battery power) on behalf of the other nodes. However, such cooperation leads to better overall quality of service for the whole network. Specifically, such cooperation enables communication between users that are far apart, which is not possible with the traditional single-hop communication. Cooperative relay networks can be divided into three subclasses depending on the direction of communication and number of user nodes involved in data exchange between users. These three subclasses are briefly summarized in the following subsections.

2.2.1 One-Way Relay Channel (OWRC)

In one-way relay channels, the source nodes are different from the destination node(s) and the communication between the source and destination nodes is performed with the help of one or more relay stations. In this setup, the data flow is in one direction from the sources to the destinations. Such one-way communication mostly takes place in broadcasting scenarios, for example, broadcast radio and television.

2.2.2 Two-Way Relay Channel (TWRC)

Communication usually involves a bidirectional exchange of information between the communicating nodes. In TWRC, two user nodes want to exchange their data through a relay. Hence, the communicating nodes serve as both source and destination. TWRC is conventionally performed in two separate one-way communications as shown in Fig. 2.1a. In the first one-way communication, user 1 sends its data to the relay and the relay forwards it to user 2. In the second one-way communication, user 2 sends its data to the relay and the relay forwards it to user 1. In this way, 4 time-slots are required in total for the two users to share data which gives the throughput of 1/4 symbol per user per channel use. Such relaying is called one-way relaying for bidirectional communication, uncoded bidirectional relaying, or simply plain routing.

Two-way relaying can be performed more spectrally efficient by using network-coded bidirectional relaying as shown in Fig. 2.1b. Here, the communicating users
send their data simultaneously to the relay in the first time-slot. In the second time-slot, the relay forwards the sum or a function or combination of both users data to them. Then each user extracts the other user data by subtracting self-information from the received signal. Hence, the total number of time-slots for the two users to share data is only two, which gives the improved throughput of $\frac{1}{2}$ symbol per user per channel use.

### 2.2.3 Multi-Way Relay Channel (MWRC)

The TWRC allows mutual data exchange among only two users. However, certain practical applications such as multimedia teleconferencing via a satellite or mutual data exchange between sensor nodes and the data fusion center in wireless sensor networks require data exchange among more than just two user nodes. A multi-way relay channel is a wireless relay network consisting of multiple interfering clusters of users where there is no direct link between them. The users within the same cluster communicate simultaneously and wish to exchange messages among themselves with the help of one or more relay nodes. MWRC has two extreme cases, namely full data exchange, in which every user wants to receive messages from all other users, and pairwise data exchange, consisting of multiple two-way relay channels.

The message exchange in MWRC is accomplished in two phases, the multiple access (MAC) phase, and the broadcast (BC) phase. Each phase can take one or
multiple time-slots depending on the transmission scheme. In the MAC phase, users transmit their messages to the relay in a pairwise or non-pairwise manner. The relay receives the sum of the signals and processes it based on different network coding schemes and relaying protocols described in the next section. In the BC phase, the relay sends the processed data to all users. A general model of an MWRC is shown in Fig. 2.2.

OWRCs have already been included in LTE-A standard and TWRCs are being studied for relay-based IMT-A systems [64]. Thus, MWRCs are also expected to be integral parts of the next generation wireless standards.

2.2.4 Synchronization in Relay Networks

A fundamental issue of relevance to many communication systems, not just relay networks, is synchronization of different transmitters at a common receiver. As with most of the cooperative relay networks, perfect synchronization between user nodes is assumed. This means that user packets arrive at the relay with the packet boundary and symbol boundary aligned. Furthermore, the frequencies used by the users are the same and their relative phase offset is zero. Packet alignment is a medium access
control layer scheduling issue and it involves a longer time scale compared to symbol synchronization. Symbol synchronization is at a finer time scale and is, therefore, more challenging. Even if the transmissions of user packets are synchronized, their symbol boundaries when arriving at the relay may be unaligned. This means that a symbol from one user may overlap with two symbols from another user.

There have been studies on how to align symbols of different transmitters at a common receiver [65]. Time synchronization in cooperative relay networks can be achieved via techniques originally developed for MIMO systems in [66, Ch. 11] and [67, Ch. 6]. To deal with asynchronous systems, a framework for decoding at the receiver based on belief propagation has been developed [68, 69]. It has often been thought that strict synchronization is needed for network coding in relay networks. However, the results in [68] indicate that asynchronies are not always bad. For example, in an un-channel-coded network coding, phase asynchrony usually leads to a performance penalty, yet in channel-coded network coding, phase asynchrony may result in a performance reward rather than a performance penalty. This suggests that the penalty due to asynchrony can be nullified to a large extent with the decoding method.

As synchronization is not the main issue of this dissertation, it is assumed that perfect network synchronization is achievable, which is a common assumption made in the literature.

2.3 Relaying Protocols

A relay station (RS) is a node which assists the transmission of data between other nodes in the network. It may be a dedicated RS, whose purpose is solely to forward data for other nodes, or a cooperative relay which assists other nodes when it does not have packets for transmission in its own queue. Like user terminals, relays are not connected to the wireline network through a back-haul connection, but have to rely on wireless transmission to communicate to the base station. For different usage scenarios, relay stations can be classified into fixed and mobile RS. In most cases, relay stations are deployed as fixed entities and are owned by infrastructure providers. Mobile RS can be deployed by mounting them on a mobile vehicle such as a bus or a train. With mobile relay stations, the wireless channels to the users will be more challenging as the channel gains may vary in shorter time durations and more frequent channel state information is required. Here, different relaying protocols are
briefly described. These relaying protocols are also illustrated in Fig. 2.3 [70].

(a) AF relaying. Here, $\alpha$ is the amplification factor.

(b) DF relaying. Here, $\oplus$ is XOR operation.

(c) CF relaying. Here, $Q$ is quantization operation.

(d) FDF relaying. Here, $f(.,.)$ denotes a function.

(e) CPF relaying. Here, $a$ and $b$ are integer coefficients.

Figure 2.3: Different Relaying Protocols.
2.3.1 Amplify and Forward (AF) Relaying

AF relays are referred to as repeaters, analog relays, or non-regenerative relays. In AF relaying, the signal received at the relay is weighted by a complex coefficient, which adjusts the amplitude and phase, and then broadcasted to all the users. The amplification is done such that the transmit power of the relay does not exceed a certain amount. This relaying protocol is attractive because it is simple in terms of relaying complexity. However, its drawback is in noise amplification due to the amplification of the received signal at the AF relays. AF relays can be integrated into the existing cellular networks as they are transparent to both base station and users [71]. This means both base station and users are unaware of the AF relays.

2.3.2 Decode and Forward (DF) Relaying

Another relaying protocol supported by the LTE standard is decode and forward (DF) relaying. DF relays are referred to as digital relays or regenerative relays. DF relays are more complex than AF relays as they decode and re-encode the messages contained in the incoming signals before broadcasting it to all users, so more processing of the received signals and larger delays are involved. The advantage of this relaying method is that the noise does not get amplified at the relay, however, it suffers from error propagation. Two main types of DF relays are considered for the LTE-Advanced standard [72].

2.3.3 Compress and Forward (CF) Relaying

In this protocol, the relay first quantizes the received signal in the uplink. Then, without attempting to decode user messages, the relay applies a lossy or lossless source coding scheme on the quantized signal and compresses it. Then it broadcasts the encoded signal to all users and they decode other user data from the noisy compressed observation. [44].

2.3.4 Functional Decode and Forward (FDF) Relaying

In functional decode and forward protocol, the relay decodes a function of the user messages instead of decoding the messages individually [52]. FDF is based on the use of nested lattice codes at the users to encode their messages. In the BC phase, the relay broadcasts the function back to all users. Following a certain order of decoding,
the users will get all other users data. For example, in a pairwise MWRC with FDF, there are \( N - 1 \) MAC and BC time-slots in the uplink and downlink phases, respectively. In the MAC time-slots, users encode their data with a class of lattice codes and a pair of users transmit their coded packets to the relay in each time-slot. The relay then decodes the sum of the users data instead of decoding them separately, and broadcasts them to the users in \( N - 1 \) downlink time-slots. Since each user has received \( N - 1 \) independent linear combinations of the other users data, it can decode all of them.

2.3.5 Compute and Forward (CPF) Relaying

In compute and forward protocol, the relay computes linear equations of the user messages according to their observed channel coefficients. Then the relay forwards these equations to the users and each user can decode the messages of the other users upon receiving a sufficient number of equations \([58, 73]\). CPF can be categorized as a special case of FDF as it can be considered a lattice-based relaying. In fact, when FDF is used for data transmission from multiple sources to a destination in a cooperative network, it is often called CPF. In general, FDF and CPF schemes beyond TWRC require a high-dimensional lattice construction which may not be practical.

2.4 Network Coding Techniques in Relay Networks

It is shown in \([26]\) that network coding (NC) in general is an operation over a finite field where the relay generates a network-coded data symbol which is a function of data symbols sent from users. Network coding operations can be either in finite field or infinite field. In finite field network coding (FFNC), the NC operations are done over a Galois field (GF) at the relay (e.g., exclusive OR mapping which is a GF(2) addition). This way of network coding is sometimes referred as Galois-field network coding (GFNC). In infinite field network coding (IFNC), the NC operations are done over an infinite field (e.g., over the real or complex number set). Regardless of whether finite or infinite field network coding is adopted, the key requirement in NC is that users must be able to uniquely extract the information of the other end users from the mapped signal transmitted by the relay. In the following subsections, different famous network coding techniques are introduced and their differences are explained.
2.4.1 Digital Network Coding (DNC)

In digital network coding protocol, the users transmit their messages in separate time-slots to the relay. The relay first decodes and combines the received packets using bitwise XOR operation or any other combining method and then broadcasts the combined messages to the users. Therefore, DNC is a bit-level operation which is done over a finite field. In order to extract the other users messages, the users perform XOR operation between their own messages and the received message from the relay. As a result, in order to exchange messages between two users, three time-slots are required with digital network coding. To perform DNC at the relay node, decode and forward (DF) relays are required. DNC does not allow user transmissions to interfere with each other. This makes it possible to have interaction between channel coding employed by the users and network coding, and hence efficient network coding schemes and decoding algorithms can be designed at the base station.

2.4.2 Analog Network Coding (ANC)

In analog network coding protocol, the users transmit their messages to the relay at the same time in the first time-slot and the relay receives their sum by exploiting the additive nature of physical electromagnetic waves. This protocol allows the analog transmissions sent by users to interfere. Here, network coding is created on the electromagnetic waves in the air rather than in the baseband at bit-level, hence ANC is a signal level operation over the infinite field of real numbers and synchronization between the user signals is required. For the downlink transmission, the AF relaying protocol is used at the relay station which means the combined signal is amplified and sent to users in the second time-slot. Therefore, in ANC two time-slots are required as compared to three time-slots in DNC. The disadvantage of ANC is that the relay does not remove receiver noise, and the noise is amplified and forwarded along with the signals to the end nodes. As a result, its fundamental performance is not as good as schemes in which the relay tries to clean up the noise.

2.4.3 Physical-layer Network Coding (PNC)

Under the assumption that bit-level synchronization is achieved, physical-layer network coding was introduced for two-way relay channels. Same as ANC, in physical-layer network coding protocol, the users transmit their messages simultaneously to
the relay in the first time-slot and the relay receives their sum. Here, the DF relaying protocol is used and the relay maps the superposition of electromagnetic signals to simple Galois field GF($2^n$) additions of digital bit streams recognizable by users. This mapping represents the joint symbols of the two combined signals and does not mean that each user is decoded separately. Therefore, PNC is a bit-level operation over finite field in which the frames of the sources are operated in a Galois field manner at the relay. For the downlink transmission, the relay broadcasts the mapped signal to the users in the second time-slot and the users extract the messages of the other users from the received signal by using self-information. Therefore, two time-slots are required with PNC in order to exchange messages between two users. PNC was inspired by the observation that it is unnecessary for the relay node to know the exact source information. The main difference between PNC and DNC is whether the network coding operation occurs at the physical layer or at a higher layer. PNC is the coding operation which directly transforms the received base band signal to a network-coded symbol for relay without separate reception of user signals. The advantage of this method is that the noise does not get amplified at the relay, however, it suffers from error propagation and the ambiguity between symbols especially for higher order modulation. Furthermore, similar to ANC, synchronization between the user signals is required.

### 2.4.4 Complex Field Network Coding (CFNC)

Complex field network coding is built on the principle of physical-layer network coding by substituting Galois field with the classical complex field approach of symbol constellations. CFNC requires two time-slots to exchange messages between users. The relay can estimate individual source symbols from the superimposed signal and combines them over complex field rather than Galois field before forwarding. Therefore, different from network coding over the Galois field, where wireless throughput may decrease as the number of sources increases, CFNC entails symbol-level operations at the physical layer and can always achieve throughput as high as 1/2 symbol per user per channel use. In CFNC, the source frames are superimposed in a symbol-wise manner at the relay to generate a network-coded frame. Here, the symbol-wise manner means that the $i$th symbol in the network-coded frame is generated by superimposing the $i$th symbols of all source frames, and it is independent on the other symbols of the sources.
Figure 2.4: Scheduling in traditional relaying and relays with network coding.

For GFNC (including PNC) to be able to match CFNC throughput, it should be able to resolve all the ambiguities in superposition of symbols from different sources received simultaneously at the relay and uniquely recover each user data by employing linear function of user messages (bits) over finite fields (e.g. XOR) which is generally impossible [56, 74]. The only setup that PNC can reach the throughput of CFNC is where \( N = 2 \) sources are considered and both of them act also as destinations but a separate destination is absent. In the presence of a separate destination, CFNC will still have higher throughput even with \( N = 2 \) sources as shown in Fig. 2.4 [56].

The use of nested lattice code or finite ring rather than finite field in network coding is also considered in [37, 74]. This generalized concept of NC is named physical-layer algebraic network coding (PANC) in some literature. The drawback with this approach is that the complexity of decoding lattice codes especially with a large alphabet cardinality is very high.
Chapter 3

Multi-Way Relay Channels with Physical-Layer Network Coding

As mentioned in Chapter 1, with the emergence of bandwidth-hungry applications, such as video streaming, increasing the throughput of wireless networks has become a serious challenge. Network coding, a technique that was originally proposed to maximize the throughput of lossless wireline networks in multicast scenarios [26], has been successfully applied to wireless relay networks in [30] and [35]. Furthermore, to simultaneously exploit the broadcast nature of the wireless environment and the superposition of electromagnetic waves, analog network coding (ANC) and physical-layer network coding (PNC) have been developed in [32] and [34], respectively. PNC was originally considered for TWRC where two user nodes communicate with the aid of an intermediate relay. It has been shown that applying this technique can increase the throughput by 100%. Significant research on PNC has been focused on TWRC. More general scenarios, in particular the case where multiple nodes broadcast packets through a single relay, were investigated in [75] and [56]. These results provide insight into the difficult and open problem of multi-node network coding. For the multi-way relay channel (MWRC) considered in [75], where $N$ user nodes are unable to hear each other and exchange data only through a relay, the throughput of plain routing (PR), conventional network coding and PNC is shown to be $\frac{1}{2N}$, $\frac{1}{2N-1}$, and $\frac{1}{2N-2}$ symbol per user node per channel use (sym/U/CU), respectively. It was concluded that as the number of user nodes increases, the performance gains of PNC over plain routing diminish.

In this chapter, an MWRC in which users want to broadcast their packets through
a relay has been considered. A decode and forward relaying scheme is proposed which works on a symbol-by-symbol basis. For every received superposition of symbols at the relay, a hard-decision is made. This method can increase the throughput of a multi-way relay channel with full data exchange to at least $\frac{1}{1.5N}$ sym/U/CU without requiring user nodes to overhear the transmissions of other nodes. Thus, for any number of nodes using binary signaling, a throughput gain of at least 33% is achieved over plain routing and conventional network coding. It is straightforward to show that the same throughput gain can be achieved when quadrature phase-shift keying (QPSK) is employed by the users. The proposed approach is not applicable to higher order modulation because of the binary nature of the protocol.

The rest of this chapter is organized as follows. In Section 3.1, the network model and notation are introduced. Then in Section 3.2, the algorithm is described and the throughput of the system analyzed. In Section 3.3, the performance of the proposed algorithm is analyzed, the average total energy consumption of the users and the power efficiency of the system are investigated and compared with the plain routing scheme. Finally, the chapter is summarized in Section 3.4. The results of this chapter are published in [76] and [77].
3.1 The Network Model

As shown in Fig. 3.1, an MWRC with $N$ user nodes, $U_1, U_2, \ldots, U_N$, and one relay is considered. Information theoretic aspects of this model is studied in [44]. Full data exchange in which every user node wants to receive the messages of all other user nodes is adopted. As in [44], no direct link between any two user nodes is assumed, so the relay is the enabler of communications. The relay uses omnidirectional transmissions to broadcast information back to the user nodes. Furthermore, the transmissions are half-duplex, i.e., communication cannot occur simultaneously in both directions. As with all cooperative relay networks, time synchronization is required as discussed in Section 2.2.4. Throughout the dissertation perfect synchronization assumption is held.

3.2 Algorithm Description and Throughput Analysis

With plain routing, $2N$ channel uses (CUs) are required for full data exchange between $N$ user nodes. In this case, the throughput is $\frac{1}{2N}$ sym/U/CU. Here we propose a network coding scheme that improves the throughput by at least 33% and achieves a throughput of $\frac{1}{1.5N}$ sym/U/CU if binary signaling (BPSK with symbols ’-1’ and ‘1’) is used. The transmission scheme consists of three main steps:

**Step 1:** This phase is a multiple access (MAC) phase in which all user nodes transmit their BPSK symbols to the relay in the same time-slot. Due to the network coding operation that naturally occurs in the air, the relay receives the superimposed electromagnetic waves, i.e., the sum of the symbols.

**Step 2:** This phase is a broadcast (BC) phase in which the relay transmits the received sum back to the user nodes. At this stage each user node will know the exact number of users that have sent ‘1’, and thus also the number that have sent ‘-1’.

**Step 3:** By exploiting this common information (i.e., the received sum signal from the relay), only some of the user nodes, called minority nodes, send their symbols to the relay for broadcasting. The goal of this step is to identify these minority nodes to all user nodes. This is accomplished by a divide-and-conquer approach in which user nodes are successively divided into smaller groups over a number of iterations. The details of this step are later illustrated with some examples.
The transmission procedure described above does not need any pre-setup and the users can follow the procedure only by knowing their user number and self-information. The key to this transmission scheme is the two specific transmissions used in the beginning (Steps 1 and 2 above). The information provided in these two steps is exploited to reduce the total number of channel uses compared to other network coding schemes. Here it is worth noting that for the case \( N = 2 \), the transmission is complete after Step 2 by using self-information as in [34].

Since binary signaling is considered for each of the \( N \) user nodes, \( N + 1 \) different sum values can be received by the relay in Step 1. If the number of users sending ‘1’ (‘-1’) is less than the number of users sending ‘-1’ (‘1’), those users are said to be ‘in minority’. If the number of users sending ‘1’ and ‘-1’ are the same, those users sending ‘-1’ are chosen to be in minority. By the end of Steps 1 and 2, each user node has the following information:

- whether it is a minority node or not,
- the number of minority nodes.

In Step 3, the objective is to identify the minority nodes, thus making available the symbol of each user to every other user. To achieve this, the users are divided into two approximately equal groups if there is any minority node among them. The splitting into two groups is repeated for the newly created groups as the algorithm proceeds. At each step, if there are \( M \) user nodes in a group, the two groups are formed as \( G_1 = \{U_1, \ldots, U_{\lfloor M/2 \rfloor}\} \) and \( G_2 = \{U_{\lfloor M/2 \rfloor + 1}, \ldots, U_M\} \). Then the minority nodes in \( G_1 \) transmit symbol ‘1’ and the minority nodes in \( G_2 \) transmit symbol ‘-1’ to the relay simultaneously, regardless of what their original symbol is. The relay then broadcasts the sum back. In this manner, the number of minority nodes in each group is known. By successively repeating this procedure, the minority nodes can be identified. This method is illustrated below for two-, three-, and four-node cases. These cases will serve as the basis for the general throughput analysis for \( N \)-node case.

### 3.2.1 Two User Nodes

With two user nodes, after the first two transmissions in Steps 1 and 2, two cases are possible. These two cases and their corresponding probabilities and the number of required transmissions in Step 3 for each case are shown in Fig. 3.2. In one
case, if both users had sent the same symbols, there are no minority nodes, and the communication is complete, i.e., both users know the information symbol of the other user and Step 3 is not required. This is shown in Fig. 3.2 (a). The case of two user nodes having different symbols is shown in Fig. 3.2 (b) with the minority node colored. To identify the minority node in this case, the two user nodes are grouped into \( G_1 = \{ U_1 \} \) and \( G_2 = \{ U_2 \} \). If the minority node is in \( G_1 \), it sends ‘1’ and if it is in \( G_2 \) it sends ‘-1’ to the relay. The relay broadcasts this information to both user nodes. Thus in this case two transmissions are needed in Step 3 to identify the minority node.

Table 3.1 shows the transmission cases for two user nodes. The cases in which the received sum at the relay in Step 1 is -2 or 2 have no minority node, and the probability of this occurring is \( \left( \begin{array}{c} 2 \\ 0 \end{array} \right) / 2^2 \). The case where the received sum at the relay in Step 1 is 0 has one minority node, and the corresponding probability of occurrence is \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) / 2^2 \). Including the two transmissions needed in Steps 1 and 2, the average number of channel uses is

\[
C(N) = 2 + \frac{1}{4} \left[ \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \times 0 + \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \times 2 + \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \times 0 \right] = 3. \tag{3.1}
\]

Compared to plain routing where 4 channel uses are required, the information exchange is done in 0.75 \( \times \) 4 = 3 channel uses on average, thus giving a 33% increase in throughput.

For the special case of two user nodes, if the self-information was considered as in [34], the information exchange could be completed in just 2 channel uses. The above two-node example is presented for illustration purposes, and more importantly to develop a general solution for an arbitrary number of user nodes.
Table 3.1: Transmission cases for two user nodes

<table>
<thead>
<tr>
<th>Sum</th>
<th>Symbols</th>
<th>Prob.</th>
<th># Channel uses</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>two (-1) and zero (1)</td>
<td>( \binom{2}{0}/2^2 )</td>
<td>2 + 0</td>
</tr>
<tr>
<td>0</td>
<td>one (-1) and one (1)</td>
<td>( \binom{1}{1}/2^2 )</td>
<td>2 + 2</td>
</tr>
<tr>
<td>2</td>
<td>zero (-1) and two (1)</td>
<td>( \binom{2}{0}/2^2 )</td>
<td>2 + 0</td>
</tr>
</tbody>
</table>

3.2.2 Three User Nodes

With three user nodes, as in the previous case, if there are no minority nodes (i.e., all users have transmitted the same symbol), the two channel uses in Steps 1 and 2 are sufficient to complete the information exchange. The only other possibility is having one minority node which is shown in Fig. 3.3 with the minority node colored. To identify the minority node in Step 3, the three user nodes are grouped into \( G_1 = \{U_1\} \) and \( G_2 = \{U_2, U_3\} \). As in the two-node case, if the minority node is in \( G_1 \), (the left side case in Fig. 3.3), it sends symbol ‘1’ to the relay and the relay broadcasts it back. After these two transmissions, the minority node is identified for all users. However, if the minority node is in \( G_2 \), (the middle and right side cases in Fig. 3.3), it sends symbol ‘-1’ to the relay for broadcasting. In this case, the minority node cannot be identified but it is clear to all user nodes that \( G_1 \) does not contain the minority node. Therefore, \( G_2 \) is divided into two groups to identify the minority node. This leads to two more transmissions giving four in total in Step 3.

The probability of having the minority node in \( G_1 \) and \( G_2 \) is \( \frac{1}{3} \) and \( \frac{2}{3} \), respectively. Thus the average number of channel uses in Step 3 is \( \frac{1}{3} \times 2 + \frac{2}{3} \times 4 = 3.33 \). Including the 2 transmissions in Steps 1 and 2, and noting the symmetry of the cases in Table 3.2, the average number of channel uses is

\[
C(N) = 2 + 2 \times \frac{1}{8} \left[ \binom{3}{0} \times 0 + \binom{3}{1} \times 3.33 \right] = 4.5. \tag{3.2}
\]

Thus the information exchange is completed in 4.5 channel uses on average which is 75% of the 6 channel uses required for plain routing. This is a 33% increase in throughput.
3.2.3 Four User Nodes

When four user nodes are communicating and all users transmit the same symbol, the information exchange is complete after only two transmissions in Steps 1 and 2. The other possibilities are having one or two minority nodes. These cases are shown in Fig. 3.4. The user nodes are divided into two groups $G_1 = \{U_1, U_2\}$ and $G_2 = \{U_3, U_4\}$ for Step 3. As indicated previously, the minority nodes in $G_1$ and $G_2$ send symbols ‘1’ and ‘-1’, respectively, and the relay transmits the sum back to all users. After this step, if there is only one minority node (e.g. $U_1$ in Fig. 3.4 (a)), all user nodes acquire knowledge that the minority node should be in group $G_1$. Then the next smaller groups are formed as $G_1 = \{U_1\}$ and $G_2 = \{U_2\}$, and the minority node is identified with two more transmissions. Thus, four transmissions are required in Step 3. When there are two minority nodes both in the same group ($U_1$ and $U_2$ in Fig. 3.4 (b)), two transmissions in Step 3 are adequate to identify both users. If on the other hand the two minority nodes are in different groups ($U_1$ and $U_3$ in Fig. 3.4 (b)), for each group two separate transmissions are required to identify the minority nodes. Therefore, in this case six transmissions are needed in Step 3.

Table 3.2: Transmission cases for three user nodes

<table>
<thead>
<tr>
<th>Sum</th>
<th>Symbols</th>
<th>Prob.</th>
<th># Channel uses</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>three (-1) and zero (1)</td>
<td>$\binom{3}{0}/2^3$</td>
<td>2 + 0</td>
</tr>
<tr>
<td>-1</td>
<td>two (-1) and one (1)</td>
<td>$\binom{3}{1}/2^3$</td>
<td>2 + 3.33</td>
</tr>
<tr>
<td>1</td>
<td>one (-1) and two (1)</td>
<td>$\binom{3}{2}/2^3$</td>
<td>2 + 3.33</td>
</tr>
<tr>
<td>3</td>
<td>zero (-1) and three (1)</td>
<td>$\binom{3}{3}/2^3$</td>
<td>2 + 0</td>
</tr>
</tbody>
</table>
If there are two minority nodes, the probability of having both of them in one group is \( \frac{1}{3} \), and the probability of two minority nodes in different groups is \( \frac{2}{3} \). Consequently, when there are two minority nodes, the average number of channel uses in Step 3 is \( (\frac{1}{3} \times 2) + (\frac{2}{3} \times 6) = 4.66 \), as shown in Table 3.3. Including the two transmissions needed in Steps 1 and 2, and noting the symmetry in Table 3.3, the average total number of channel uses is

\[
C(N) = 2 + 2 \times \frac{1}{16} \left[ \binom{4}{0} \times 0 + \binom{4}{1} \times 4 \right] + \frac{1}{16} \left[ \binom{4}{2} \times 4.66 \right] = 5.75. \tag{3.3}
\]

In comparison to plain routing, where 8 channel uses are required, the information exchange is done in \( 0.719 \times 8 = 5.75 \) channel uses on average, a 39% increase in throughput.
Table 3.3: Transmission cases for four user nodes

<table>
<thead>
<tr>
<th>Sum</th>
<th>Symbols</th>
<th>Prob.</th>
<th># Channel uses</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>four (-1) and zero (1)</td>
<td>(\binom{4}{0}/2^4)</td>
<td>2 + 0</td>
</tr>
<tr>
<td>-2</td>
<td>three (-1) and one (1)</td>
<td>(\binom{4}{1}/2^4)</td>
<td>2 + 4</td>
</tr>
<tr>
<td>0</td>
<td>two (-1) and two (1)</td>
<td>(\binom{4}{2}/2^4)</td>
<td>2 + 4.66</td>
</tr>
<tr>
<td>2</td>
<td>one (-1) and three (1)</td>
<td>(\binom{4}{1}/2^4)</td>
<td>2 + 4</td>
</tr>
<tr>
<td>4</td>
<td>zero (-1) and four (1)</td>
<td>(\binom{4}{0}/2^4)</td>
<td>2 + 0</td>
</tr>
</tbody>
</table>

3.2.4 The General \(N\) User Nodes Case

In the general case of \(N\) user nodes operating in a multi-way relay channel, by induction on \(N\) it can be shown that the required number of channel uses is at most 0.75 times that of plain routing. The throughput gain will therefore not be less than 33%.

The results for \(N = 2\) and \(N = 3\) given previously are used as the base for a proof by induction. In the inductive step, we assume \(N\) user nodes require at most \(0.75 \times 2^N\) channel uses, and then prove that \(N + 2\) user nodes will need at most \(0.75 \times 2(N + 2)\) channel uses, i.e., a throughput gain no worse than 33%.

\textit{Proof.} After Steps 1 and 2, and determining the total number of minority nodes, the \(N + 2\) user nodes are divided into groups of two and \(N\) users. The minority nodes that are placed in the two-node group transmit ‘1’, and the minority nodes that are placed in the \(N\) node group transmit ‘-1’. At this stage, the number of minority nodes in each group is known. If there are no or two minority nodes in the two-node group, they are determined at this stage. If there is one minority node in the two-node group, another two transmissions are needed. Therefore, on average one channel use is required for the two-node group in Step 3. Since we have assumed the \(N\) nodes can complete transmissions in \(0.75 \times 2^N\) channel uses, \(N + 2\) user nodes require \(2 + 1 + 0.75 \times 2N = 0.75 \times 2(N + 2)\) channel uses. Thus, on average, \(N + 2\) user nodes require at most 0.75 of channels uses needed with plain routing.

The average number of channel uses and throughput gain are computed analytically for five to eight user nodes, and the results are given in Table 3.4. Figure 3.5 also shows the channel use ratio of the proposed algorithm to plain routing for up to 40 user nodes. Each point was generated by averaging the number of channel uses over \(6 \times 10^5\) runs. The ratio is always less that 0.75, confirming the result above.
Figure 3.5: The ratio of the number of channel uses with the proposed algorithm to that with plain routing (solid line) is always less than or equal to 0.75 (dashed line).

### 3.3 Performance Analysis and Comparison

In the following subsections, first the error performance of the proposed PNC relaying scheme is analyzed. Then the average total energy consumption of the users for the PNC algorithm is analyzed and compared with plain routing. The power efficiency is a direct indicator of error performance. A system with higher power efficiency shows a better error rate for a given signal to noise ratio (SNR). Hence, the power efficiency and error performance of both PNC and PR relaying schemes are calculated and compared as well.

<table>
<thead>
<tr>
<th>Number of Users</th>
<th>Ratio</th>
<th>Throughput Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.737</td>
<td>35.7%</td>
</tr>
<tr>
<td>6</td>
<td>0.733</td>
<td>36.4%</td>
</tr>
<tr>
<td>7</td>
<td>0.736</td>
<td>35.8%</td>
</tr>
<tr>
<td>8</td>
<td>0.71</td>
<td>40.8%</td>
</tr>
</tbody>
</table>
3.3.1 Error Performance of PNC

In this part the error performance of the proposed PNC relaying scheme is analyzed. The channels between the user nodes and relay are assumed to be additive white Gaussian noise (AWGN) with power spectral density (PSD) $\frac{N_0}{2}$. The channels are assumed to have the same coefficients and are symmetric. If this is not the case, pre-equalization can be performed before transmission by providing channel state information at the transmitter (CSIT) as in [33]. The algorithm has three steps with symbol error probabilities $P_{e1}$, $P_{e2}$ and $P_{e3}$, respectively. For the transmission to be successful, there must be no error in any step. Therefore, the probability that a user receives the symbol of at least one other user node in error is

$$P_{e,PNC} = 1 - (1 - P_{e1})(1 - P_{e2})(1 - P_{e3}). \quad (3.4)$$

For $N$ users, the probability of error in Step 1 (the uplink step), $P_{e1}$, is the error probability of an $M$-ary pulse-amplitude modulation ($M$-PAM) with unequal symbol probabilities where $M = N + 1$. This error probability is always lower than that of $M$-PAM modulation with equiprobable symbols. To prove this, the symbol error probability (SEP) of $M$-PAM modulation with unequal symbol probabilities is derived. The signal space is shown in Fig. 3.6 where $E$ is the BPSK symbol energy and symbol $s_j$ has probability $p_j$ for $j = 1, \ldots, M$.

Applying the maximum a posteriori probability (MAP) rule [78, Ch. 4], the optimal decision region boundaries are

$$\gamma_j = \frac{-N_0}{4\sqrt{E}} \ln \frac{p_{j+1}}{p_j} - (M - 2j)\sqrt{E} \quad 1 \leq j \leq M - 1. \quad (3.5)$$

When the received signal is between $\gamma_{j-1}$ and $\gamma_j$, $j = 1, \ldots, M$, we declare it to be $s_j$, where $\gamma_0 = -\infty$ and $\gamma_M = \infty$. To calculate the SEP, note that there are $M - 2$ inner points and 2 outer points in the constellation. The error probabilities for the outer
and inner points are

\[ P_{e, outer} = \begin{cases} 
  p_1 \left[ \Pr \left( n < - (M-1) \sqrt{E} - \gamma_1 \right) \right] \\
  + p_M \left[ \Pr \left( n > (M-1) \sqrt{E} - \gamma_{M-1} \right) \right] 
\end{cases} 
\]

\[ = p_1 \cdot Q \left( \frac{(M-1) \sqrt{E} + \gamma_1}{\sqrt{\frac{N_0}{2}}} \right) 
+ p_M \cdot Q \left( \frac{(M-1) \sqrt{E} - \gamma_{M-1}}{\sqrt{\frac{N_0}{2}}} \right), \quad (3.6) \]

and

\[ P_{e, inner} = \sum_{j=2}^{M-1} p_j \left[ \Pr \left( n < \gamma_{j-1} - (2j - M - 1)\sqrt{E} \right) \right] 
+ \Pr \left( n > \gamma_j - (2j - M - 1)\sqrt{E} \right) \]

\[ = \sum_{j=2}^{M-1} p_j \left[ Q \left( \frac{-\gamma_{j-1} + (2j - M - 1)\sqrt{E}}{\sqrt{\frac{N_0}{2}}} \right) \right] 
\]

\[ + Q \left( \frac{\gamma_j - (2j - M - 1)\sqrt{E}}{\sqrt{\frac{N_0}{2}}} \right), \quad (3.7) \]

respectively, where \( n \) corresponds to the AWGN with variance \( \frac{N_0}{2} \), and \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp(-t^2/2) dt \). Therefore, the error probability is Step 1 is

\[ P_{e,1} = P_{e, outer} + P_{e, inner}. \quad (3.8) \]

In the proposed algorithm, the probability distribution of the symbols is given by

\[ p_j = \frac{(M-1)_{j-1}}{2^{M-1}} \quad \text{for} \quad j = 1, \ldots, M. \quad (3.9) \]

For Step 2 (the downlink step), since each user receives the same signal as the relay in Step 1, the probability of error \( P_{e,2} \) is the same as in Step 1. Therefore, (3.4) can be approximated as \( P_{e, PNC} \approx 2P_{e,1} + P_{e,3} \) if higher order terms are ignored. In Step 3, only the minority nodes (at most half of the total number of users) will transmit, thus the error probability will be much lower than in the previous steps and can be ignored. This is confirmed analytically for three values of \( N \) as shown in Fig. 3.7.
where the SNR is the ratio of the average bit energy to $N_0$. Consequently, the error performance is well approximated by $P_{e,PNC} \approx 2P_{e1}$ where $P_{e1}$ is given by (3.8). Fig. 3.8 shows this approximation for various numbers of users. For comparison purposes, the SEP of BPSK modulation and some equiprobable PAMs are also shown.

### 3.3.2 Energy Consumption Analysis

As it has been shown, the proposed PNC algorithm provides a throughput of $\frac{1}{1.5N}$ sym/U/CU using binary signaling, which is a 33% increase compared to plain routing which has a throughput of $\frac{1}{2N}$. However, a high throughput is not the only desirable factor for a wireless communications system. From both the environmental and economic perspectives, low power consumption is also desirable.

Plain routing (PR) or traditional relaying (TR) consists of the following two main steps:

**Step 1:** This is the MAC phase (uplink transmissions) in which all user nodes transmit their BPSK symbols to the relay sequentially in $N$ time-slots.

**Step 2:** This is the BC phase (downlink transmission) in which the relay transmits the received BPSK signals back to the user nodes sequentially in $N$ time-slots.
With both PR and PNC relaying schemes it is assumed that users employ BPSK symbols with average symbol energy of $E_s$. Fig. 3.9 shows the constellations employed by the relay in Step 1 and 2 for both PR and PNC. In the signal space with PNC relaying, the probability of symbol $s_j$ being received at the relay is given by (3.9).

The probability of error is a function of the minimum Euclidean distance of the signal constellation. Hence, it is assumed that the constellations employed by the users and relay in both relaying schemes have the same minimum Euclidean distance $d_{\text{min}}$ as shown in Fig. 3.9. With this assumption, the average energy consumption of the users in the uplink and the average energy consumption of the relay in the downlink is the same for both relaying schemes. This is readily verified for plain routing as the average energy consumption of the users in Step 1 (uplink) is $N \times E_s$, which equals the average energy consumption of the relay in Step 2 (downlink). For the proposed PNC relaying, the average energy consumption of the users in Step 1 is
also $N \times E_s$. The average energy consumption of the relay in Step 2 is given by

$$E_{\text{PNC, Step 2}} = \sum_{j=1}^{N+1} p_j s_j^2 = \sum_{j=1}^{N+1} \frac{(j - 1)}{2^N} (2j - N - 2)^2 E_s$$

$$= N E_s. \quad (3.10)$$

Therefore, the energy consumption in Step 3 of PNC relaying can be considered additional energy consumption compared to plain routing.

An upper bound for the average energy consumption of users in Step 3 with PNC relaying can be found by assuming the worst case minority node locations in the subgroups during the divide-and-conquer algorithm. This occurs in the following case. Suppose there are $m$ minority nodes and $N$ user nodes, and suppose the divide-and-conquer algorithm continues until $m$ subgroups are formed. The worst case is when each of these $m$ subgroups contains exactly 1 minority node, as then the maximum number of user transmissions will be required. Then the upper bound on the average

Figure 3.9: The signal constellations received and employed by the relay in Steps 1 and 2 of both relaying schemes.
The average total user energy consumption of users in Step 3 is given by

\[
E_{\text{PNC, Step3, UL}} = \sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} 2 \times \left( \frac{N}{m} \right) \times \left[ m \log_2(m) + (a_2 - m) \log_2 \left( \frac{N}{a_1} \right) + (2m - a_2) \log_2 \left( \frac{N}{a_2} \right) \right] E_s, \tag{3.11}
\]

where \( m \) is the number of minority nodes in the \( N \) user nodes, \( a_1 \) is the largest power of two less than or equal to \( m \), and \( a_2 \) is the smallest power of two greater than \( m \). The term \( m \log_2(m) \) is the number of user transmissions required until \( m \) subgroups are formed in the divide-and-conquer algorithm. These \( m \) subgroups are basically \((a_2 - m)\) and \((2m - a_2)\) subgroups with \( \frac{N}{a_1} \) and \( \frac{N}{a_2} \) user nodes, respectively.

Considering the user transmissions in Step 1 for plain routing and Steps 1 and 3 for PNC relaying, Fig. 3.10 shows the average total user energy consumption. It can be seen that as the number of users increases, the difference in energy consumption between PR and PNC relaying also increases.
3.3.3 Power Efficiency and Performance Comparison

In this section, the power efficiency and symbol error probability (SEP) of the proposed PNC relaying schemes and TR are compared. The power efficiency of the relay constellation is defined as

\[ \eta_p = \frac{d_{\text{min}}^2}{2E_b}, \]  

(3.12)

where \( E_b \) is the average bit energy at the relay constellation. This can be calculated using Fig. 3.9 for PR and PNC relaying. For plain routing, \( \eta_{p, \text{PR}} = 2 \), and for PNC relaying with \( N \) users

\[ \eta_{p, \text{PNC}} = \frac{2 \log_2(N + 1)}{N}, \]  

(3.13)

which shows PR has a higher power efficiency compared to PNC. This is actually reflected in comparing the error performance of the two relaying schemes as well.

As shown in Section 3.3.1 the error performance of proposed PNC relaying is well approximated by \( P_{e, \text{PNC}} \approx 2P_{e_1} \) where \( P_{e_1} \) is given by (3.8).

For \( N \) users with plain routing, there are \( N \) uplink and \( N \) downlink transmissions of BPSK signals. Therefore, the probability that a node receives the symbol of at least one other node in error is given by

\[ P_{e, \text{PR}} = 1 - (1 - P_{e, \text{BPSK}})^N (1 - P_{e, \text{BPSK}})^N \approx 2NP_{e, \text{BPSK}}, \]  

(3.14)

where \( P_{e, \text{BPSK}} \) is the symbol error probability of BPSK in AWGN.

The SEP with both relaying schemes for different numbers of users is given in Fig. 3.11. This shows that choosing between PNC and plain routing is a trade-off between spectral and power efficiencies, since although PNC relaying provides a better throughput than plain routing, it has worse performance.

3.4 Summary

In this chapter, full data exchange in a multi-way relay channel was considered. An algorithm was proposed which provides a throughput of \( \frac{1}{3N} \) sym/U/CU, which is 33% improvement over plain routing. This shows that physical-layer network coding
Figure 3.11: Error performance with PNC and plain routing.

can also be beneficial in systems with more than two user nodes. Besides having low complexity, this algorithm can easily be scaled to higher numbers of user nodes. It can also be employed with QPSK modulation, which also provides a 33% gain. This is achieved by separately (and concurrently) dealing with the inphase and quadrature components of QPSK symbols. For higher order modulation, since it is not possible to define minority nodes as in this chapter, the proposed approach is not directly applicable.

Furthermore, in this chapter, the average total user energy consumption with PNC relaying was studied, and its power efficiency and error performance were compared with plain routing. It was shown that although plain routing has a lower throughput than PNC relaying, there is performance degradation with PNC relaying. Therefore, there is a trade-off between power and spectral efficiencies in selecting a relaying scheme.

The idea of restraining some users from transmitting in order to increase throughput may be applicable to other multi-source scenarios and can be explored further. Moreover, the concept of using common knowledge in relay communications should find applications in other communication systems.
Chapter 4

Complex Field Network Coding in Multi-Way Relay Channels

The wireless medium has two unique features that must be considered when designing communication protocols. First, communications are inherently broadcast, meaning that signals from a transmitter can be heard at multiple users. Second, a user can receive the superposition of signals from multiple transmitters [79]. Although these features can make wireless communications challenging, they can also be exploited. As mentioned in the previous chapter, analog network coding [32] and physical-layer network coding [34] use the concept of network coding [26] in wireless channels to increase the throughput of two-way relay channels. By employing the broadcast and superposition features of the wireless medium, both techniques can increase the throughput of a user in TWRC from $1/4$ to $1/2$ sym/U/CU.

A more general approach taken in relay networks is called complex field network coding (CFNC) which is introduced in Section 2.4. In CFNC the network coding operations are performed in the complex field and it provides a throughput of $1/2$ sym/U/CU in a multi-way relay channel where several users exchange information through a relay. This gain is achieved by using precoding vectors to make the different combinations of user symbols distinguishable at the relay. In this approach, no structure is imposed on the constellation points received at the relay. This may lead to small distances between some adjacent constellation points and hence give a constellation with non-equidistant adjacent points which results in performance degradation. Therefore, the main drawback of this technique is that as the number of users increases (e.g. $N \geq 4$), the performance deteriorates dramatically, thus limiting
its applicability.

This chapter considers CFNC in an MWRC where channel state information are available at the users and they can be fully compensated, hence the channels from users to relays can be modeled as AWGN. The precoding vectors for CFNC are designed such that a rectangular quadrature amplitude modulation (QAM) is received at the relay and the minimum distance between constellation points is preserved. The throughput is shown to be $1/2 \text{ sym}/U/CU$ and the error performance of each user is analyzed. The precoding vectors introduced here have the flexibility to accommodate users with different signal constellations and also allow them to join or leave the network at any time.

The rest of this chapter is organized as follows. Section 4.1 introduces the system model. In Section 4.2, the design of the precoding vectors is first discussed for users employing BPSK. This design is then generalized to higher order modulations. The proposed technique is analyzed in Section 4.3, and some performance results are presented and compared with other relaying schemes. The optimality of the precoding vector is then discussed in Section 4.4, and in Section 4.5 a decoding technique at the relay is considered to further improve the system performance. Finally, the chapter is summarized in Section 4.6. The results of this chapter are published in [80], [81], and [82].

### 4.1 System Model

Consider an MWRC in which communication between users is enabled through a relay node. There are $L$ clusters of users and cluster $l$, for $l = 1, \ldots, L$, contains $k_l$ users as shown in Fig. 4.1. Each user is equipped with one antenna and is interested in the information from all other users in its own cluster, i.e., full data exchange in each cluster is desired. Note that the locations of the users in a cluster is arbitrary and the grouping in Fig. 4.1 is only a possible representation. As in [44], it is assumed that the users cannot overhear transmissions from other users and the relay node is the enabler of the communication. The transmissions are half-duplex, i.e., communication cannot occur simultaneously in both directions. Additionally, user nodes are assumed to send independent messages at a common symbol rate. Similar to all cooperative relay networks, time synchronization is required as discussed in Section 2.2.4. Perfect synchronization is assumed throughout this chapter.
4.2 Full Data Exchange Algorithm

The focus here is on information exchange between users in a single cluster. The number of users collaborating in the cluster needs to be known in advance. Assume there are $K$ users in this cluster, each using an $M$-QAM modulation for transmission. All users transmit their symbols $s_i$, for $1 \leq i \leq K$, simultaneously. Due to the superposition of electromagnetic waves, the relay receives the sum of the transmitted signals. The relay then broadcasts a signal to the users using the decode and forward protocol. To ensure unique decodability of the symbols of every user and achieve a throughput of $1/2 \text{sym}/U/\text{CU}$, i.e., one uplink and one downlink transmission, the relay must be able to distinguish between the $M^K$ possible constellation points. This can be achieved by multiplying the symbol vector $\mathbf{s} = (s_1, \ldots, s_K)^T$ by a suitable precoding vector $\phi^T = (\phi_1, \ldots, \phi_K)$, where the elements of $\mathbf{s}$ are the user symbols. The precoding vector has a major impact on decoding performance at the relay and thus the overall symbol error probability (SEP). One approach in designing the precoding vector is based on algebraic number theory [56]. In this approach, no structure is imposed on the $M^K$ constellation points at the relay. This may lead to a constellation with non-equidistant adjacent points resulting in poor performance. Here, we propose a precoding vector such that reception of a rectangular QAM constellation at the relay is ensured in AWGN channels and the minimum Euclidean distance between constellation points is preserved.
4.2.1 BPSK Modulation

First, the $K$ users are assumed to transmit BPSK symbols. The generalization to $M$-QAM will be presented later. The $2^K$ different combinations of symbols from the $K$ users are placed in the rows of a matrix $A$, resulting in a $2^K \times K$ matrix containing ‘1’s and ‘-1’s. The set of all possible received points at the relay is a $2^K \times 1$ vector $b$ given by

$$b_{(2^K \times 1)} = A_{(2^K \times K)} \phi_{(K \times 1)},$$

(4.1)

where $b$ is predetermined according to the constellation desired at the relay. Equation (4.1) is an overdetermined system of linear equations which may not have a solution in general. Since in the studied case always $\text{rank}(A) = K$, a unique solution exists only when $\text{rank}(A) = \text{rank}([A \ b])$, where $[A \ b]$ is the augmented matrix obtained by appending the columns of $A$ and $b$. This can occur when $b$ represents the points of a rectangular $2^K$-QAM constellation.

The average transmitted power of a rectangular QAM constellation is only slightly greater than that of an optimal constellation, and it is easier to generate and demodulate. The optimal (in terms of performance) 2-D signal constellation obtained from a hexagonal lattice provides an asymptotic gain of only 0.82 dB over a rectangular constellation [78, Ch. 4]. Thus the constellation received at the relay is assumed to be rectangular $M$-QAM with $M = 2^K = 2^{K_1} \times 2^{K_2}$, where $K_1$ and $K_2$ are the numbers of inphase and quadrature components, respectively. If $b$ represents a $2^K$-QAM constellation, it can be shown that a vector $\phi$ always exists that satisfies (4.1).

**Proposition 1.** For $K$ users each employing BPSK, the precoding vector

$$\phi = (1, 2, 2^2, \ldots, 2^{K_1-1}, j, j2, j2^2, \ldots, j2^{K_2-1})^T,$$

satisfies $b = A\phi$ where $K_1 + K_2 = K$ and $b$ represents the points of a rectangular $(2^{K_1} \times 2^{K_2})$-QAM constellation with preserved minimum Euclidean distance.

**Proof.** Since a rectangular QAM constellation can be constructed using two PAM signal sets, it is sufficient to consider the real elements of $\phi$ only. The $M$-PAM signal amplitudes are assumed to take values from $\pm 1, \pm 3, \ldots, \pm(M - 1)$. The proof is by mathematical induction. Consider the base case with 2 users, so that

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^T.$$
and $\phi = (1, 2)^T$, which gives a $2^2$-PAM constellation $b = (-3, -1, 1, 3)^T$. Thus the proposition holds for the base case. In the inductive step we assume the proposition holds for $n$ users with $\phi = (1, \ldots, 2^n-1)^T$, and it gives $b = (L R)^T$, where $L = (-2^n+1, \ldots, -3, -1)$ and $R = (1, \ldots, 2^n-1)$ are the negative and positive parts of $b$, respectively. This means $b$ is a $2^n$-PAM constellation with a distance of 2 between the adjacent points.

Now for $n+1$ users, $2^n$ is added to $\phi$, i.e., $\phi = (1, \ldots, 2^n-1, 2^n)^T$. The newly generated vector $\hat{b}$ can be written as $\hat{b} = (L_{\text{out}}, L_{\text{in}}, R_{\text{in}}, R_{\text{out}})^T$ with

$$
\begin{align*}
R_{\text{out}} &= 2^n \cdot +R \\
R_{\text{in}} &= 2^n \cdot +L \\
L_{\text{in}} &= -2^n \cdot +R \\
L_{\text{out}} &= -2^n \cdot +L
\end{align*}
$$

where $\cdot +$ is element-wise addition. It can be readily verified that $R_{\text{in}} = R$, $L_{\text{in}} = L$, $R_{\text{out}}$ is the shifted version of $R$ by $2^n$, and $L_{\text{out}}$ is the shifted version of $L$ by $-2^n$. Since none of these four parts overlap, $\hat{b}$ is a $2^{n+1}$-PAM constellation with the preserved distance of 2 between its adjacent points.

Consider a $K$-user relay system with each user transmitting information using BPSK, and a precoding vector $\phi$ constructed according to Proposition 1. The signal received by the relay is from a $2^K$-QAM constellation from which the symbols of each user can be derived. The relay broadcasts the decoded constellation point to the users. By decoding the signal sent from the relay, each user can obtain the information of all other users.

### 4.2.2 Higher Order Modulation

In this section, the solution for BPSK user modulation is generalized to $M$-QAM modulation. This can be accomplished by first generalizing BPSK to QPSK and $M$-PAM separately, and then combining the results to give $M$-QAM. Suppose $K$ users employ QPSK modulation to communicate. The relay must then distinguish between $4^K$ different combinations of user symbols. In this case, the number of different signal combinations is equivalent to that for $2^K$ users communicating with BPSK. For these $2K$ users, the matrix $A = (a_1, \ldots, a_{2K})$ has size $2^{2K} \times 2K$ where $a_i$ for $i = 1, \ldots, 2K$ are the columns of $A$, and the precoding vector $\phi = (1, \ldots, 2^{K-1}, j, \ldots, j2^{K-1})^T$ has
size $2K$. Combining pairs of columns of $A$ gives a matrix $A_Q = (a_{1q}, \ldots, a_{Kq})$ where $a_{iq} = a_i + ja_{K+i}$, for $i = 1, \ldots, K$, are the columns of $A_Q$. The rows of $A_Q$ contain all combinations of $K$-user QPSK symbols. The corresponding precoding vector $\phi_Q$ for $K$ users employing QPSK can be obtained by choosing the first $K$ elements of vector $\phi$. Consequently $A_Q \phi_Q$ generates $4^K$ distinguishable points in a square $4^K$-QAM constellation. The following propositions provide the precoding vectors for the cases where users employ $M$-PAM and $M$-QAM modulation.

**Proposition 2.** For $K$ users each employing $M$-PAM,

$$\phi = (1, M, \ldots, M^{K-1}, j, jM, \ldots, jM^{K-1})^T$$

satisfies $b = A\phi$ where $K_1 + K_2 = K$ and $b$ represents the points of a rectangular $(M^{K_1} \times M^{K_2})$-QAM constellation with preserved minimum Euclidean distance.

**Proof.** The proof is by induction. Since a rectangular QAM constellation can be constructed using two PAM signal sets, it is sufficient to consider only the real elements of $\phi$. We begin with the proof for the 2-user $M$-PAM case. With two users, each using $M$-PAM, the precoding vector will be $\phi = (1, M)^T$ and $M^2$ points are generated at the relay in the range $-(M^2 - 1)$ to $(M^2 - 1)$ which has length $2(M^2 - 1)$. We show that employing this vector $\phi$, the $M^2$ points will be distinct and equally spaced by a distance of 2. Consider two arbitrarily distinct points at the relay, $a_1 + Mb_1$ and $a_2 + Mb_2$, where $a_1$, $a_2$, $b_1$ and $b_2$ are $M$-PAM symbols. If these points are equal, then $a_1 + Mb_1 = a_2 + Mb_2$ and rearranging gives $a_1 - a_2 = M(b_2 - b_1)$. Ignoring the trivial solution, since the minimum value of $| M(b_2 - b_1) |$, which is $2M$, is greater than the maximum value of $| a_1 - a_2 |$, which is $2(M - 1)$, equality cannot occur and therefore, the $M^2$ points are distinct. The distance between the two points is $d = | (a_1 - a_2) + M(b_1 - b_2) |$. Clearly the minimum value of $d$ is 2, and with $M^2$ points equally spaced by the minimum distance 2, their range is $2(M^2 - 1)$. This completes the proof for the 2 user case.

As the induction hypothesis, suppose that $N$ users employ $M$-PAM with the real part of the precoding vector given by $\phi = (1, M, \ldots, M^{N-1})^T$. Thus there are $M^N$ distinct equally spaced points with distance 2 in the range $-(M^N - 1)$ to $(M^N - 1)$. Adding the $(N + 1)$th user with coefficient $M^N$ increases the number of points to $M^{N+1}$. Each point in the new constellation can be expressed as $a_i + M^Nb_i$ where $a_i$ is a point in the previous constellation and $b_i$ is an $M$-PAM symbol. Again suppose $a_1 + M^Nb_1$, and $a_2 + M^Nb_2$ are two arbitrarily generated points in the new constellation.
If these points are equal then \( a_1 - a_2 = M^N(b_2 - b_1) \). Ignoring the trivial solution, since the minimum value of \( |M^N(b_2 - b_1)| \), which is \( 2M^N \), is greater than the maximum value of \( |a_1 - a_2| \), which is \( 2(M^N - 1) \), equality cannot occur and therefore, the \( M^{N+1} \) points are distinct. Additionally, the distance between the two points is \( d = |(a_1 - a_2) + M^N(b_1 - b_2)| \). Since we have assumed the \( N \)-user \( M \)-PAM constellation is equally spaced by 2, the minimum value of \( d \) is 2. The \( M^{N+1} \) points equally spaced by a distance of 2 cover a range of length \( 2(M^{N+1} - 1) \), which completes the proof of the induction step.

**Proposition 3.** For \( K \) users each employing \( M \)-QAM where \( M = M_1 \times M_2 \) and \( M_1 \leq M_2 \), the precoding vector \( \phi = (1, M_2, \ldots, M_2^{K-1})^T \) satisfies \( b = A\phi \) where \( b \) represents the points of a rectangular \( (M_1^K \times M_2^K) \)-QAM constellation.

**Proof.** By a similar approach to that for extending BPSK to QPSK, it is straightforward to generalize PAM to QAM.

### 4.3 Performance Analysis

In this section, the performance of an MWRC with full data exchange is evaluated. The precoding design and the uplink performance analysis presented here are also applicable to fading channels. In slow fading, channel state information (CSI) can be obtained via training or pilot symbols, and this can be used to perform pre-equalization [83]. Therefore, by using the precoding vector presented here along with pre-equalization, the links from users to the relay can be modeled as AWGN channels. Consequently, the uplink performance analysis in a slow fading environment with pre-equalization will be the same as that presented here.

It is assumed that white Gaussian noise with variance \( \frac{N_0}{2} \) is present at the relay in the uplink and at the receiver of each user in the downlink. We first evaluate the case where four users, each using BPSK, exchange data through a relay, and then generalize the results to \( K \) users, each using \( M \)-QAM.

#### 4.3.1 Four Users Employing BPSK

For four users using BPSK with \( \phi = (1, 2, j, 2j)^T \), the signal space at the relay is shown in Fig. 4.2. The relay may denoise the received symbol or simply amplify and forward it back to the users. Taking the former approach, a symbol error occurs if
there is an error in the uplink and/or downlink (ignoring the case where an uplink error is corrected by a downlink error that follows it). Thus the probability that a user receives the symbol of at least one other user in error is \( P_{e,\text{tot}} = 1 - (1 - P_{e,U})(1 - P_{e,D}) \) where \( P_{e,U} \) and \( P_{e,D} \) are the SEPs of the uplink and downlink, respectively. Since the constellations at the relay and user nodes are the same, \( P_{e,U} = P_{e,D} \), and with four users, \( P_{e,D} = P_{e,16\text{-}QAM} \) and therefore, \( P_{e,\text{tot}} \approx 2P_{e,16\text{-}QAM} \). To obtain the SEP, a nearest-neighbor approximation is employed which gives \( P_{e,16\text{-}QAM} \approx 3Q \left( \frac{4E_b}{5N_0} \right) \) where \( E_b \) is the average energy per bit [78, Ch. 4]. The nearest neighbors of a given constellation point are defined as the points with minimum Euclidean distance from that point.

To determine the probability that a user symbol is received in error by the other users, consider Fig. 4.2. Without loss of generality, consider the second user symbol (second from the left shown in bold red color). This symbol is received in error only if the signal value crosses the boundary between the two shown regions (the boundary in this example is the \( Q \) axis). There are 4 nearest-neighbor symbol pairs at this boundary. The total number of nearest-neighbor symbol pairs in this constellation is 24, so the probability that the other users receive the symbol of the second user...
erroneously is \( P_{e,s_2} = \frac{4}{21} \times P_{e,tot} \). A similar analysis gives \( P_{e,s_1} = P_{e,s_3} = \frac{1}{2} P_{e,tot} \) and \( P_{e,s_4} = P_{e,s_2} = \frac{1}{6} P_{e,tot} \), where \( P_{e,tot} \approx 2P_{e,16-QAM} \) and \( P_{e,s_1}, P_{e,s_3} \) and \( P_{e,s_4} \) are the first, third and fourth user SEPs, respectively, at the other users. Note that users which are assigned larger precoding values (users 2 and 4 in this example), experience lower symbol error probabilities than the other users.

### 4.3.2 \( K \) Users Employing \( M \)-PAM

In the general case, \( K \) users each employing \( M \)-PAM modulation communicate through a relay. If the precoding vector is \( \phi = (1, M, \ldots, M^{K_1-1}, j, jM, \ldots, jM^{K_2-1})^T \) where \( K_1 + K_2 = K \), a symbol from an \((M^{K_1} \times M^{K_2})\)-QAM constellation is received at the relay, and this symbol is transmitted back to the users. To determine the symbol error probability of a user at the other user nodes, the total number of nearest-neighbor symbol pairs and the number of nearest-neighbor symbol pairs in which the symbol of user \( i \) changes, must be determined. In an \((M^{K_1} \times M^{K_2})\)-QAM constellation, there are \((M^{K_1} - 1)M^{K_2}\) and \((M^{K_2} - 1)M^{K_1}\) nearest-neighbor pairs in the inphase and quadrature directions, respectively, giving a total of \(2M^K - M^{K_1} - M^{K_2}\) symbol pairs.

Let \( s_i, i \in \{1, \ldots, K_1\} \), be the symbol of the \( i \)-th user with real precoding coefficient \( M^{i-1} \), and \( s'_l, l \in \{1, \ldots, K_2\} \), be the symbol of the \( l \)-th user with imaginary precoding coefficient \( jM^{l-1} \). Symbol \( s_i \) only changes every \( M^{i-1} \) symbol points in the inphase direction, therefore, there are only \( M^{K_2} \lfloor \frac{M^{K_1-1}}{M^{i-1}} \rfloor \) nearest-neighbor symbol changes in this direction. Similarly, for symbol \( s'_l \) there are \( M^{K_1} \lfloor \frac{M^{K_2-1}}{M^{l-1}} \rfloor \) nearest-neighbor symbol changes in the quadrature direction. Consequently, the symbol error probability for the \( i \)-th user with real coefficient \( M^{i-1} \) at the other user nodes is

\[
P_{e,s_i} = \frac{M^{K_2} \lfloor \frac{M^{K_1-1}}{M^{i-1}} \rfloor}{2M^K - M^{K_1} - M^{K_2}} P_{e,tot}, \quad i = 1, \ldots, K_1,
\]

and the symbol error probability for the \( l \)-th user with imaginary coefficient \( jM^{l-1} \) at the other user nodes is

\[
P_{e,s'_l} = \frac{M^{K_1} \lfloor \frac{M^{K_2-1}}{M^{l-1}} \rfloor}{2M^K - M^{K_1} - M^{K_2}} P_{e,tot}, \quad l = 1, \ldots, K_2
\]
where \( P_{e,\text{tot}} \approx 2P_{e,(M^{K_1} \times M^{K_2})-QAM} \) and

\[
P_{e,(M^{K_1} \times M^{K_2})-QAM} = \frac{N_{d_{\min}}}{M^K} Q \left( \sqrt{\frac{d_{\min}^2}{2N_0}} \right),
\]

in which \( d_{\min} \) is the minimum Euclidean distance between the signal points and \( N_{d_{\min}} \) is the total number of nearest neighbors for all signal points in the constellation (e.g. for a 16-QAM constellation \( N_{d_{\min}} = 48 \)) [78, Ch. 4].

### 4.3.3 \( K \) Users Employing \( M \)-QAM

For \( K \) users each employing \( M \)-QAM with \( M = M_1 \times M_2 \) and \( M_1 \leq M_2 \), the precoding vector \( \phi = (1, M_2, \ldots, M_2^{K-1})^T \) is assigned. Using the same arguments as above, the total number of nearest-neighbor symbol pairs in an \((M_1^K \times M_2^K)\)-QAM constellation is given by \( 2M^K - M_1^K - M_2^K \). A change in symbol \( s_i \) with precoding value \( M_i^{-1} \) will result in a symbol at least \( M_i^{-1} \) constellation points away from \( s_i \) either in the inphase or quadrature directions. Therefore, a symbol change occurs in \( M_2^K \left\lfloor \frac{M_1^K - 1}{M_2^{-1}} \right\rfloor \) places in the inphase direction and \( M_1^K \left\lfloor \frac{M_2^K - 1}{M_2^{-1}} \right\rfloor \) places in the quadrature direction. As a result, the symbol error probability for the \( i \)th user at the other user nodes is given by

\[
P_{e,s_i} = \frac{M_2^K \left\lfloor \frac{M_1^K - 1}{M_2^{-1}} \right\rfloor + M_1^K \left\lfloor \frac{M_2^K - 1}{M_2^{-1}} \right\rfloor}{2M^K - M_1^K - M_2^K} P_{e,\text{tot}}, \quad i = 1, \ldots, K,
\]

where \( P_{e,\text{tot}} \approx 2P_{e,M^K-QAM} \).

Fig. 4.3 presents the symbol error probability from (4.3) and (4.4) for 4 users using BPSK and 4-PAM constellations. This shows that in both cases, the users with greater precoding values experience lower error probabilities. Fig. 4.4 depicts the SEP of each user for the 3-user and 4-user cases when they employ QPSK. In Fig. 4.5, the SEP from (4.5) is shown for an increasing number of users. For clarity, only users with the best and worst SEPs are shown.

### 4.3.4 Multiple Clusters and Quality of Service

Consider the system model in Fig. 4.1 where cluster \( l \) has \( k_l \) users, \( 1 \leq l \leq L \). Each cluster can be assigned a specific precoding vector and the clusters can share the chan-
Figure 4.3: SEP for four users employing BPSK and 4-PAM constellations with one relay.

e and communicate with the relay by time division multiple access (TDMA) method. In this way the throughput of each cluster is 1/2 sym/U/CU and the throughput of all clusters is 1/2L sym/U/CU.

In the proposed algorithm, it is not necessary that all users use the same constellation. For example, with 3 users and one relay, users 1 and 2 can employ BPSK while user 3 employs QPSK. With the appropriate precoding vector $\phi = (1, j, 2)^T$, a square 16-QAM constellation similar to that in Fig. 4.2 will be received at the relay. Following the same procedure as in Section 4.3, the SEPs of users 1, 2 and 3 are $\frac{1}{2}P_{e,tot}$, $\frac{1}{2}P_{e,tot}$ and $\frac{1}{3}P_{e,tot}$, respectively, where $P_{e,tot} \approx 2P_{e,16-QAM}$. In general, a suitable precoding vector is found to form a unique constellation point for every combination of user symbols.

The algorithm offers flexibility in the sense that the number of users can be controlled to provide quality of service. As an example, consider the SEPs of the 4-user case shown in Fig. 4.4, and suppose the maximum acceptable SEP for communications is $10^{-2}$. This imposes a threshold signal to noise ratio (SNR) for each user, e.g. for user 1 this value is 17.5 dB. If the SNR falls below this threshold, user 1 should leave the network. Therefore, by changing the precoding values from $\phi = (1, 2, 4, 8)$
Figure 4.4: SEP for three and four users employing QPSK with one relay. The acceptable SEP threshold for remaining in the network is also shown.

to $\phi' = (0, 1, 2, 4)$ (a zero coefficient denotes that the user has left the network), the performance of the remaining three users is improved as shown in Fig. 4.4.

The proposed MWRC communications technique and higher order modulation with plain routing (PR) can provide similar performance. However, a transmission scheme employing TR may not be the most appropriate. For example, consider a power constraint transmission such that for a certain error performance, at most a 16-QAM signal can be transmitted by a user in one symbol period. The highest throughput that can be achieved with TR occurs when each user sends a 16-QAM signal in a time-slot while the other users are silent. There is no way to increase the throughput in this case. However, with the proposed scheme all users can send 16-QAM signals simultaneously. Although the total power is higher, the power constraint for each user is met and the throughput is increased. Another example involves a sensor network. Suppose several sensors are communicating with each other via a relay node. The sensors communicate via BPSK signals, and only transmit to inform the other sensors that an event has occurred. The nature of this problem is such that sensors must transmit in a timely fashion and cannot wait to use higher order modulation or until other sensors have finished transmitting. In addition, the proposed scheme allows each user to transmit in a time-slot, and also to receive
symbols from every other user in a time-slot. Thus it is a fair and low latency solution to communications between multiple user nodes.

### 4.3.5 CFNC, PNC, and PR Performance Comparison

As it has been discussed before, plain routing relaying throughput is $\frac{1}{2N}$. In Chapter 3, it was shown that the proposed PNC relaying algorithm had a throughput of $\frac{1}{1.5N}$ which was at least 33% improvement over plain routing for any number of nodes using binary signaling. Additionally, in this chapter, it was shown that CFNC relaying provides a throughput of $\frac{1}{2}$ which is better than both PNC and PR.

Choosing between these relaying schemes is actually a trade-off between spectral and power efficiencies, which means a relaying scheme with a better throughput shows a worse error performance. This can be seen in Fig. 4.6 where four users communicate via CFNC, PNC, and PR relaying schemes. In all three cases, the users employ BPSK signalling which leads to 16-QAM, 5-PAM with unequal symbol probabilities, and BPSK modulations received at the relay and broadcasted back to the users, respectively.
4.4 On the Optimality of Precoding for Users with Different Constellations

In the previous section, an MWRC, in which all users apply the same QAM modulation, was considered and a precoding vector for users was designed such that a rectangular QAM constellation is received at the relay. In this section, a more general problem is studied where each user can employ an arbitrary PAM or rectangular QAM constellation and a precoding vector for the users is designed. Furthermore, it is proved that the proposed precoding vector is optimal in the sense that the power efficiency of the received rectangular QAM constellation at the relay is maximized over all possible rectangular QAM constellations. In [84], optimized constellations for two-way wireless relaying systems were investigated to improve power efficiency. For our purpose, the precoding values should have the following two properties to result in an optimum rectangular QAM constellation at the relay. First, they should guarantee that every superposed combination of user symbols is mapped to a unique constellation point at the relay and hence distinguishable. Second, all adjacent points in the rectangular QAM constellation should be equally spaced. An equidistant QAM

Figure 4.6: Error performance comparison of CFNC, PNC and PR relaying schemes with four users employing BPSK.
(or PAM) constellation always shows a better power efficiency than a QAM (or PAM) constellation with different spacings between its points because it is packed more efficiently.

Since a rectangular QAM constellation can be constructed using two PAM signal sets, it suffices to consider users that employ PAM and focus on one-dimensional signal sets. Hence, instead of considering $K$ users employ $(M_i 	imes P_i)$-QAM modulation, it is assumed that they employ $M_i$-PAM modulation with $M_i$ being an even integer. Then a precoding vector should be designed such that an $(M_1 \times M_2 \times \cdots \times M_K)$-PAM constellation is received at the relay with maximum power efficiency (i.e., an equidistant PAM). The problem is first considered for two users and then extended to $K$ users with PAM signal sets.

Consider two users transmitting PAM symbols to the relay. User 1 transmits $s_1$ from an $M_1$-PAM signal set with precoding value $\phi_1$, and user 2 transmits $s_2$ from an $M_2$-PAM signal set with precoding value $\phi_2$, where $\phi_1$ and $\phi_2$ are positive real numbers. Without loss of generality, it is assumed that $\phi_2 \geq \phi_1$. Further, the $M_i$-PAM signal amplitudes are assumed to take values from odd multiples of $\frac{d_{\min}}{2}$ (i.e. $\pm 1, \pm 3, \ldots, \pm (M_i - 1) \times \frac{d_{\min}}{2}$), where $d_{\min}$ is the minimum Euclidean distance of user constellations. With a suitable choice of $\phi_1$ and $\phi_2$, the superposition of these two signals at the relay is a point from a set of $M_1 \times M_2$ distinct points. Disregarding noise, the signal at the relay is then given by

$$y = \phi_1 s_1 + \phi_2 s_2,$$

where $y$ is a point from an $(M_1 \times M_2)$-PAM constellation. Now if $\phi_1$ is an arbitrary precoding value assigned to user 1, a precoding value for user 2, $\phi_2$, must be found such that the $(M_1 \times M_2)$-PAM constellation received at the relay has maximum power efficiency. This means that all adjacent constellation points should be equidistant. Assuming $\phi_2 = b \phi_1$, with $b \geq 1$, it can be shown that the last two adjacent points at each far end of the PAM constellation received at the relay always have the distance of $\phi_1 d_{\min}$. Therefore, in order to have a maximum power efficiency, the distance between all adjacent points in the constellation should be equal to $\phi_1 d_{\min}$. In the following, the power efficiency of the relay constellation is derived and its value with respect to $b$ is discussed.

As mentioned above, user $i$ employs $M_i$-PAM with precoding value $\phi_i$ for trans-
mission. The average transmit energy of user $i$ can be calculated as

$$E_{s_i} = \frac{\phi_i^2 d_{\text{min}}^2 (1^2 + 3^2 + 5^2 + \cdots + (M_i - 1)^2)}{2 M_i}, \quad i = 1, 2.$$  \hspace{1cm} (4.7)

Therefore, the average energy of the received constellation at the relay is given by

$$E_s = E_{s_1} + E_{s_2} = \frac{d_{\text{min}}^2}{2} \left( \frac{\phi_1^2 (1^2 + 3^2 + 5^2 + \cdots + (M_1 - 1)^2)}{M_1} + \frac{\phi_2^2 (1^2 + 3^2 + 5^2 + \cdots + (M_2 - 1)^2)}{M_2} \right).$$  \hspace{1cm} (4.8)

If

$$\gamma_i = \frac{(1^2 + 3^2 + 5^2 + \cdots + (M_i - 1)^2)}{M_i}, \quad \text{for } i = 1, 2 \hspace{1cm} (4.9)$$

then (4.8) can be written as

$$E_s = \frac{d_{\text{min}}^2}{2} \left( \phi_1^2 \gamma_1 + \phi_2^2 \gamma_2 \right) = \gamma_3 E_b,$$  \hspace{1cm} (4.10)

where $\gamma_3 = \log_2(M_1 \times M_2)$ is the average number of bits per symbol and $E_b$ is the average bit energy received at the relay.

The minimum Euclidean distance of the relay constellation, $d_{\text{min}}$, is the smallest distance between two adjacent points. Consider two adjacent points in the relay constellation given by

$$y_1 = \phi_1 s_1^{(1)} + \phi_2 s_2^{(1)},$$

$$y_2 = \phi_1 s_1^{(2)} + \phi_2 s_2^{(2)},$$  \hspace{1cm} (4.11)

where $s_i^{(j)}$ is the $j$th symbol of user $i$. Then $d_{\text{min}}$ can be written as

$$d_{\text{min}} = |y_2 - y_1| = \left| \phi_1 \left( s_1^{(2)} - s_1^{(1)} \right) + \phi_2 \left( s_2^{(2)} - s_2^{(1)} \right) \right|$$

$$= |\phi_1 a_1 d_{\text{min}} + \phi_2 a_2 d_{\text{min}}|,$$  \hspace{1cm} (4.12)

with

$$a_i \in \{0, \pm 1, \pm 2, \ldots, \pm (M_i - 1)\} \quad \text{for } i = 1, 2.$$

$$\hspace{1cm} (4.13)
Table 4.1: Possible Cases for \((a_1, a_2)\) and the Corresponding Values of \(d_{\text{min}, r}\) and \(\eta_p\)

<table>
<thead>
<tr>
<th>Case</th>
<th>((a_1, a_2))</th>
<th>(d_{\text{min}, r}) from (4.12)</th>
<th>(\eta_p) from (4.14)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_1 &gt; 0, \ a_2 &gt; 0) (a_1 &lt; 0, a_2 &lt; 0) (a_1 &lt; 0, a_2 &gt; 0) (a_1 = 0, a_2 = 0)</td>
<td>((\phi_1 + \phi_2) d_{\text{min}}) (\phi_1 d_{\text{min}}) (\phi_2 d_{\text{min}}) (</td>
<td>-\phi_1</td>
</tr>
</tbody>
</table>

and \(a_1\) and \(a_2\) are not equal to 0 simultaneously. From (4.10) and (4.12), the power efficiency of the relay constellation can be written as

\[
\eta_p = \frac{E_b}{2} = \frac{\gamma_3 (\phi_1 a_1 + \phi_2 a_2)^2}{\phi_1^2 \gamma_1 + \phi_2^2 \gamma_2}.
\]  

(4.14)

Considering different cases for \(a_1\) and \(a_2\), Table 4.1 can be formed which shows the minimum Euclidean distance and the corresponding power efficiency for each case. \(a_1^*\) and \(a_2^*\) in the table are values of \(a_1\) and \(a_2\) that lead to the minimum Euclidean distance among all possible combinations of \(a_1\) and \(a_2\). Table 4.1 shows that \(d_{\text{min}, r}\) in Cases 1 and 3 is always greater than \(d_{\text{min}, r}\) in Case 2. Therefore, only Cases 2 and 4 need to be considered to determine the minimum Euclidean distance and hence the maximum value of \(\eta_p\).

Case 2 always leads to the minimum Euclidean distance if for all possible combinations of \(a_1\) and \(a_2\), the inequality

\[
| -\phi_1 |a_1^*| + \phi_2 |a_2^*| | \geq \phi_1,
\]  

(4.15)
satisfies. Considering the worst case combination of \(a_1\) and \(a_2\) from (4.13), (4.15) is always true if

\[
-\phi_1 (M_1 - 1) + \phi_2 \geq \phi_1.
\]  

(4.16)

By replacing \(\phi_2 = b \phi_1\), (4.16) gives \(b \geq M_1\). Thus, if \(b \geq M_1\), then (4.15) is satisfied and Case 2 always leads to the minimum Euclidean distance. The corresponding
power efficiency is given by

\[ \eta_p = \frac{\gamma_3 \phi_1^2}{\phi_1^2 \gamma_1 + \phi_2^2 \gamma_2} = \frac{\gamma_3}{\gamma_1 + b^2 \gamma_2}, \]  

(4.17)

which shows that the maximum power efficiency occurs when \( b \) is minimum, i.e. \( b = M_1 \), or equivalently \( \phi_2 = M_1 \phi_1 \).

For \( b < M_1 \), (4.15) may not be satisfied for all values of \( a_1 \) and \( a_2 \). In fact, values \( a_1^* \) and \( a_2^* \) may be found such that

\[ | -\phi_1 |a_1^*| + \phi_2 |a_2^*| | < \phi_1. \]  

(4.18)

Therefore, Case 4 can lead to a minimum Euclidean distance which is less than \( \phi_1 d_{\text{min}} \). This means that the resulting PAM constellation is not made of equidistant constellation points since the last two adjacent points at each far end of the PAM constellation always have the distance of \( \phi_1 d_{\text{min}} \). Therefore, if case 4 leads to the minimum Euclidean distance, it does not result in a constellation with maximum power efficiency.

Another way to show that \( b = M_1 \) leads to a \((M_1 \times M_2)\)-PAM with maximum power efficiency is as follows. There should be \((M_1 \times M_2)\) distinct constellation points at the relay with the minimum Euclidean distance of \( \phi_1 d_{\text{min}} \) between all adjacent points.

The most negative constellation point at the relay which is formed by the minimum values of \( a_1 \) and \( a_2 \) in (4.13) is \( P_1 = (-\phi_1(M_1 - 1) - \phi_2(M_2 - 1)) \frac{d_{\text{min}}}{2} \). In the same way, the most positive constellation point is \( P_2 = (\phi_1(M_1 - 1) + \phi_2(M_2 - 1)) \frac{d_{\text{min}}}{2} \).

This gives

\[ \phi_1 d_{\text{min}} = \frac{P_2 - P_1}{M_1 \times M_2 - 1} = \frac{\phi_1(M_1 - 1) + \phi_2(M_2 - 1)}{M_1 \times M_2 - 1} d_{\text{min}}. \]  

(4.19)

By considering \( \phi_2 = b \phi_1 \), solving (4.19) results in \( b = M_1 \).

**Proposition 4.** Consider an MWRC with \( K \) users, and user \( i \) employing \( M_i \)-PAM modulation. The modulation sizes of users may not follow any particular order. The optimal precoding value for user \( i \) such that an \((M_1 \times M_2 \times \cdots \times M_K)\)-PAM constellation with maximum power efficiency is received at the relay is given by

\[ \phi_i = \left( \prod_{j=1}^{i-1} M_j \right) \phi_1, \quad \text{for } i = 2, \ldots, K, \]  

(4.20)
where $\phi_1$ is the precoding value assigned to user 1.

Proof. The proof is by induction. It has been shown that the proposition is true for the case $K = 2$. Thus for 2 users employing $M_1$-PAM and $M_2$-PAM, if $\phi_1$ is the precoding value assigned to user 1, then the optimal precoding value for user 2 that leads to the optimum $(M_1 \times M_2)$-PAM constellation at the relay is given by $\phi_2 = M_1 \phi_1$.

As the induction hypothesis, suppose that for $N$ users employing $M_i$-PAM, $i = 1, \ldots, N$, if $\phi_1$ is the precoding value assigned to user 1, then the optimal precoding value for user $i$, $i = 2, \ldots, N$, that leads to the optimum $(M_1 \times M_2 \times \cdots \times M_N)$-PAM constellation at the relay is given by $\phi_i = \left(\prod_{j=1}^{i-1} M_j\right) \phi_1$. Adding the $(N + 1)$th user employing $(M_{N+1})$-PAM, a precoding value must be found such that it leads to the optimum $(M_1 \times M_2 \times \cdots \times M_N \times M_{N+1})$-PAM constellation at the relay. This precoding value can be found by considering two users, where the first user employs an $(M_1 \times M_2 \times \cdots \times M_N)$-PAM constellation (representing the first $N$ users) with precoding value $\phi_1$ and the second user employs $(M_{N+1})$-PAM. From the base case of the induction, the precoding value for the second user such that the optimum $(M_1 \times M_2 \times \cdots \times M_N \times M_{N+1})$-PAM constellation is received at the relay should be
Table 4.2: $d_{\text{min}}$, $\eta_p$ and SEP for different precoding values including the optimum values of $(\phi_1, \phi_2) = (1, 2)$ for the studied example.

<table>
<thead>
<tr>
<th>$(\phi_1, \phi_2)$</th>
<th>$d_{\text{min}}$</th>
<th>$\eta_p$</th>
<th>SEP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2)$</td>
<td>2</td>
<td>0.3846</td>
<td>$Q\left(\sqrt{0.3846\frac{E_b}{N_0}}\right)$</td>
</tr>
<tr>
<td>$(1, 1.4)$</td>
<td>0.8</td>
<td>0.1163</td>
<td>$Q\left(\sqrt{0.1163\frac{E_b}{N_0}}\right)$</td>
</tr>
<tr>
<td>$(1, 2.5)$</td>
<td>2</td>
<td>0.2532</td>
<td>$Q\left(\sqrt{0.2532\frac{E_b}{N_0}}\right)$</td>
</tr>
</tbody>
</table>

$(\prod_{j=1}^{N} M_j)^{\phi_1}$. This completes the proof.

4.4.1 Simulation Results

In this section, the SEP for different precoding values is evaluated. Two users are considered where the first user employs $(2 \times 2)$-QAM and the second user employs $(4 \times 2)$-QAM. According to the results in Section 4.4, the optimal precoding values for users 1 and 2 such that the power efficiency of the $(8 \times 4)$-QAM constellation received at the relay is maximized, are given by $\phi_1$ and $\phi_2 = 2\phi_1$, where $\phi_1$ is an arbitrary precoding value. Here it is assumed $\phi_1 = 1$. In addition to the optimal precoding value, $(\phi_1, \phi_2) = (1, 2)$, two non-optimal precoding values $(\phi_1, \phi_2) = (1, 1.4)$ and $(\phi_1, \phi_2) = (1, 2.5)$ are also considered. The constellations employed by the users and the relay constellation for these precoding values are shown in Fig. 4.8. Table 4.2 shows the minimum Euclidean distance $d_{\text{min}}$, the power efficiency $\eta_p$, and SEP formula of the relay constellation for each precoding value. The SEPs from Table 4.2 are plotted in Fig. 4.7. This figure shows that the optimal precoding values $(\phi_1, \phi_2) = (1, 2)$ provide the best performance among other precoding values.

4.5 Exploiting Self-information to Improve the Performance of MWRC

In Section 4.2 it was shown that for $K$ users employing $M$-QAM modulation there would be $M^K$ distinguishable signal points at the relay and unique decodability was achievable with one uplink multicast and one downlink broadcast transmission if an appropriate precoding vector was applied. By decoding the uplink transmission differently, it is possible to decrease the constellation size on the downlink broadcast
(a) The constellations employed by users 1 and 2.

(b) The relay constellation with the optimum precoding values \((\phi_1, \phi_2) = (1, 2)\)

(c) The relay constellation with precoding values \((\phi_1, \phi_2) = (1, 1.4)\)

(d) The relay constellation with precoding values \((\phi_1, \phi_2) = (1, 2.5)\)

Figure 4.8: The user constellations and the relay constellations for different precoding values.
transmissions and hence, improve the error performance of the system while maintaining unique decodability. This can be accomplished by grouping symbols appropriately and decoding them to new constellation points in the uplink which will be discussed in detail below.

4.5.1  **Full Data Exchange Algorithm Using Self-Information**

Suppose $K$ users, each employing an $M$-ary constellation, transmit their symbols to the relay using a precoding scheme that guarantees unique decodability at the relay as discussed in Sections 4.2 and 4.4. These $M^K$ constellation points are arranged into $M^{K-1}$ groups such that each group $R_i$, with $i \in \{1, 2, \ldots, M^{K-1}\}$, consists of $M$ constellation points given by

$$R_i = \{R_i^1, \ldots, R_i^M\}. \quad (4.21)$$

Each constellation point $R_i^j$ is the linear combination of $K$ user symbols, that is

$$R_i^j = \sum_{k=1}^{K} s_{j,k}^i \phi_k, \quad 1 \leq j \leq M, \quad (4.22)$$

where $s_{j,k}^i$ is the symbol of the $k$th user contributing to the $j$th received constellation point in group $R_i^j$ and $\phi_k$ is the precoding value assigned to $k$th user as discussed in Sections 4.2 and 4.4. Then each group $R_i^j$ is decoded to a new constellation point in the uplink transmission and the new constellation is used for the downlink transmission. This reduces the constellation size to $M^{K-1}$. For the downlink constellation points to be uniquely decodable at each user, i.e., each user can determine the information symbols of all other users, it suffices to have

$$s_{j,k}^i \neq s_{l,k}^i, \quad \text{for } j, l \in \{1, 2, \ldots, M\}, \text{ and } j \neq l. \quad (4.23)$$

Decoding user symbols in the downlink transmission is performed as follows. For a symbol sent from the relay, the corresponding group $R_i^j$ is known. From (4.21), the received constellation point corresponds to one of the $M$ constellation points in $R_i^j$, and the user can determine the correct point. Since $i$ and $k$ are known by the user, from (4.23) and exploiting the self-information, $j$ can be uniquely determined and hence $R_i^j$ and consequently the symbols of all other users can be obtained.
The procedure given above is illustrated with the following examples. Consider four users in an MWRC employing BPSK with precoding values $\phi = (1, j, 2, 2j)^T$. Here $K = 4$ and $M = 2$, so the relay receives a signal from a 16-QAM constellation as shown in Fig. 4.9. The four user symbols corresponding to each constellation point are also shown in the figure (in green). These points are arranged in $M^{K-1} = 8$ groups as in Table 4.3. Different precoding values will change the constellation received at the relay, but as long as the points are distinguishable at the relay (i.e., there is no overlap between the points), the grouping in Table 4.3 can be performed. This grouping satisfies the unique decodability criterion in (4.23). Each group can then be mapped to an arbitrary point in an 8-point constellation for downlink transmission. When a user demodulates a downlink signal point, its corresponding group in Table 4.3 is known. Then by using the self-information, it can decide between the two signal points in the group, and thus determines the symbols of all other users. It is also worth noting that in a non-homogeneous transmission in which users employ different constellation sizes, similar grouping can be applied in which the constellation size reduction at the relay will be equal to the size of the smallest user constellation.

Now consider two users employing QPSK with precoding values $\phi = (1, 2)^T$. Here $K = 2$ and $M = 4$, so the received constellation at the relay is the same as in Fig.
Table 4.3: Grouping received constellation at relay with four users employing BPSK.

<table>
<thead>
<tr>
<th>$R_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^i_1$</td>
<td>3+3j</td>
<td>-3-j</td>
<td>-1+3j</td>
<td>-1-j</td>
<td>3+j</td>
<td>3-j</td>
<td>-1+j</td>
<td>-1-3j</td>
</tr>
<tr>
<td>$R^i_2$</td>
<td>-3-3j</td>
<td>3+j</td>
<td>-3+j</td>
<td>1-j</td>
<td>3-j</td>
<td>-1-3j</td>
<td>1+j</td>
<td>1+3j</td>
</tr>
<tr>
<td>$s^i_{1,1}$, $s^i_{1,2}$, $s^i_{1,3}$, $s^i_{1,4}$</td>
<td>1,1,1,1</td>
<td>1,1,-1,1</td>
<td>1,1,1,-1</td>
<td>1,-1,1,1</td>
<td>1,-1,-1,1</td>
<td>1,1,-1,1</td>
<td>1,1,1,-1</td>
<td>1,-1,1,1</td>
</tr>
</tbody>
</table>

Table 4.4: Grouping received constellation at relay with two users employing QPSK.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^i_1$</td>
<td>3+3j</td>
<td>3-j</td>
<td>-1-j</td>
<td>-1+3j</td>
</tr>
<tr>
<td>$R^i_2$</td>
<td>-3-3j</td>
<td>3+j</td>
<td>-1+j</td>
<td>-1-3j</td>
</tr>
<tr>
<td>$R^i_3$</td>
<td>-3-j</td>
<td>-3+j</td>
<td>1+j</td>
<td>1-3j</td>
</tr>
<tr>
<td>$R^i_4$</td>
<td>-3-j</td>
<td>3-j</td>
<td>1+j</td>
<td>1-3j</td>
</tr>
<tr>
<td>$s^i_{1,1}$, $s^i_{1,2}$</td>
<td>1+j, 1+j</td>
<td>1+j, 1-j</td>
<td>1+j, -1-j</td>
<td>1+j, 1+j</td>
</tr>
<tr>
<td>$s^i_{2,1}$, $s^i_{2,2}$</td>
<td>1-j, 1-j</td>
<td>1-j, 1+j</td>
<td>1-j, -1+j</td>
<td>1-j, 1-j</td>
</tr>
<tr>
<td>$s^i_{3,1}$, $s^i_{3,2}$</td>
<td>-1-j, -1-j</td>
<td>-1-j, 1-j</td>
<td>-1-j, 1-j</td>
<td>-1-j, -1-j</td>
</tr>
<tr>
<td>$s^i_{4,1}$, $s^i_{4,2}$</td>
<td>-1+j, -1+j</td>
<td>-1+j, 1-j</td>
<td>-1+j, 1-j</td>
<td>-1+j, 1+j</td>
</tr>
</tbody>
</table>

4.9. The user symbols corresponding to each constellation point are also shown in the figure (in blue). For this example, the 16 constellation points can be arranged in $M^{K-1} = 4$ groups as shown in Table 4.4. By mapping each group to a constellation point, a 4-point constellation is sufficient for downlink transmission. To decode a downlink transmission, each user decides between the four signal points in the group by exploiting the self-information.

4.5.2 Downlink Performance

In this section, the downlink SEP of a user, which is defined as the probability of decoding at least one of the symbols of the other users in error, is investigated. Since the proposed scheme reduces the constellation size from $M^K$ to $M^{K-1}$, the downlink SEP of each user is improved compared to the case when no grouping is used at the relay. For the above examples of MWRC with four users employing BPSK and two users employing QPSK, the constellations shown in Fig. 4.10 are used for downlink transmissions in which the groups in Tables 4.3 and 4.4 are mapped to these constellation points.

The downlink SEPs with the proposed scheme are shown in Fig. 4.11 for the two examples given above in AWGN channels with PSD of $N_0/2$. The SEPs when no grouping is used on the downlink (original scheme) are given in Section 4.3. Figure
Figure 4.10: Downlink constellation for an MWRC with (a) four users employing BPSK with $R^i$ from Table 4.3, and (b) two users employing QPSK with $R^i$ from Table 4.4.

4.12 compares the downlink SEP of the proposed scheme with the original scheme for MWRCs with three users employing 4-level symmetric PAM and four users employing QPSK. These results show that reducing the downlink constellation can provide significant performance gains in AWGN channels.

4.5.3 Uplink Performance Improvement

In the previous section, user self-information was exploited to improve the downlink performance of an MWRC. Here, it is shown that the proposed transmission scheme can also improve uplink performance because of the different decoding employed for the uplink transmission. Consider the example of two users employing QPSK modulation with precoding values $\phi = (1, 2)^T$. The channels between the users and the relay are assumed to be AWGN with PSD $N_0/2$. For fading channels, pre-equalization can be performed before transmission by providing channel state information at the transmitter. With the original transmission scheme in Section 4.2, the received sum symbol at the relay is ML detected and decoded to a point from the 16-QAM constellation shown in Fig. 4.9. Hence, the uplink error performance is the SEP of a 16-QAM constellation in AWGN. To evaluate the SEP, a nearest-neighbor approximation is considered. The nearest neighbors of a constellation point are defined as the points...
Figure 4.11: Downlink SEP of the improved scheme compared to the original scheme in Section 4.3 for MWRCs with four users employing BPSK and two users employing QPSK, and AWGN channels.

with minimum Euclidean distance from that point. For the 16-QAM constellation in Fig. 4.9, there are 24 symbol pairs with the smallest nearest-neighbor minimum distance. Using the grouping of the 16-QAM signal points in Table 4.4, the received sum symbol at the relay is decoded to a signal point from the QPSK constellation in Fig. 4.10(b). The groups in the 16-QAM signal points are shown in the same color in Fig. 4.13. Since groups of four signal points in the 16-QAM constellation are decoded to a signal point in the QPSK constellation, incorrect detection of a received symbol will result in correct decoding if the detected signal point is in the same group as the correct point. The signal point grouping in Fig. 4.13 indicates that 8 (out of 24) of the minimum distance nearest-neighbor symbol pairs do not result in a decoding error. Hence, the uplink error performance of the proposed scheme is $\frac{2}{3}$ of the uplink SEP in the original scheme where the received sum symbol at the relay is decoded to a 16-QAM constellation point. This is shown in Fig. 4.14 for the example of two users employing QPSK. The same approach can be applied to other cases with different numbers of users and/or modulation schemes.
Figure 4.12: Downlink SEP of the improved scheme compared to the original scheme in Section 4.3 for MWRCs with three users employing 4-PAM and four users employing QPSK, and AWGN channels.

4.6 Summary

In this chapter, a multi-way relay channel in which clusters of users perform full data exchange has been considered. Using complex field network coding, a throughput of $1/2$ sym/U/CU can be achieved in each cluster. To implement CFNC, precoding must be employed with each user. The average transmit power of a rectangular QAM constellation is only slightly greater than that of an optimal $M$-ary QAM constellation, and the corresponding signals are easier to demodulate. In Section 4.2 a precoding vector for full data exchange was developed to allow a rectangular QAM constellation to be received at the relay. These vectors have the flexibility to accommodate users and allow them to join or leave the network at any time.

In Section 4.3 the error performance of each user in an MWRC was analyzed. It was shown that the symbols of users employing larger precoding values would experience lower error probabilities. Furthermore, the performance of CFNC was compared with plain routing and PNC relaying introduced in Chapter 3. It was shown that from PR to PNC and CFNC, although the throughput of the network increases, its error performance degrades.
Figure 4.13: The received 16-QAM constellation at the relay in which constellation points in the same group are shown with similar color. Each group is mapped to a signal point from the QPSK constellation in Fig. 4.10(b).

Figure 4.14: Uplink SEP of the improved scheme compared to the original scheme in Section 4.2 for an MWRC with two users employing QPSK, and AWGN channels.

In Section 4.4, a more general problem was considered where each user employs an arbitrary PAM or rectangular QAM constellation. The optimal precoding values were derived such that the power efficiency of the relay QAM constellation is highest
compared to the efficiency with other precoding values.

In Section 4.5 it was shown that by employing CFNC and exploiting user self-information, the size of constellation for the downlink broadcast transmission could be decreased which led to both downlink and uplink error performance improvement.

In the wireless medium, it is also important to evaluate the performance of MWRC in fading environment which is the subject of next chapter.
Chapter 5

Error Performance of Multi-Way Relay Channels

5.1 Introduction

In Chapter 4, a precoding vector was designed for an MWRC such that a rectangular QAM constellation is received at the relay when CSI are available at the users, and the performance of this system was given. The error performance of network coding in relay networks has been studied for different network topologies and relaying techniques. The symbol error rate of PNC in TWRC with AWGN channels is investigated in [85], and closed-form SEP results for BPSK and QPSK are provided using a nearest-neighbor approximation. The performance of BPSK modulation in physical-layer network coded TWRC with Rayleigh fading is analyzed in [86], in which upper and lower bounds on the bit error rate (BER) are derived. The instantaneous BER for PNC with BPSK modulation over fading channels is also given in [87]. In [88], the SEP for $M$-QAM modulated PNC over AWGN channels is considered along with the effect of phase errors on the performance. The error performance of AF and DF MWRC with pairwise data exchange protocol using BPSK modulation in both AWGN and Rayleigh fading channels is also studied in [53] and [89]. Most of the previous results on MWRCs have focused on the transmission scheme, capacity and achievable rate regions for different network coding techniques and relaying protocols while the error performance of MWRCs has not been adequately studied.

In this chapter, a multi-way relay channel with CFNC relaying is considered in which the users employ arbitrary modulation schemes and the links between the users
and the relay can have different SNRs. For instance, in a Rayleigh fading environment, users most likely have channels with different gains resulting in different received SNRs. Then the error performance of such a network is analyzed. Particularly, analytical expressions for the exact pairwise error probability (PEP) of an MWRC with independent Rayleigh fading channels are derived, and the effect of the precoding vector on the PEP is considered. Then, a tight approximation for the minimum Euclidean distance of the received constellation points at the relay is given. Using this result, a tight upper bound on the SEP of MWRCs with Rayleigh fading channels is obtained.

The rest of this chapter is organized as follows. The system model and full data exchange in MWRC network with fading channels are presented in Section 5.2. The exact PEP in Rayleigh fading is given in Section 5.3. The probability distribution function (PDF) of the minimum Euclidean distance between constellation points at the relay is closely approximated in Section 5.4. This PDF is used in Section 5.5 to obtain tight upper bounds on the SEP over Rayleigh fading using a nearest-neighbor approximation. Finally, the chapter summary is given in Section 5.6. The results of this chapter are published in [90].

5.2 System Model and Full Data Exchange Algorithm

In this chapter, the same MWRC model and information exchange algorithm as described in Sections 4.1 and 4.2, respectively, are considered. However, in this chapter the signals are assumed to be transmitted over independent Rayleigh fading channels. The channel path gains are assumed to be reciprocal and quasi-static, so they do not change within a duration of $T$ time-slots, but may vary afterward. Channel state information (CSI) are also assumed to be known at the relay and users. This can be achieved by sending known training symbols or pilots embedded in the packets.

The Rayleigh fading channel gain between user $i$ and the relay is denoted by $h_i$ and is modeled as a zero-mean, circularly symmetric complex Gaussian random variable with variance 1, i.e., $h_i \sim \mathcal{C}\mathcal{N}(0, 1)$. Then the channel vector is defined as $h = (h_1, h_2, \ldots, h_K)^T$ which has distribution $h \sim \mathcal{C}\mathcal{N}(0, I_K)$ where $I_K$ is a $K \times K$
identity matrix. Hence, the received signal at the relay can be written as

\[ y = h_1 \phi_1 s_1 + h_2 \phi_2 s_2 + \cdots + h_K \phi_K s_K + n, \]  

(5.1)

where \( n \) is a complex Gaussian noise with zero mean and variance \( \sigma^2 = \frac{N_0}{2} \) in each dimension. After \( y \) is received, the relay makes a decision based on the maximum likelihood (ML) rule to determine the transmitted signal.

If each user employs an \( M \)-ary constellation, then the relay must be able to distinguish between \( M^K \) possible constellation points. This was achieved in Chapter 4 for AWGN channels by employing a precoding vector \( \phi \) which ensures the reception of a rectangular QAM constellation at the relay. However, without pre-equalization in fading channels, the reception of a rectangular QAM constellation at the relay cannot be guaranteed using this precoding vector. In fact, the shape of the received constellation at the relay cannot be controlled since the channel gains rotate and scale the user symbols independently. In the following sections, the MWRC performance with fading is analyzed, and the effect of the precoding vector is examined.

## 5.3 Exact Pairwise Error Probability in Rayleigh Fading

In this section, the PEP of the MWRC system described in Section 5.2 is analyzed. The PEP is defined as the probability that the ML detector outputs a symbol \( r' \) that differs from the superimposed transmitted symbol to the relay \( r \), and is denoted by \( P(r \rightarrow r') \). The focus here is on uplink transmission performance which is more challenging as the signals from independent fading channels are combined. The downlink performance analysis for each user is straightforward since it is a point-to-point communication over a fading channel.

Consider an MWRC with \( K \) users each employing an \( M \)-ary constellation. The analysis to be presented here is general enough to let the users employ different constellations, but for simplicity, it is assumed all users employ the same constellation. The superimposed transmitted symbol that should be ideally detected at the relay is then given by

\[ r = \sum_{i=1}^{K} h_i \phi_i s_i, \]  

(5.2)
where $s_i$ is the symbol transmitted by user $i$ and modeled as a discrete uniform random variable. An error occurs if the detected symbol at the relay, $r'$, is a different symbol from $r$, i.e.

$$r' = \sum_{i=1}^{K} h_i \phi_i s'_i \neq r,$$

(5.3)

which gives $s'_i \neq s_i$ for at least one $i \in \{1, \ldots, K\}$. In this case, random variables $r$ and $r'$ are different signal points from an $M^K$-ary constellation at the relay. Given a channel vector realization $\mathbf{h}$, the conditional PEP with respect to Rayleigh fading can be written as

$$\Pr(r \rightarrow r' \mid \mathbf{h}) = \Pr \left( n > \frac{d_{st}}{2} \right) = Q \left( \sqrt{\frac{d_{st}^2}{2N_0}} \right),$$

(5.4)

where $d_{st}$, the distance between $r$ and $r'$, is a random variable given by

$$d_{st} = |r - r'| = \left| \sum_{i=1}^{K} h_i \phi_i e_i \right| = \left| \sum_{i=1}^{K} h_i \phi_i a_i \right| d_{\text{min}},$$

(5.5)

in which

$$e_i = s_i - s'_i = a_i d_{\text{min}},$$

(5.6)

and the deterministic variable $d_{\text{min}}$ is the minimum Euclidean distance of the user constellation. Note that the $e_i$ and therefore $a_i$ are discrete random variables whose statistics are completely known given the constellation for $s_i$.

Define the power efficiency as

$$\eta_p = \frac{d_{\text{min}}^2}{2E_b},$$

(5.7)

where $E_b$ is the average energy per bit for the relay constellation. Combining (5.4),
(5.5) and (5.7), the conditional PEP can be written as

\[
Pr(r \rightarrow r'| h) = Q \left( \sqrt{\sum_{i=1}^{K} h_i \phi_i a_i} \right)^2 \frac{\eta_p E_b}{N_0} = Q \left( \sqrt{\sum_{i=1}^{K} \sqrt{\eta_p \phi_i a_i} h_i} \right)^2 \frac{E_b}{N_0}
\]

\[
= Q \left( \sqrt{\eta_p} \cdot \sqrt{\phi_i a_i} \cdot \sqrt{h_i} \right) E_b \frac{E_b}{N_0} = Q \left( \sqrt{\eta_p} \cdot \sqrt{\phi_i a_i} \cdot \sqrt{h_i} \right) E_b \frac{E_b}{N_0},
\]

(5.8)

where \((.)^H\) denotes conjugate-transpose and \(\tilde{c} = (c_1, \ldots, c_K)\) is a discrete random vector with

\[
c_i = \sqrt{\eta_p \phi_i a_i}, \quad i \in \{1, \ldots, K\}.
\]

(5.9)

Using the well-known expression

\[
Q(x) = \frac{1}{\pi} \int_{0}^{\infty} \exp \left( -\frac{x^2}{2 \sin^2 \theta} \right) d\theta,
\]

(5.10)

(5.8) can be written as

\[
Pr(r \rightarrow r'| h) = \frac{1}{\pi} \int_{0}^{\infty} \exp \left( -\frac{h^H \tilde{c}^H \tilde{c} h \cdot E_b}{2 \sin^2 \theta \cdot N_0} \right) d\theta.
\]

(5.11)

The PEP can be calculated by averaging (5.11) over the channel gain vector \(h\), i.e.

\[
Pr(r \rightarrow r') = \mathbb{E}_h \left[ Q \left( \sqrt{\eta_p} \cdot \sqrt{\phi_i a_i} \cdot \sqrt{h_i} \right) E_b \frac{E_b}{N_0} \right] d\theta,
\]

(5.12)

where \(\mathbb{E}(.)\) denotes expectation. In order to calculate (5.12) the following two mathematical properties [91] are exploited.

First, for a Hermitian matrix \(A\) and a zero-mean complex Gaussian random vector \(g\) with covariance matrix \(\Pi\), i.e., \(g \sim \mathcal{CN}(0, \Pi)\),

\[
\mathbb{E}_g \left[ \exp \left( -g^H Ag \right) \right] = \frac{1}{\det(I + \Pi A)}
\]

(5.13)

where \(I\) is an identity matrix of appropriate size and \(\det(.)\) denotes determinant.
Second, if $u$ and $v$ are column vectors, then

$$\det(I + uv^H) = 1 + u^H v. \quad (5.14)$$

Exploiting the above properties, Eq. (5.12) can now be written as

$$\Pr(r \rightarrow r') = \frac{1}{\pi} \int_0^{\pi} \mathbb{E}_h \left[ \exp \left( -h^H \left( \frac{\tilde{c}^H \tilde{c} E_b}{2 \sin^2 \theta N_0} \right) h \right) \right] \, d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{\det(I + I \tilde{c}^H \tilde{c} E_b / 2 \sin^2 \theta N_0)} \, d\theta = \frac{1}{\pi} \int_0^{\pi} \frac{1}{1 + \tilde{c}^H \tilde{c} E_b / 2 \sin^2 \theta N_0} \, d\theta, \quad (5.15)$$

where the second and third equalities are obtained from (5.13) and (5.14), respectively. Defining a new discrete random variable

$$c = \tilde{c}^H = \sum_{i=1}^{K} |c_i|^2 = \sum_{i=1}^{K} \left| \sqrt{\eta_b \phi_i} a_i \right|^2, \quad (5.16)$$

(5.15) can be expressed as

$$\Pr(r \rightarrow r') = \frac{1}{\pi} \int_0^{\pi} \frac{1}{1 + \frac{\rho c}{2 \sin^2 \theta N_0}} \, d\theta = \frac{1}{2} \left( 1 - \sqrt{\frac{\rho c}{2 + \rho c}} \right), \quad (5.17)$$

where $\rho = \frac{E_b}{N_0}$ is the SNR. The exact PEP is given by (5.17), and depends on the discrete random variable $c$ specified by (5.16). Therefore, (5.17) must be averaged over $c$ to obtain the exact PEP. The distributions of $c$, $a_i$ and $|a_i|^2$ must thus be considered. These are derived in the following subsections for several important cases.

### 5.3.1 Two Users Employing BPSK

Consider $K = 2$ users communicating in an MWRC using BPSK symbols and precoding values $\phi_1$ and $\phi_2$. The average symbol energy of the constellation at user $i$ is

$$E_{s_i} = \frac{1}{4} d_{\min}^2 |\phi_i|^2 = E_{b_i}, \quad i = 1, 2, \quad (5.18)$$

where $E_{b_i}$ is the average bit energy of the constellation at user $i$. For the 4-ary constellation at the relay, we have $E_{b_1} + E_{b_2} = 2E_b$. Therefore, the power efficiency
Table 5.1: Values of $a_i$ and $|a_i|^2$ for $i = 1, 2$, and their probabilities for 2-user BPSK

| $a_i$ | $|a_i|^2$ | Probability |
|-------|----------|-------------|
| 0     | 0        | 1/2         |
| 1     | 1        | 1/4         |
| -1    | -1       | 1/4         |

<table>
<thead>
<tr>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
</tr>
<tr>
<td>1/4</td>
</tr>
<tr>
<td>1/4</td>
</tr>
</tbody>
</table>

Table 5.2: Possible expressions for $c$ and their probabilities for 2-user BPSK

| $(|a_1|^2, |a_2|^2)$ | Probability | $c$ for $\phi = (1, j)^T$ | $c$ for $\phi = (1, 2j)^T$ |
|---------------------|-------------|---------------------------|---------------------------|
| (0,1)               | 1/3         | $4|\phi_2|^2/(|\phi_1|^2 + |\phi_2|^2)$ | 2                         | 16/5                     |
| (1,0)               | 1/3         | $4|\phi_1|^2/(|\phi_1|^2 + |\phi_2|^2)$ | 2                         | 4/5                      |
| (1,1)               | 1/3         | 4                         | 4                         | 4                        |

is given by

$$\eta_p = \frac{d_{\min}^2}{2E_b} = \frac{4}{|\phi_1|^2 + |\phi_2|^2}. \quad (5.19)$$

Using this value of $\eta_p$ in (5.16) results in

$$c = \frac{4(|a_1|^2|\phi_1|^2 + |a_2|^2|\phi_2|^2)}{|\phi_1|^2 + |\phi_2|^2}. \quad (5.20)$$

Considering (5.6) with $s_i$ and $s'_i$ as BPSK symbols, the possible values of $a_i$ and $|a_i|^2$ and their probabilities can be calculated. These values are shown in Table 5.1.

Given Table 5.1 and (5.20), the distribution of $c$ can be calculated. Table 5.2 shows the expressions for $c$ as a function of the precoding values $\phi_1$ and $\phi_2$. Values of $c$ are also given for two precoding vectors $\phi = (1, j)^T$ and $\phi = (1, 2j)^T$. Note that $(a_1, a_2) = (0, 0)$ is not considered a possible outcome in calculating the PEP since it does not lead to $r \neq r'$.

By averaging (5.17) over $c$, the exact PEP as a function of the precoding values is obtained as

$$\Pr(r \rightarrow r') = \frac{1}{2} - \frac{1}{6} \left( \sqrt{\frac{2\rho}{1 + 2\rho}} + \sqrt{\frac{2\rho|\phi_1|^2}{(1 + 2\rho)|\phi_1|^2 + |\phi_2|^2}} + \sqrt{\frac{2\rho|\phi_2|^2}{|\phi_1|^2 + (1 + 2\rho)|\phi_2|^2}} \right). \quad (5.21)$$
Figure 5.1: Uplink transmission PEP in an MWRC with 2 users each using BPSK. Two different precoding vectors are considered.

With precoding vector $\phi = (1, j)^T$, (5.21) reduces to

$$Pr(r \to r') = \frac{1}{2} - \frac{1}{3} \sqrt{\frac{\rho}{1 + \rho}} - \frac{1}{6} \sqrt{\frac{2\rho}{1 + 2\rho}},$$

(5.22)

and for $\phi = (1, 2j)^T$ it reduces to

$$Pr(r \to r') = \frac{1}{2} - \frac{1}{6} \sqrt{\frac{2\rho}{1 + 2\rho}} - \frac{1}{6} \sqrt{\frac{2\rho}{5 + 2\rho}} - \frac{1}{3} \sqrt{\frac{2\rho}{5 + 8\rho}}.$$  

(5.23)

Fig. 5.1 gives the PEP from (5.22) and (5.23) along with the simulation results. This figure shows that the PEP for $\phi = (1, j)^T$ is less than that for $\phi = (1, 2j)^T$. Equation (5.21) can be optimized with respect to $\phi$. Since (5.21) is a convex function with respect to the precoding values as shown in Fig. 5.2, its minimum can be found by setting its gradient to zero, i.e.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial \phi_1} \\ \frac{\partial}{\partial \phi_2} \end{bmatrix} Pr(r \to r') = 0.$$  

(5.24)
Figure 5.2: PEP from (5.21) with respect to the precoding values for \( \rho = 1 \), shown in two different perspectives.

Solving (5.24) shows that the best PEP is achieved when the precoding values have the same magnitude, i.e., when \( |\phi_1| = |\phi_2| \).

## 5.3.2 Two Users Employing BPSK and 8-QAM

Consider an MWRC with \( K = 2 \) users employing BPSK and rectangular 8-QAM constellations with precoding values \( \phi_1 \) and \( \phi_2 \), respectively. Throughout this chapter, by 8-QAM constellation we mean a zero-mean 4 by 2 (i.e., 4 inphase and 2 quadrature components) rectangular constellation with equiprobable symbols. Hence, the average user symbol energies are

\[
\bar{E}_{s_1} = \frac{1}{4} d_{\text{min}}^2 |\phi_1|^2 = E_{b_1} \\
\bar{E}_{s_2} = \frac{3}{2} d_{\text{min}}^2 |\phi_2|^2 = 3 E_{b_2}. \tag{5.25}
\]

For the 16-ary constellation at the relay, we have \( E_{b_1} + 3 E_{b_2} = 4 E_b \). Therefore, the power efficiency is given by

\[
\eta_p = \frac{d_{\text{min}}^2}{2 E_b} = \frac{8}{|\phi_1|^2 + 6 |\phi_2|^2}. \tag{5.26}
\]
Table 5.3: Values of $a_2$ and $|a_2|^2$ and their probabilities for 2-user BPSK and 8-QAM

<table>
<thead>
<tr>
<th>$a_2$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>j</th>
<th>1+j</th>
<th>1-j</th>
<th>2+j</th>
<th>2-j</th>
<th>3+j</th>
<th>3-j</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob.</td>
<td>$\frac{8}{64}$</td>
<td>$\frac{12}{64}$</td>
<td>$\frac{8}{64}$</td>
<td>$\frac{4}{64}$</td>
<td>$\frac{8}{64}$</td>
<td>$\frac{6}{64}$</td>
<td>$\frac{6}{64}$</td>
<td>$\frac{4}{64}$</td>
<td>$\frac{4}{64}$</td>
<td>$\frac{2}{64}$</td>
<td>$\frac{2}{64}$</td>
</tr>
</tbody>
</table>

| $|a_2|^2$ | 0 | 1 | 2 | 4 | 5 | 9 | 10 |
|---------|---|---|---|---|---|----|----|
| Probability | $\frac{8}{64}$ | $\frac{20}{64}$ | $\frac{12}{64}$ | $\frac{8}{64}$ | $\frac{8}{64}$ | $\frac{4}{64}$ | $\frac{4}{64}$ |

Table 5.4: Possible expressions for $c$ and their probabilities for 2-user BPSK and 8-QAM

| $|a_1|^2$, $|a_2|^2$ | Probability | $c$ | $c$ for $\phi = (1, 1)^T$ | $c$ for $\phi = (1, 2)^T$ |
|----------------|-------------|-----|----------------|----------------|
| ($0, 1$) | 5/30 | $8|\phi_2|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{8}{7}$ | $\frac{32}{25}$ |
| ($0, 2$) | 3/30 | $16|\phi_2|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{16}{7}$ | $\frac{64}{25}$ |
| ($0, 4$) | 2/30 | $32|\phi_2|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{32}{7}$ | $\frac{128}{25}$ |
| ($0, 5$) | 2/30 | $40|\phi_2|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{40}{7}$ | $\frac{160}{25}$ |
| ($0, 9$) | 1/30 | $72|\phi_2|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{72}{7}$ | $\frac{288}{25}$ |
| ($0, 10$) | 1/30 | $80|\phi_2|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{80}{7}$ | $\frac{320}{25}$ |
| ($1, 0$) | 2/30 | $8|\phi_1|^2/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{8}{7}$ | $\frac{8}{25}$ |
| ($1, 1$) | 5/30 | $8(|\phi_1|^2 + |\phi_2|^2)/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{16}{7}$ | $\frac{40}{25}$ |
| ($1, 2$) | 3/30 | $8(|\phi_1|^2 + 2|\phi_2|^2)/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{24}{7}$ | $\frac{72}{25}$ |
| ($1, 4$) | 2/30 | $8(|\phi_1|^2 + 4|\phi_2|^2)/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{40}{7}$ | $\frac{136}{25}$ |
| ($1, 5$) | 2/30 | $8(|\phi_1|^2 + 5|\phi_2|^2)/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{48}{7}$ | $\frac{168}{25}$ |
| ($1, 9$) | 1/30 | $8(|\phi_1|^2 + 9|\phi_2|^2)/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{80}{7}$ | $\frac{296}{25}$ |
| ($1, 10$) | 1/30 | $8(|\phi_1|^2 + 10|\phi_2|^2)/(|\phi_1|^2 + 6|\phi_2|^2)$ | $\frac{88}{7}$ | $\frac{328}{25}$ |

Using this value of $\eta_p$ in (5.16) results in

$$c = \frac{8(|a_1|^2|\phi_1|^2 + |a_2|^2|\phi_2|^2)}{|\phi_1|^2 + 6|\phi_2|^2}. \quad (5.27)$$

Considering (5.6) with $s_1$ and $s_1'$ as BPSK symbols and $s_2$ and $s_2'$ as 8-QAM symbols, the possible values for $a_1$ and $|a_1|^2$ are given in Table 5.1, and the possible values for $a_2$ and $|a_2|^2$ are given in Table 5.3, along with their probabilities.

The distribution of $c$ can be calculated using (5.27) and Tables 5.1 and 5.3. Table 5.4 shows the expressions for $c$ and the corresponding values for the precoding vectors $\phi = (1, 1)^T$ and $\phi = (1, 2)^T$. Note that $(a_1, a_2) = (0, 0)$ is not considered a possible outcome in calculating the PEP since it does not lead to $r \neq r'$. The exact PEP as a function of the precoding values for this example can be calculated by averaging
Figure 5.3: Uplink transmission PEP in an MWRC with 2-user BPSK and 8-QAM. Two different precoding vectors are considered.

(5.17) over the values of \( c \) given in Table 5.4. For an MWRC with 2 users employing BPSK and 8-QAM with \( \phi = (1, 1)^T \), the exact PEP is given by

\[
\Pr(r \rightarrow r') = \frac{1}{2} - \frac{7}{30} \sqrt{\frac{\rho}{7 + 4\rho}} - \frac{4}{15} \sqrt{\frac{2\rho}{7 + 8\rho}} - \frac{1}{10} \sqrt{\frac{3\rho}{7 + 12\rho}} - \frac{2}{15} \sqrt{\frac{\rho}{7 + 16\rho}} \\
- \frac{2}{15} \sqrt{\frac{5\rho}{7 + 20\rho}} - \frac{1}{15} \sqrt{\frac{6\rho}{7 + 24\rho}} - \frac{1}{10} \sqrt{\frac{\rho}{7 + 36\rho}} - \frac{1}{15} \sqrt{\frac{10\rho}{7 + 40\rho}} - \frac{1}{30} \sqrt{\frac{11\rho}{7 + 44\rho}}.
\]

(5.28)

The PEP can be optimized with respect to \( \phi \) to obtain the minimum PEP. Here also the best PEP is achieved when the precoding values have the same magnitude, i.e., when \( |\phi_1| = |\phi_2| \). Fig. 5.3 presents the PEP from (5.28) and the PEP for precoding vector \( \phi = (1, 2)^T \), along with the simulation results to confirm the analysis.

\subsection*{5.3.3 Four Users Employing BPSK}

Consider \( K = 4 \) users communicating in an MWRC using BPSK symbols and precoding values \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \). The average symbol energy of user constellation \( i \)
Therefore, the power efficiency is given by

\[ \eta_p = \frac{d_{\text{min}}^2}{2E_b} = \frac{8}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2}. \quad (5.30) \]

Using this value of \( \eta_p \) in (5.16) results in

\[ c = \frac{8 \sum_{i=1}^{4} |a_i|^2 |\phi_i|^2}{\sum_{i=1}^{4} |\phi_i|^2}. \quad (5.31) \]

Considering (5.6) with \( s_i \) and \( s'_i \) being BPSK symbols, the possible values for \( a_i \) and \( |a_i|^2 \) and their probabilities are given in Table 5.1. From the distribution of \( a_i \) given in Table 5.1 and (5.31), the distribution of \( c \) can be obtained. Table 5.5 shows the possible expressions for \( c \) as a function of \( \phi_i \) and the corresponding values for \( \phi = (1, j, 1, j)^T \) and \( \phi = (1, j, 2, 2j)^T \). Since \( (a_1, a_2, a_3, a_4) = (0, 0, 0, 0) \) does not lead to \( r \neq r' \), it was not considered in calculating the PEP.
Figure 5.4: Uplink transmission PEP in an MWRC with 4 users each using BPSK. Two different precoding vectors are considered.

By averaging (5.17) over $c$ given in Table 5.5, the exact PEP as a function of the precoding values can be found. For $\phi = (1, j, 1, j)^T$, the exact PEP is given by

$$\Pr(r \rightarrow r') = \frac{1}{2} - \frac{2}{15}\sqrt{\frac{\rho}{1 + \rho}} - \frac{1}{15}\sqrt{\frac{2\rho}{1 + 2\rho}} - \frac{2}{15}\sqrt{\frac{3\rho}{1 + 3\rho}} - \frac{1}{15}\sqrt{\frac{\rho}{1 + 4\rho}}, \quad (5.32)$$

and for $\phi = (1, j, 2, 2j)^T$, it is given by

$$\Pr(r \rightarrow r') = \frac{1}{2} - \frac{2}{15}\sqrt{\frac{2\rho}{1 + 2\rho}} - \frac{1}{15}\sqrt{\frac{2\rho}{5 + 2\rho}} - \frac{1}{15}\sqrt{\frac{2\rho}{1 + 4\rho}} - \frac{1}{15}\sqrt{\frac{\rho}{5 + 4\rho}}$$

$$- \frac{2}{15}\sqrt{\frac{2\rho}{5 + 8\rho}} - \frac{2}{15}\sqrt{\frac{3\rho}{5 + 12\rho}} - \frac{2}{15}\sqrt{\frac{\rho}{5 + 16\rho}} - \frac{1}{15}\sqrt{\frac{2\rho}{5 + 18\rho}}. \quad (5.33)$$

Fig. 5.4 shows the PEP from (5.32) and (5.33) along with the corresponding simulation results. The PEP can be optimized with respect to $\phi$. Here also the best PEP is achieved when the precoding values have the same magnitude, i.e., when $|\phi_1| = |\phi_2| = |\phi_3| = |\phi_4|$. 
5.4 Average Minimum Distance

In an MWRC, the minimum Euclidean distance of the relay constellation, \( d_{\text{min}_r} \), is a random variable (rv) which is defined as \( d_{\text{min}_r} = \min_{i \neq j} \| r_i - r_j \| \), where \( r_i \) and \( r_j \) are signal points in the relay constellation. In this section, a tight lower bound on the average of \( d_{\text{min}_r} \) is presented which will be used in Section 5.5 to obtain an upper bound on the SEP.

Consider an MWRC with \( K \) users each employing an \( M \)-ary constellation. The distance between two arbitrary constellation points at the relay, \( d_{rf} \), is given by (5.5). The possible values for \( a_i \) and their probabilities depend on the user constellations. Suppose the sample space of \( a_i \) has \( A \) members. This gives \( A^K \) possible \( a_i \) combinations in (5.5), each resulting in a random variable \( d_{rf} \). These \( A^K \) dependent rvs must be considered to determine the minimum distance. Fortunately, some of these rvs are equal, and some are always larger than others. Let \( N \) be the number of rvs (from the \( A^K \) rvs), that determine the minimum distance of the constellation. Then \( d_{\text{min}_r} \) can be obtained by finding the minimum of these \( N \) dependent random variables, denoted by \( X_n \) (1 \( \leq \) \( n \) \( \leq \) \( N \)), giving

\[
d_{\text{min}_r} = \min (X_1, X_2, \ldots, X_N), \tag{5.34}
\]

where

\[
X_n = \left| \sum_{i=1}^{K} h_i \phi_i a_{i,n} \right| d_{\text{min}_r}, \quad 1 \leq n \leq N, \tag{5.35}
\]

and \( a_{i,n} \) is a realization of random variable \( a_i \). From (5.35), the probability density function (PDF) of \( X_n \) is

\[
f_{X_n}(x) = \frac{x}{v_n^2} e^{-\frac{x^2}{2v_n^2}}, \tag{5.36}
\]

where \( v_n^2 = \frac{d_{\text{min}_r}^2}{2} \sum_{i=1}^{K} |\phi_i|^2 |a_{i,n}|^2 \). Equation (5.34) is the minimum of \( N \) dependent Rayleigh random variables. Here we claim that the average minimum of \( N \) independent Rayleigh rvs is always smaller than (and actually is a tight lower bound for), the average minimum of \( N \) dependent rvs with the same distributions. Although a formal proof of this result does not seem achievable, a computer-assisted proof shows it is true for the case considered here. This claim is presented in Conjecture 1 for two
Rayleigh random variables, and the generalization to $N$ Rayleigh random variables is given in Corollary 1.

**Conjecture 1.** Consider two independent complex Gaussian random variables $h_1$ and $h_2$ with distributions $h_1 \sim CN(0, 2\sigma_1^2)$ and $h_2 \sim CN(0, 2\sigma_2^2)$. Define two independent Rayleigh random variables $X$ and $Y$ as

$$X = |h_1| \Rightarrow f_X(x) = \frac{x}{\sigma_1^2} e^{-\frac{x^2}{2\sigma_1^2}},$$

$$Y = |h_2| \Rightarrow f_Y(y) = \frac{y}{\sigma_2^2} e^{-\frac{y^2}{2\sigma_2^2}}.$$  

Now define a third rv $Z$ which is dependent on $X$ and has the same distribution as $Y$

$$Z = \frac{\sigma_2}{\sqrt{|a_1|^2 \sigma_1^2 + |a_2|^2 \sigma_2^2}} |a_1 h_1 + a_2 h_2| \Rightarrow f_Z(z) = \frac{z}{\sigma_2^2} e^{-\frac{z^2}{2\sigma_2^2}},$$

where $a_1$ and $a_2$ are arbitrary complex numbers (not simultaneously equal to zero).

Next, define random variables $M_1$ and $M_2$ as

$$M_1 = \min(X, Y),$$

$$M_2 = \min(X, Z).$$

(5.37)

The expected value of the minimum of two independent Rayleigh rvs is always less than or equal to that of two dependent Rayleigh rvs, i.e.

$$\mathbb{E}[M_1] \leq \mathbb{E}[M_2].$$

(5.38)

Furthermore, $\mathbb{E}[M_1]$ is a tight lower bound on $\mathbb{E}[M_2]$.

**Proof.** The proof provided here is computer-assisted. Without loss of generality, we can reduce the number of unknown parameters from four to two by assuming that $a_2 = \sigma_2 = 1$, and let $a_1$ be a positive real number. Hence, $h_2 \sim CN(0, 2)$ and

$$Z = \frac{1}{\sqrt{a_1^2 \sigma_1^2 + 1}} |a_1 h_1 + h_2|.$$  

(5.39)

We also assume $h_1 = \sigma_1 \hat{h}_1$ where $\hat{h}_1 \sim CN(0, 2)$. Equation (5.38) was verified by extensive simulations with tens of thousands values of $(\sigma_1^2, a_1)$. For clarity, Fig.
Figure 5.5: The average minimum of 2 dependent Rayleigh rvs (the black dots), \( E[M_2] \), and 2 independent Rayleigh rvs (the red surface), \( E[M_1] \) for 1800 values of \((\sigma_1^2, a_1)\).

5.5 shows the results for 1800 values of \((\sigma_1^2, a_1)\). More precisely, 10 equally spaced values in the range \([0,0.9)\) and 50 equally spaced values in the range \([1,50]\) for \(\sigma_1^2\), and 10 equally spaced values in the range \([0,0.9)\), and 20 equally spaced values in the range \([1,20]\) for \(a_1\), were considered. For each of these values, \(M_1\) and \(M_2\) in (5.37) were calculated for 2\(^{20}\) samples of the random variables \(X\), \(Y\) and \(Z\). Then the expected values in (5.38) were computed. The results presented in Fig. 5.5 show that the average minimum of two independent Rayleigh rvs is smaller than that of two dependent Rayleigh rvs. Note that for \(\sigma_1^2\) very small and very large, the averages are close.

To further verify Conjecture 1, the behaviour of (5.38) is considered for very large and very small values of \(\sigma_1\). For \(\sigma_1 >> \sigma_2 = 1\) and \(\sigma_1 << \sigma_2 = 1\), (5.38) is satisfied with equality regardless of the value of \(a_1\). Without loss of generality, assume \(a_1 = 1\). If \(\sigma_1 >> 1\), then \(Z \simeq |\hat{h}_1|\) and we have

\[
E[M_1] = E[\min(X, Y)] = E[\min(|h_1|, |h_2|)] = E\left[\min(\sigma_1 |\hat{h}_1|, |h_2|)\right] = E[|h_2|],
\]
\[
E[M_2] = E[\min(X, Z)] = E\left[\min(\sigma_1 |\hat{h}_1|, Z)\right] = E[Z].
\]
Figure 5.6: The average minimum of 2 dependent Rayleigh rvs, $E[M_2]$, and 2 independent Rayleigh rvs, $E[M_1]$, for 500 values of $\sigma_1^2$ and $a_1 = 1$.

Since $|h_2|$ and $Z = |\hat{h}_1|$ are two random variables with the same distribution, $E[|h_2|] = E[Z]$. If $\sigma_1 << 1$, then $Z \simeq |h_2|$ and we have

$$E[M_1] = E\left[\min\left(\sigma_1 |\hat{h}_1|, |h_2|\right)\right] = \sigma_1 E\left[|\hat{h}_1|\right],$$
$$E[M_2] = E\left[\min\left(\sigma_1 |\hat{h}_1|, Z\right)\right] = \sigma_1 E\left[|\hat{h}_1|\right].$$

Therefore, in both cases $E[M_1] = E[M_2]$. Figure 5.6 compares $E[M_1]$ and $E[M_2]$ for 500 different equally spaced values of $\sigma_1^2$ in the range 0 to 50. For this figure, the general form of $Z$ given in (5.39) is considered with $a_1 = 1$. This figure shows that although the results converge for very small and large values of $\sigma_1$, the average with the independent rvs is still smaller.

We next show that if $\sigma_1$ is comparable to $\sigma_2 = 1$, then for very large and small values of $a_1$, that is $a_1 >> a_2 = 1$ and $a_1 << a_2 = 1$, (5.38) is satisfied. For simplicity, let $\sigma_1 = 1$ so that

$$E[M_1] = E[\min(X,Y)] = E[\min(|h_1|, |h_2|)] = E\left[\min\left(|\hat{h}_1|, |h_2|\right)\right].$$
Figure 5.7: The average minimum of 2 dependent Rayleigh rvs, $\mathbb{E}[M_2]$, and 2 independent Rayleigh rvs, $\mathbb{E}[M_1]$ for 400 values of $a_1$ and $\sigma_1 = 1$.

If $a_1 >> 1$, then $Z \simeq |\hat{h}_1|$ and we have

$$\mathbb{E}[M_2] = \mathbb{E}[\min(X, Z)] = \mathbb{E}\left[\min\left(|\hat{h}_1|, |\hat{h}_1|\right)\right] = \mathbb{E}\left[|\hat{h}_1|\right],$$

which is clearly greater than $\mathbb{E}[M_1]$. If $a_1 << 1$, then $Z \simeq |h_2|$ and

$$\mathbb{E}[M_2] = \mathbb{E}[\min(X, Z)] = \mathbb{E}\left[\min\left(|\hat{h}_1|, |h_2|\right)\right],$$

which is equal to $\mathbb{E}[M_1]$. Therefore, in both cases (5.38) is satisfied. Figure 5.7 compares $\mathbb{E}[M_1]$ and $\mathbb{E}[M_2]$ for 400 equally spaced values of $a_1$ in the range 0 to 4. For this figure, the general form of $Z$ given in (5.39) with $\sigma_1 = 1$ was considered. This shows that for very small and very large values of $a_1$, (5.38) is satisfied with equality and strict inequality, respectively.

\[ \square \]

**Corollary 1.** Suppose $X_i = |h_i|$ with $h_i \sim CN(0, 2\sigma^2_i)$ and $1 \leq i \leq N$ are $N$ independent Rayleigh distributed random variables and $Y_i$s are $N$ dependent Rayleigh distributed random variables which have the same distributions as $X_i$s. The expected value of the minimum of $N$ independent Rayleigh rvs $X_i$ is less than or equal to that
of \( N \) dependent Rayleigh rvs \( Y_i \), i.e.

\[
\mathbb{E}[M_x] \leq \mathbb{E}[M_y].
\]  

(5.40)

where

\[
M_x = \min(X_1, X_2, \ldots, X_N),
\]

\[
M_y = \min(Y_1, Y_2, \ldots, Y_N).
\]  

(5.41)

Proof. The proof is by induction on Conjecture 1 which indicates that (5.40) is true for the base case of \( N = 2 \). For the inductive step, it can be shown that if (5.40) holds for \( N \) Rayleigh rvs, then it also holds for \( N + 1 \) Rayleigh rvs. This is represented as

\[
\mathbb{E}[\min(X_1, X_2, \ldots, X_N, X_{N+1})] = \mathbb{E}[\min(M_x, X_{N+1})]
\]

\[
\leq \mathbb{E}[\min(M_x, Y_{N+1})] \leq \mathbb{E}[\min(M_y, Y_{N+1})]
\]

\[
= \mathbb{E}[\min(Y_1, Y_2, \ldots, Y_n, Y_{n+1})].
\]  

(5.42)

where \( X_{N+1} \) is Rayleigh distributed and independent of \( M_x \), and \( Y_{N+1} \) can be dependent on \( M_x \) and \( M_y \) and has the same distribution as \( X_{N+1} \). As will be shown in (5.48), the minimum of \( N \) independent Rayleigh distributed random variables has a Rayleigh distribution. Therefore, \( M_x \) is Rayleigh distributed. The first inequality in (5.42) is from Conjecture 1, and the second inequality is from the inductive hypothesis.

Determining the PDF of the random variable \( d_{min_{rf}} \) given in (5.34) is not tractable. Therefore, Conjecture 1 and Corollary 1 are used to approximate this distribution by substituting \( X_i \)s with independent random variables \( Y_i \)s. This is denoted by

\[
d_{min_{rf}} \approx d_{\min} \times d,
\]  

(5.43)

where \( d \) is a random variable defined as

\[
d = \min(Y_1, Y_2, \ldots, Y_N).
\]  

(5.44)
The $Y_n$ are independent Rayleigh distributed random variables given by

$$Y_n = \left| \sum_{i=1}^{K} h_{i,n} \phi_i a_{i,n} \right|, \quad 1 \leq n \leq N,$$

(5.45)
in which the channel gains $h_i$ in (5.35) are replaced by channel gains $h_{i,n}$ which are assumed independent for different values of $n$ and have the same distributions as $h_i$. Hence, the cumulative distribution function (CDF) and PDF of $Y_n$ are given by

$$F_{Y_n}(y) = 1 - e^{-\frac{y^2}{2\nu_n^2}}, \quad 1 \leq n \leq N,$$

$$f_{Y_n}(y) = \frac{y}{\nu_n^2} e^{-\frac{y^2}{2\nu_n^2}}, \quad 1 \leq n \leq N,$$

$$\nu_n^2 = \frac{1}{2} \sum_{i=1}^{K} |\phi_i|^2 |a_{i,n}|^2 = \frac{v_n^2}{d_{\min}^2}. \quad (5.46)$$

The CDF of $d$ in (5.44) is then [92, Ch. 6]

$$F_d(y) = 1 - \prod_{i=1}^{N} (1 - F_{Y_i}(y)). \quad (5.47)$$

Taking the derivative of (5.47), the PDF of $d$ is given by

$$f_d(y) = 2 \beta y e^{-\beta y^2}, \quad (5.48)$$

where

$$\beta = \left( \sum_{i=1}^{N} \frac{1}{2\nu_i^2} \right). \quad (5.49)$$

From (5.43) and (5.48), the distribution of the minimum distance of the relay constellation is then

$$f_{d_{\min}}(y) \approx \frac{2 \beta y}{d_{\min}^2} e^{-\frac{\beta y^2}{d_{\min}^2}}. \quad (5.50)$$
Table 5.6: Possible values of \((a_1, a_2)\) and the corresponding \(d_{rf}\) for 2-user BPSK

<table>
<thead>
<tr>
<th>((a_1, a_2))</th>
<th>(d_{rf})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0)</td>
<td>(</td>
</tr>
<tr>
<td>(-1, 0)</td>
<td>(</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(</td>
</tr>
<tr>
<td>(0, -1)</td>
<td>(</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(</td>
</tr>
<tr>
<td>(-1, -1)</td>
<td>(</td>
</tr>
<tr>
<td>(1, -1)</td>
<td>(</td>
</tr>
<tr>
<td>(-1, 1)</td>
<td>(</td>
</tr>
</tbody>
</table>

Table 5.7: The random variables in Table 5.6 needed to find \(d_{\min,rf}\) for 2-user BPSK

<table>
<thead>
<tr>
<th>(n)</th>
<th>((a_{1,n}, a_{2,n}))</th>
<th>(X_n)</th>
<th>(v_n^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 0)</td>
<td>(</td>
<td>h_1\phi_1</td>
</tr>
<tr>
<td>2</td>
<td>(0, 1)</td>
<td>(</td>
<td>h_2\phi_2</td>
</tr>
<tr>
<td>3</td>
<td>(1, 1)</td>
<td>(</td>
<td>h_1\phi_1 + h_2\phi_2</td>
</tr>
<tr>
<td>4</td>
<td>(1, -1)</td>
<td>(</td>
<td>h_1\phi_1 - h_2\phi_2</td>
</tr>
</tbody>
</table>

From (5.50), the average minimum Euclidean distance is

\[
\bar{d}_{\min,rf} \approx \int_0^\infty 2 \beta y^2 e^{-\frac{\beta y^2}{d_{\min}^2}} dy = \frac{d_{\min} \sqrt{\pi}}{2\sqrt{\beta}}. \tag{5.51}
\]

To illustrate this procedure, consider the following three examples.

### 5.4.1 Two Users Employing BPSK

If \(K = 2\) users communicate in an MWRC network with BPSK modulation, \(a_i\) can take three different values as shown in Table 5.1. Therefore, 9 different combinations of \((a_1, a_2)\) are possible in (5.5), each resulting in a random variable \(d_{rf}\). Ignoring the \((0, 0)\) case, the remaining combinations of \((a_1, a_2)\) and their corresponding random variables \(d_{rf}\) are shown in Table 5.6. As mentioned previously, some of these random variables are equal and some are always larger than others. Ignoring the identical rvs, only \(N = 4\) remain that must be considered to find the minimum distance of the relay constellation given in (5.34). These 4 dependent rvs, \(X_n, 1 \leq n \leq 4\), and the corresponding values of \(v_n^2\) and \(a_{i,n}\) are given in Table 5.7.

Instead of considering the minimum of the 4 dependent rvs given in Table 5.7, \(d_{\min,rf}\) can be approximated from (5.43) by assuming independent random variables
Table 5.8: Actual and approximate values of $\bar{d}_{\text{min},t}$ in 2-user BPSK for two precoding vectors

<table>
<thead>
<tr>
<th></th>
<th>$(\phi_1, \phi_2) = (1, j)$</th>
<th>$(\phi_1, \phi_2) = (1, 2j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 dependent rvs (from simulation)</td>
<td>$\bar{d}_{\text{min},t} = 1.069$</td>
<td>$\bar{d}_{\text{min},t} = 1.422$</td>
</tr>
<tr>
<td>4 independent rvs (from (5.51))</td>
<td>$\bar{d}_{\text{min},t} \approx 1.023$</td>
<td>$\bar{d}_{\text{min},t} \approx 1.379$</td>
</tr>
</tbody>
</table>

and taking their minimum. From Table 5.7 and (5.46), $\beta$ in (5.49) then becomes

$$\beta = \frac{1}{|\phi_1|^2} + \frac{1}{|\phi_2|^2} + \frac{2}{|\phi_1|^2 + |\phi_2|^2}. \quad (5.52)$$

Table 5.8 shows the approximate values of $\bar{d}_{\text{min},t}$ given by (5.51) for two different precoding vectors, as well as the actual values of $\bar{d}_{\text{min},t}$ obtained by simulation with 4 dependent rvs. As expected, the average minimum of the independent rvs is a tight lower bound for the corresponding average for dependent rvs.

### 5.4.2 Two Users Employing BPSK and 8-QAM

If $K = 2$ users communicate in an MWRC with the first employing BPSK and the second employing rectangular 8-QAM, then $a_1$ and $a_2$ can take 3 and 21 different values as shown in Tables 5.1 and 5.3, respectively. Therefore, 63 different combinations of $(a_1, a_2)$ are possible in (5.5), each resulting in a random variable $d_{\text{rf}}$, but only $N = 22$ rvs need to be considered to find the minimum distance. These 22 dependent rvs, $X_n$, $1 \leq n \leq 22$, and the corresponding values of $v^2_n$ and $a_{i,n}$ are shown in Table 5.9. From Table 5.9, (5.46), and (5.49) we have

$$\beta = \frac{1}{|\phi_1|^2} + \frac{1}{|\phi_2|^2} + \frac{4}{|\phi_1|^2 + |\phi_2|^2} + \frac{4}{|\phi_1|^2 + 2|\phi_2|^2} + \frac{4}{|\phi_1|^2 + 4|\phi_2|^2} + \frac{2}{|\phi_1|^2 + 5|\phi_2|^2} + \frac{2}{|\phi_1|^2 + 9|\phi_2|^2} + \frac{2}{|\phi_1|^2 + 10|\phi_2|^2}. \quad (5.53)$$

Table 5.10 shows the approximate values of $\bar{d}_{\text{min},t}$ given by (5.51) for two different precoding vectors. It also presents the actual values of $\bar{d}_{\text{min},t}$ obtained by simulation with dependent rvs. This also verifies that the average minimum of the independent random variables is a tight lower bound for the corresponding average of the dependent rvs.
### Four Users Employing BPSK

For $K = 4$ users employing BPSK symbols in an MWRC, $a_i$ can take 3 different values as shown in Table 5.1. Therefore, 81 different combinations of $(a_1, a_2, a_3, a_4)$ are possible in (5.5), each resulting in a random variable $d_{ri}^2$, but only $N = 40$ of these dependent rvs need to be considered to obtain the minimum distance. By assuming these 40 random variables are independent, $d_{\text{min}_r}$ can be approximated by considering their minimum. The distribution and mean of $d_{\text{min}_r}$ in an MWRC with
Table 5.10: Actual and approximate values of $\bar{d}_{\text{min}_r}$ in 2-user BPSK and 8-QAM for two precoding vectors

<table>
<thead>
<tr>
<th></th>
<th>$(\phi_1, \phi_2) = (1, 1)$</th>
<th>$(\phi_1, \phi_2) = (1, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22 dependent rvs (from simulation)</td>
<td>$\bar{d}_{\text{min}_r} = 0.724$</td>
<td>$\bar{d}_{\text{min}_r} = 1.094$</td>
</tr>
<tr>
<td>22 independent rvs (from (5.51))</td>
<td>$\bar{d}_{\text{min}_r} \approx 0.672$</td>
<td>$\bar{d}_{\text{min}_r} \approx 1.031$</td>
</tr>
</tbody>
</table>

Table 5.11: Actual and approximate values of $\bar{d}_{\text{min}_r}$ in 4-user BPSK for two precoding vectors

<table>
<thead>
<tr>
<th></th>
<th>$\phi = (1, j, 1, j)$</th>
<th>$\phi = (1, j, 2, 2j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 dependent rvs (from simulation)</td>
<td>$\bar{d}_{\text{min}_r} = 0.441$</td>
<td>$\bar{d}_{\text{min}_r} = 0.643$</td>
</tr>
<tr>
<td>40 independent rvs (from (5.51))</td>
<td>$\bar{d}_{\text{min}_r} \approx 0.427$</td>
<td>$\bar{d}_{\text{min}_r} \approx 0.612$</td>
</tr>
</tbody>
</table>

4 users employing BPSK are given by (5.50) and (5.51), respectively, with

$$\beta = \sum_{i=1}^{4} \frac{1}{|\phi_i|^2} + \sum_{i=1}^{3} \frac{2}{|\phi_i|^2 + |\phi_{i+1}|^2} + \sum_{i=1}^{2} \frac{2}{|\phi_i|^2 + |\phi_{i+2}|^2} + \frac{2}{|\phi_1|^2 + |\phi_4|^2}$$

$$+ \sum_{i=1}^{2} \frac{4}{|\phi_i|^2 + |\phi_{i+1}|^2 + |\phi_{i+2}|^2} + \frac{4}{|\phi_1|^2 + |\phi_3|^2 + |\phi_4|^2} + \frac{4}{|\phi_1|^2 + |\phi_2|^2 + |\phi_4|^2}$$

$$+ \frac{8}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2}. \quad (5.54)$$

Table 5.11 shows the approximate values of $\bar{d}_{\text{min}_r}$ given by (5.51) for two precoding vectors. It also presents the actual values of $\bar{d}_{\text{min}_r}$ obtained by simulation with 40 dependent rvs.

### 5.5 Bounds on the Symbol Error Probability in Rayleigh Fading

By exploiting the average minimum distance of the relay constellation provided in Section 5.4, a tight upper bound on the SEP for an MWRC with Rayleigh fading channels is obtained in this section. This is achieved using a nearest-neighbor approximation. The effect of the precoding vectors on the SEP is also examined.
Constellation points $r$ and $r'$ are said to be nearest neighbors if the distance between them equals $d_{\text{min}}$. The nearest-neighbor approximation for the SEP is then given by [78, Ch. 4]

$$P_e \approx \bar{N}_{d_{\text{min}}r} \Pr \left( r \xrightarrow{d_{\text{min}}r} r' \right),$$  \hspace{1cm} (5.55)

where $\Pr(r \xrightarrow{d_{\text{min}}r} r')$ is the PEP between the nearest neighbor points of the constellation and $\bar{N}_{d_{\text{min}}r}$ denotes the average number of nearest neighbors for a signal point. The PEP for the nearest neighbor points can be written as

$$\Pr(r \xrightarrow{d_{\text{min}}r} r') = \Pr \left( n > \frac{d_{\text{min}}r}{2} \right) = Q \left( \sqrt{\frac{d_{\text{min}}r^2}{2N_0}} \right) \approx Q \left( \sqrt{\frac{d_{\text{min}}r^2}{2N_0} d^2} \right) = Q \left( \sqrt{\eta_p \rho d^2} \right),$$  \hspace{1cm} (5.56)

where the approximation and last equality are from (5.43) and definition of $\eta_p$, respectively, and $d$ is a random variable with distribution given by (5.48). By averaging (5.56) over $d$, the PEP for the nearest neighbor points is given by

$$\Pr(r \xrightarrow{d_{\text{min}}r} r') = \mathbb{E}_d \left[ Q \left( \sqrt{\eta_p \rho d^2} \right) \right] = \int_0^\infty Q \left( \sqrt{\eta_p \rho y^2} \right) 2 \beta y e^{-\beta y^2} dy$$

\begin{align*}
&= \frac{1}{2} \left( 1 - \sqrt{\frac{\eta_p \rho}{2 \beta + \eta_p \rho}} \right), \hspace{1cm} (5.57)
\end{align*}

with $\beta$ given in (5.49). The three examples are now considered.

### 5.5.1 Two Users Employing BPSK

With $K = 2$ users and BPSK signaling in an MWRC, a 4-ary constellation is formed at the relay. $\eta_p$ and $\beta$ for this constellation are given by (5.19) and (5.52), respectively. Substituting these values into (5.57) gives the PEP for the nearest neighbor points as

$$\Pr(r \xrightarrow{d_{\text{min}}r} r') = \frac{1}{2} \left( 1 - \sqrt{\frac{2|\phi_1|^2|\phi_2|^2 \rho}{(|\phi_1|^4 + 4|\phi_1|^2|\phi_2|^2 + |\phi_2|^4) + 2|\phi_1|^2|\phi_2|^2 \rho}} \right).$$  \hspace{1cm} (5.58)

$\bar{N}_{d_{\text{min}}r}$ can be obtained by simulation. For each constellation point at the relay, the number of points with minimum Euclidean distance from it was calculated. Then this number was averaged over all constellation points for a large number of iterations to
Figure 5.8: Exact uplink transmission SEP and its nearest-neighbor approximation in an MWRC with 2 users employing BPSK for two different precoding vectors.

get \( \bar{N}_{d_{\text{min}_r}} \). This was done for two different precoding vectors. For \( \phi = (1, j)^T \), \( \bar{N}_{d_{\text{min}_r}} = 0.855 \) was obtained. Therefore, (5.55) and (5.58) give

\[
P_e \approx \frac{0.855}{2} \left( 1 - \sqrt{\frac{\rho}{3 + \rho}} \right).
\]

For \( \phi = (1, 2j)^T \), \( \bar{N}_{d_{\text{min}_r}} = 0.896 \) was obtained which gives

\[
P_e \approx \frac{0.896}{2} \left( 1 - \sqrt{\frac{8\rho}{33 + 8\rho}} \right).
\]

Fig. 5.8 presents the SEP approximation from (5.59) and (5.60) along with the exact SEP obtained by simulation. The SEP of QPSK in Rayleigh fading is also included for comparison.

### 5.5.2 Two Users Employing BPSK and 8-QAM

With \( K = 2 \) users with BPSK and 8-QAM signalings in an MWRC, a 16-ary constellation is received at the relay. Substituting (5.26) and (5.53) in (5.57) gives the
PEP for the nearest neighbor points as

\[ \Pr(r \xrightarrow{d_{\text{min}rf}} r') = \frac{1}{2} \left( 1 - \sqrt{\frac{\lambda_1 \rho}{\lambda_2 + \lambda_1 \rho}} \right), \]

(5.61)

where \( \lambda_1 = 4|\phi_1|^2|\phi_2|^2(|\phi_1|^2+|\phi_2|^2)(|\phi_1|^2+2|\phi_2|^2)(|\phi_1|^2+5|\phi_2|^2)(|\phi_1|^2+9|\phi_2|^2)(|\phi_1|^2+10|\phi_2|^2) \) and \( \lambda_2 = (|\phi_1|^2+6|\phi_2|^2)(|\phi_1|^4+3600|\phi_2|^4+14|\phi_1|^2|\phi_2|^2(13|\phi_1|^6+9975|\phi_2|^{10}) + 8|\phi_1|^4|\phi_2|^4(115|\phi_1|^6+7011|\phi_2|^6) + 2|\phi_1|^6|\phi_2|^6(3736|\phi_1|^2 + 14937|\phi_2|^2)). \)

For a given precoding vector, \( \bar{N}_{d_{\text{min}rf}} \) can be obtained by simulation. For \( \phi = (1, 1)^T \), we get \( \bar{N}_{d_{\text{min}rf}} = 0.861 \), and the corresponding SEP from (5.55) and (5.61) is

\[ P_e \approx \frac{0.861}{2} \left( 1 - \sqrt{\frac{220\rho}{2681 + 220\rho}} \right). \]

(5.62)

For \( \phi = (1, 2)^T \), \( \bar{N}_{d_{\text{min}rf}} = 0.863 \) is obtained and the corresponding SEP is

\[ P_e \approx \frac{0.863}{2} \left( 1 - \sqrt{\frac{25995312\rho}{479968135 + 25995312\rho}} \right). \]

(5.63)

Fig. 5.9 presents the SEP from (5.62) and (5.63) along with the exact SEP obtained by simulation. The SEP of 16-QAM in fading is also included for comparison.

### 5.5.3 Four Users Employing BPSK

With \( K = 4 \) users and BPSK signaling in an MWRC, a 16-ary constellation is received at the relay. By substituting (5.30) and (5.54) in (5.57), the PEP of the nearest neighbor points as a function of the precoding vector is obtained. For \( \phi = (1, j, 1, j)^T \), we get \( \bar{N}_{d_{\text{min}rf}} = 0.515 \) and the corresponding SEP is

\[ P_e \approx \frac{0.515}{2} \left( 1 - \sqrt{\frac{3\rho}{52 + 3\rho}} \right). \]

(5.64)

For \( \phi = (1, j, 2, 2j)^T \), \( \bar{N}_{d_{\text{min}rf}} = 0.581 \) is obtained and the corresponding SEP is

\[ P_e \approx \frac{0.581}{2} \left( 1 - \sqrt{\frac{72\rho}{1507 + 72\rho}} \right). \]

(5.65)

Fig. 5.10 shows the nearest-neighbor approximation of the SEP from (5.64) and (5.65) as well as the exact SEP obtained by simulation. The SEP of 16-QAM in fading is
Figure 5.9: Exact uplink transmission SEP and its nearest-neighbor approximation in an MWRC with 2 users employing BPSK and 8-QAM for two different precoding vectors.

also included for comparison.

The examples above show that the nearest-neighbor approximation provided is a tight upper bound on the SEP at sufficiently high SNRs, and this holds for all of the precoding vectors considered. In addition, the SEP expressions (e.g. (5.58) or (5.61)) are convex functions of the precoding values. Therefore, they can be minimized with respect to the precoding vector by setting the gradient equal to zero. This gives that the lowest average SEP is achieved when the precoding vector values have the same magnitude.

5.6 Summary

In this chapter, the performance of a multi-way relay channel communication network, where the links between the users and the relay could have different SNRs, was considered. Analytical expressions for exact pairwise error probability of an MWRC in Rayleigh fading was derived. Then the probability density function and the mean of the minimum Euclidean distance of the relay constellation were accurately approximated. Furthermore, it was debated that this approximation would always be a lower
bound to the actual value. Finally, by exploiting the approximation of minimum Euclidean distance, a tight upper bound on the symbol error probability was developed using a nearest-neighbor approximation.
Chapter 6

Conclusion and Future Work

The main focus of this dissertation was proposing transmission schemes for full data exchange in multi-way relay channel networks and improving the throughput and the performance of transmission in such networks. The performance of MWRCs for different channel types including AWGN and Rayleigh fading channels was also analyzed.

Designing a new transmission scheme for BPSK modulation in MWRCs was the focus of Chapter 3. The proposed algorithm provides a throughput of $\frac{1}{3N}$ sym/U/CU which is a 33% improvement over plain routing. This shows that physical-layer network coding can also be beneficial in systems with more than two user nodes. Besides having low complexity, this algorithm can easily be scaled to higher numbers of users. It can also be employed with QPSK modulation, which also provides a 33% gain. This is achieved by separately (and concurrently) dealing with the inphase and quadrature components of QPSK symbols. Furthermore, in this chapter, the average total user energy consumption with PNC relaying was studied, and its power efficiency and error performance were compared with plain routing. It was shown that although plain routing has a lower throughput than PNC relaying, there is performance degradation with PNC relaying. Therefore, there is a trade-off between power and spectral efficiencies in selecting a relaying scheme.

The idea of restraining some users from transmitting in order to increase throughput can be applicable to other multi-source scenarios and can be explored further. Additionally, the concept of using common knowledge in relay communications should find applications in other communication systems.

In Chapter 4, a multi-way relay channel in which clusters of users exchange data was considered. It was shown that a throughput of $\frac{1}{2}$ sym/U/CU can be achieved
in each cluster by applying complex field network coding. To implement CFNC, precoding must be employed with each user. Some precoding vectors could lead to small Euclidean distances between constellation points resulting in poor performance. A precoding vector for full data exchange in MWRC was developed such that the reception of a rectangular QAM constellation would be ensured at the relay. The average transmit power of a rectangular QAM constellation is only slightly greater than that of an optimal $M$-ary QAM constellation and the corresponding signals are easier to demodulate. These vectors have the flexibility to accommodate users with different constellations and allow them to join or leave the network at any time.

Then the error performance of each user in an MWRC was analyzed. It was debated that the symbols of users employing larger precoding values experience lower error probabilities. Furthermore, the performance of CFNC was compared with plain routing and PNC relaying, and it was shown that from PR to PNC and CFNC, although the throughput of the network increases, its error performance degrades.

Later in the chapter, a more general problem was considered where each user employed an arbitrary PAM or rectangular QAM constellation, and the optimal precoding values were derived such that the power efficiency of the relay QAM constellation would be highest compared to the efficiency with other precoding values.

Finally, it was shown that by employing CFNC and exploiting user self-information, the size of constellation for the downlink broadcast transmission could be decreased, which led to both downlink and uplink error performance improvement in both AWGN and Rayleigh fading channels.

In Chapter 5, the performance of an MWRC with fading channels was considered. The exact average pairwise error probability of an MWRC in Rayleigh fading was obtained and it was verified through different examples. The problem of evaluating the minimum Euclidean distance of the relay constellation in an MWRC is a hard problem as lots of dependent random variables are involved. This problem was tackled by closely approximating the probability density function and the mean of the minimum Euclidean distance of the relay constellation. Simulation results showed that the analytical approximation of the minimum distance was very close to its actual value. Furthermore, it was debated that this approximation would always be a lower bound to the actual value. Finally, by exploiting the lower bound approximation of minimum Euclidean distance, a tight upper bound on the symbol error probability was developed using a nearest-neighbor approximation followed by different examples which showed its validity.
6.1 Future Research Directions

In the following, some research problems are proposed to be studied to extend the scope of the presented work.

6.1.1 MWRC with Amplify and Forward Relaying in an MWRC with Nakagami Fading

In Chapter 5, the uplink performance of an MWRC was studied in which the relay demodulates the received signal. Another scenario can be considered in which the relay simply amplifies the received signal and forwards it back to the user nodes without denoising the signal. Therefore, the received signal by each user $i$ is given by

$$r_i = \left( \sum_{i=1}^{K} h_i \phi_is_i + n_1 \right) h_i + n_{2,i}, \quad (6.1)$$

where $n_1$ and $n_{2,i}$ are the uplink and downlink AWGN at the relay and user $i$, respectively. The SEP of the users in this amplify and forward scenario can be studied.

If an MWRC with $K$ users each employing an $M$-ary constellation is considered, then the size of the constellation received at the relay and each user will be $M^K$. To improve the performance, each user can exploit the self-information and subtract it from the received signal. In this way, the constellation size for signal detection will be reduced to $M^{K-1}$ for each user node.

Rayleigh distribution cannot always capture the underlying physical properties of the channel models, e.g. when one strong direct line of sight component exists in propagation paths. A more general fading distribution which can be adjusted to fit a variety of empirical measurements is the Nakagami fading distribution. Therefore, the performance in an MWRC with a Nakagami fading distribution should be studied.

6.1.2 MWRC with Power Constraint Equalization

In Chapter 4, it was assumed that in a slow fading environment the channel state information is provided at the user nodes via training or pilot symbols. It was also assumed that the users perform perfect pre-equalization before transmitting their signals to the relay. In this way, the channels between the users and relay can be modeled as AWGN. The assumption of perfect pre-equalization is not very realistic
since the transmitter should devote an unlimited amount of power to compensate
the channel effect in deep fades. Therefore, the average transmitted power will be
unbounded. More precisely, in a point-to-point communication with a fading gain
\( h \sim CN(0, 1) \), the received signal is given by

\[
y = c(h) \times h \times s,
\]

(6.2)

where \( s \) is the signal to be transmitted and \( c(h) \) is the pre-equalization coefficient. If
the pre-equalization coefficient corrects the channel phase shift, or the channel phase
shift is corrected at the receiver by using the channel state information (CSI), then
(6.2) can be written as

\[
y = c(g) \times g \times s,
\]

(6.3)

where \( g = |h| \).

The average transmitted power can then be calculated as

\[
\bar{P} = \int_{0}^{\infty} c^2(g) \ast f_G(g) \, dg,
\]

(6.4)

where \( f_G(g) = 2ge^{-g^2} \) is the probability distribution of \( g \), and the average power of \( s \)
is supposed 1 for simplicity. Considering the perfect pre-equalization case \( c(g) = \frac{1}{g} \),
the integral in (6.4) becomes unbounded which implies that infinite average transmit
power is required.

A more realistic scenario is to consider a power constraint equalization, where
the average transmit power of a user cannot exceed \( P \). Time-division duplex multi-
carrier code-division multiple-access (TDD/MC-CDMA) channel pre-compensation
techniques with power constraint condition are studied in [93]. Considering a single
point-to-point channel with average transmit power \( P \), the results in [93] suggest the
pre-equalization coefficient should be \( c(g) = \sqrt{P} \). This means the transmitter should
use its maximum power at all times to compensate the fading as much as it can.
However, a more power efficient way is to devote more transmit power to a poorer
channel condition, i.e., smaller \( g \). Therefore, \( c(g) \) should be a decreasing function
with respect to \( g \) with \( c(\infty) = 0 \). Furthermore, \( c(g) \) must satisfy (6.4) with \( \bar{P} = P \).
Many of such functions may be found. Some examples are
\[ c(g) = \sqrt{mP} e^{-ng}, \]  
(6.5)
or a piece-wise function
\[
c(g) = \begin{cases} 
\sqrt{mP}, & 0 \leq g \leq \alpha \\
\sqrt{mP} \frac{\alpha}{g}, & g > \alpha 
\end{cases} 
\]  
(6.6)
where \(m\), \(n\), and \(\alpha\) are appropriately selected positive real numbers to satisfy (6.4) with \(\bar{P} = P\). The average transmitted power of (6.6) is calculated from (6.4) as
\[
\bar{P} = mP x(\alpha) = mP \left( 1 - e^{-\alpha^2} + \alpha^2 \Gamma(0, \alpha^2) \right), 
\]  
(6.7)
where \(\Gamma(0, \alpha^2)\) is the incomplete gamma function.

As an example, consider that \(s\) is a BPSK signal. From (6.6) and (6.7), \(c(g)\) is given by
\[
c(g) = \begin{cases} 
c_1(g) = \sqrt{\frac{P}{x(\alpha)}}, & 0 \leq g \leq \alpha \\
c_2(g) = \sqrt{\frac{P}{x(\alpha)} \frac{\alpha}{g}}, & g > \alpha 
\end{cases} 
\]  
(6.8)
such that \(\bar{P} = P\). Due to the power constraint condition residual fading remains. By assuming a coherent receiver, the SEP for the BPSK system with the power constraint pre-equalization factor given by (6.8) can be calculated as
\[
P_s = \int_0^\alpha Q \left( \sqrt{\frac{2E_b}{N_0} c_1^2(g) g^2} f_G(g) \right) dg + \int_\alpha^\infty Q \left( \sqrt{\frac{2E_b}{N_0} c_2^2(g) g^2} f_G(g) \right) dg
\]
\[
= 0.5 \left( 1 - \sqrt{\frac{P E_b}{x(\alpha) + P E_b N_0}} + \sqrt{\frac{P E_b N_0}{x(\alpha) + P E_b N_0}} Q \left( \alpha \sqrt{2 \left( 1 + \frac{P E_b}{N_0} \right)} \right) \right). 
\]  
(6.9)

If the constant pre-equalization factor \(c(g) = \sqrt{P}\) is used instead of (6.8), the same average transmit power is obtained and the corresponding SEP is given by
\[
P_s = 0.5 \left( 1 - \sqrt{\frac{P E_b N_0}{1 + P E_b N_0}} \right). 
\]  
(6.10)
Fig. 6.1 shows the SEP from (6.9) for different values of $\alpha$ and also from (6.10), both with $P = 4$. From the figure it can be seen that by selecting an appropriate pre-equalization factor, $c(g)$, that depends on the channel condition, a better performance can be achieved in most SNRs compared to the case with the constant pre-equalization factor.

The performance of an MWRC with power constraint pre-equalization can also be studied in a similar way.

### 6.1.3 Exploiting Constellation Rotation in MWRC

The diversity order of a signal set is the minimum number of distinct components between any two constellation points. In other words, the diversity order is the minimum Hamming distance between any two coordinate vectors of constellation points. This kind of diversity is termed modulation diversity or signal space diversity [94]. Rotating QAM constellations with the aim of increasing the diversity order of the signal set has been studied in [94]. To gain from modulation diversity, interleaver/deinterleaver pair is required to assure that the inphase and quadrature components of the received symbol are affected by independent fading. This idea is illustrated in Fig. 6.2. After
performing interleaving/deinterleaving, a deep fade hits only one of the components of the transmitted signal. Then the compressed constellation in Fig. 6.2(b) (empty circles) offers more protection against noise than the one in Fig. 6.2(a), since no two points collapse together [94].

It has been shown that this simple operation results in a gain of 8 dB at SEP of $10^{-3}$ over the traditional 4-PSK. This uncoded modulation scheme enables one to trade diversity for system complexity at no power or bandwidth expense.

The performance of the rotate and forward protocol in the two-hop relay channels has been studied in [95]. The question of how constellation rotation can be benefitted in an MWRC to improve the performance is an open problem to be studied.

6.1.4 Diversity Gain in MWRC

Another research problem that can be considered is the diversity gain in the uplink transmission of an MWRC. As an example suppose $K$ user nodes $U_1, U_2, \ldots, U_K$ want to transmit their independent data messages to the destination relay node $R$. Considering the conventional point-to-point transmission scheme, each user sends its message to the relay $R$ in a different time-slot, therefore, totally $K$ time-slots are required to transmit all user messages. As discussed in the previous chapters, with CFNC, all users transmit their signals to the relay simultaneously. If all users employ an $M$-ary constellation, then the size of the received constellation at the relay will be $M^K$. The relay node detects a point in this constellation and it can decode all user messages from it. Therefore, only one channel use is enough to transmit all user messages to the relay. Although the throughput of the transmission with CFNC is $K$-fold better than with conventional transmission, its performance is worse.

In order to have a fair comparison with the conventional transmission, CFNC with
the same throughput should be considered. This can be achieved by transmitting with diversity order $K$, e.g. all users transmit their signals simultaneously to the relay and they repeat it $K$ times. In this way, each user sends the same message to the relay $K$ times and hence the throughput of the transmission with CFNC will be the same as the throughput of the conventional transmission. By performing interleaving/deinterleaving, channel gains at each transmission phase can be assumed independent from another phase. Now the question is how much improvement in the error performance of the signal detection can be achieved in this way for CFNC and if the performance can be made better than the conventional transmission.

By increasing $K$, there is one factor that helps to improve the performance and one factor that worsens it. When $K$ is increased, the diversity order increases which improves the performance. On the other hand, increasing $K$ increases the constellation size received at the relay which makes the detection performance worse (by assuming the same received energy).

As an alternative to time diversity, another kind of diversity can be considered. For example, instead of transmitting in $K$ different time-slots, spatial diversity can be employed in which the relay applies $K$ different antennas to receive the transmitted signals. This problem can be studied in a more general way. Consider $P$ users, with $1 \leq P \leq K$. They transmit their signals simultaneously to the relay and this is repeated in a circular fashion $K$ times. More specifically, in the $i$th round of transmission, users $U_i, U_{i+1}, \ldots, U_{i+P-1}$ transmit their messages to the relay simultaneously for $1 \leq i \leq K$. In this way each user sends the same message to the relay $P$ times. For $P = 1$, the conventional transmission case and for $P = K$, the CFNC transmission example discussed above is modeled.
Bibliography


